RQF LEVEL 4

GENBA402

FOOD AND BEVERAGES OPERATIONS,TOURISM

Applied Mathematics

TRAINEE'S MANUAL

October, 2024

BASIC MATHEMATICAL ANALYSIS

2024

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Under Rwanda TVET Board (RTB) guiding policies and directives

Under Financial and Technical support of

COORDINATION TEAM

RWAMASIRABO Aimable

MARIA Bernadette M. Ramos

MUSABYIMANA Marie Jeanne

Production Team

Authoring and Review

UFITINEMA Bonaventure

Validation

NYANDWI Fidele

Conception, Adaptation and Editorial Works

HATEGEKIMANA Olivier

GANZA Jean Francois Regis

NIKUZE Bernadette

MANISHIMWE Marc

Formatting, Graphics, Illustrations, and Infographics

NYINAWUMUNTU Gaudence

Financial and Technical Support

KOICA through TQUM Project

TABLE OF CONTENT

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

CBET: Competence Base Education and Training

PPE: Personal Protective Equipment

RQF: Rwanda Qualification Framework

RS: Rwandan Standard

RSB: Rwanda Standards Board

RTB: Rwanda TVET Board

TVET: Technical and Vocational Education and Training

TQUM: TVET Quality Management Project

$\boldsymbol{R(f)}$**:** Range of function

$\boldsymbol{Domf}$**:** Domain of function

$\boldsymbol{V.A}$**:** Vertical asymptote

$\boldsymbol{H.A}$**:** Horizontal asymptote

O.A: Oblique asymptote

T: Tangent line

N: Normal line

INTRODUCTION

This trainee's manual encompasses all necessary skills, knowledge and attitudes required to Apply Basic Mathematical Analysis. Students undertaking this module shall be exposed to practical activities that will develop and nurture their competences. The writing process of this training manual embraced competency-based education and training (CBET) philosophy by providing practical opportunities reflecting real life situations.

The trainee's manual is subdivided into units, each unit has got various topics, you will start with a self-assessment exercise to help you rate yourself on the level of skills, knowledge and attitudes about the unit.

A discovery activity is followed to help you discover what you already know about the unit.

After these activities, you will learn more about the topics by doing different activities by reading the required knowledge, techniques, steps, procedures and other requirements under the key facts section, you may also get assistance from the trainer. The activities in this training manual are prepared such that they give opportunities to students to work individually and in groups.

After going through all activities, you shall undertake progressive assessments known as formative and finally conclude with your self-reflection to identify your strengths, weaknesses and areas for improvement.

Do not forget to read the point to remember the section, which provides the overall key points and takeaways of the unit.

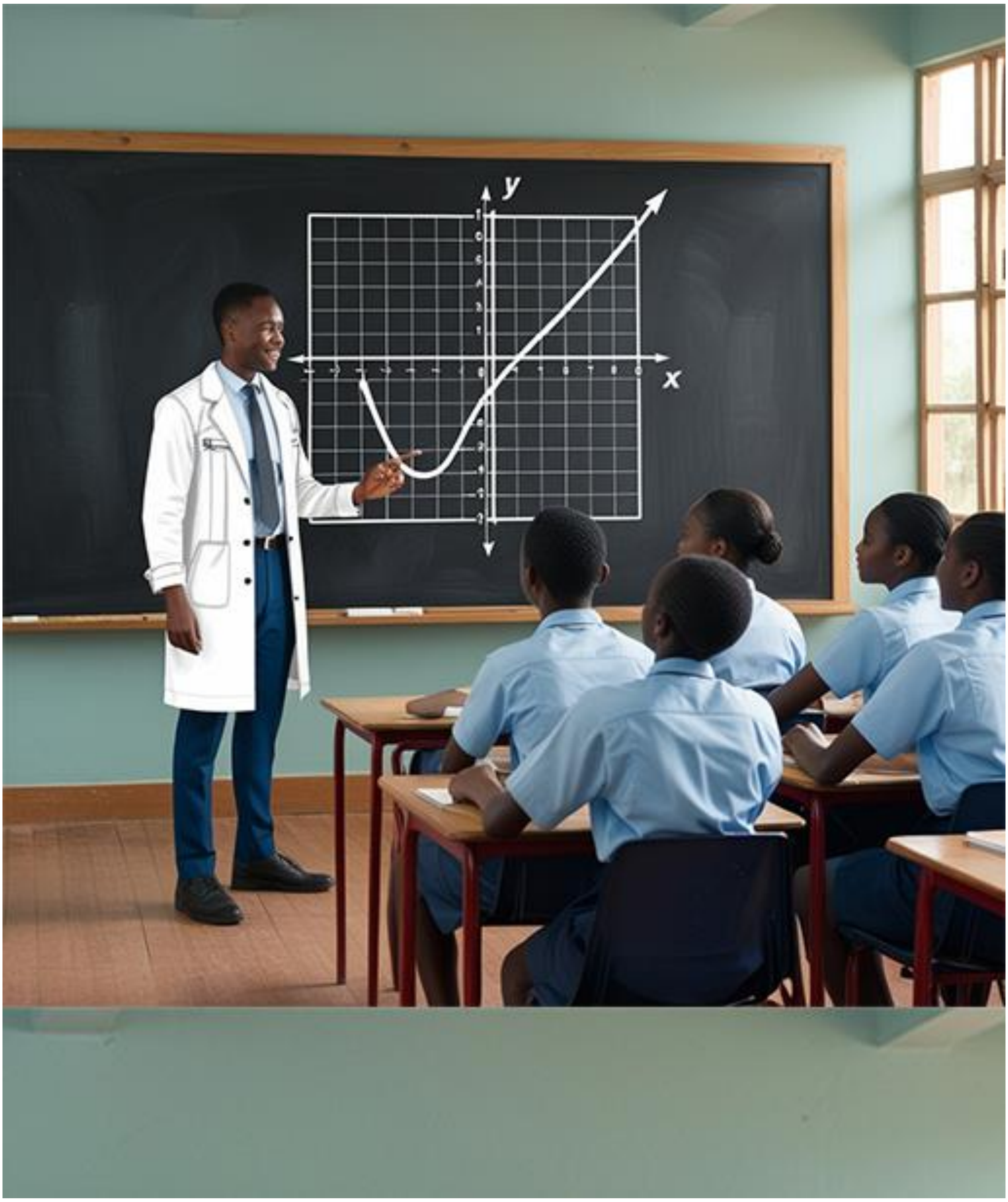
Module Units:

Unit 1: Solve algebraically or graphically linear and quadratic equations or inequalities.

Unit 2: Analyze algebraic functions

Unit 3: Apply fundamentals of differentiation.

UNIT 1: Solve Algebraically or Graphically Linear and Quadratic Equation



Self-Assessment: Unit 1

Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Describe linear equations and inequality					
Solve linear equations and inequality using algebraic and graphical method					
Critical thinking in describing linear equations and inequality					
Analytical spirit and effectiveness in Solving linear equations and inequality.					
Describe simultaneous linear equations					
Solve the simultaneous linear equations using different methods such substitution, elimination, comparison.					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Critical thinking and cooperation in describe simultaneous linear equations.					
Attention in using different methods for Solving the simultaneous linear equations					
Describe quadratic equations					
Solve the quadratic equation using various convenient methods such factoring, Square root property, completing the square, or using quadratic formula.					
Critical thinking in describing quadratic equations					

Key Competencies:

Knowledge	Skills	Attitudes
1. Describe linear equations and inequality	**1.** Solve linear equations and inequality using algebraic and graphical method	**1.** Critical thinking in describing linear equations and inequality **2.** Analytical spirit in solving linear equations and inequality **3.** Effectiveness in solving linear equations and inequality
2. Describe simultaneous linear equations	**2.** Solving the simultaneous linear equations using different methods such substitution, elimination, comparison.	**1.** Critical thinking in describing simultaneous linear equations. **2.** Attention in using different methods for solving the simultaneous linear equations. **3.** Cooperation in describing simultaneous linear equations.
3.Describe quadratic equations	**3.** Solve the quadratic equation using various convenient methods such factoring, Square root property, completing the square, or using quadratic formula.	**1.** Critical thinking in describing quadratic equations **2.** Analytical spirit in solving in solving quadratic equations **3.** Effectiveness in solving quadratic equations.

Discovery activity:

Task 1:

1. What do you understand by a linear equation?
2. The sum of two numbers is 10, and their difference is 6. Make a pair of equations.
3. What did you know about solving equations and inequalities using different methods?
4. Are your algebraic and graphical solutions the same?
5. Which method did you prefer for solving the quadratic? Why?
6. Given the quadratic equation: $\mathrm{f(x)} = x^2 + 5x + 6$ State the function whether it is up word and down and why?
7. Distinguish between linear equation and linear inequality.
8. What do you understand by two simultaneously linear equations?
9. Distinguish between quadratic equation and quadratic inequality

Topic 1.1: Solving linear equations and inequalities

Activity 1: Problem Solving

Task 2:

1. What does it mean a term linear equation?
2. State a degree of variable in linear inequality.
3. What does it mean a linear inequality?
4. Distinguish between linear equation and linear inequality.

Key Facts 1.1a: solving Linear equations and inequalities

✓ **linear equations**

A linear equation is an algebraic equation in which each term has an exponent of one. The graphing of the equation results is a straight line when graphed on a coordinate plane and one or more variables in equation.

In general, linear equation in one unknown is of the form $ax+b=0$, with a and b are constants,x is the unknown variable and a is a coefficient (constant) that is not equal to zero.

Examples of linear equations with one variable:

- $2x + 6 = 0$
- $3x - 9 = -15$

✓ **linear inequality**

The statement $x+3=10$ is true only when $x=7$. If x is replaced by another number (for example 5), the statement is false. To be true we may say that 5 + 3 is less than 10 or in symbol $5+3<10$. In this case, we no longer have equality but **inequality**. Suppose that we have the inequality $x+3<10$, in this case, we have an inequality with one unknown. Here the real value of x to satisfy this inequality is not unique. For example, 1 is a solution but 3 is also a solution. In general, all real numbers less than 7 are solutions. In this case

Now, the solution set of $x+3<10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. Solving the inequality $x+3<10$, one can easily find that the values of x are less than 7. Mathematically is written as follow: $S=]-\infty,7[$

In general, inequalities in one unknown are of the forms: $ax+b>0$, $ax+b<0$, $ax+b\geq 0$, $ax+b\leq 0$, with a and b constant and $a\neq 0$

.

Recall that:

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.
- The direction of the inequality is not **changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

Examples of linear inequality with one variable:

- $2x + 6 > 0$
- $3x - 9 < -15$

✓ **Parametric equation**

In case certain coefficients of linear equations contain one or several letter variables, the equation is called **parametric** and the letters are called **real parameters.** In this case, we discuss the equation (for parameters only).

Examples of parametric linear equations with one variable:

- $mx + 6 = 0$
- $ax - 9 = -15$

Key Facts 1.1b: Solving linear equations and inequalities

✓ **Linear equations**

Algebraic method

An equation is a statement in which the values of two mathematical expressions are equal. Consider the statement $ax + b = 0$, where $a, b \in \mathbb{R}$ and $a \neq 0$.

This statement is true when $x = -\frac{b}{a}$ (the solution or the root of the equation $ax + b = 0$). Thus, to find a solution to the given equation is to find the value that satisfy that equation.

Examples: Solve in set of real numbers

a) $x + 6 = 14$

b) $4x + 5 = 20 + x$

c) $x = 14 - x$

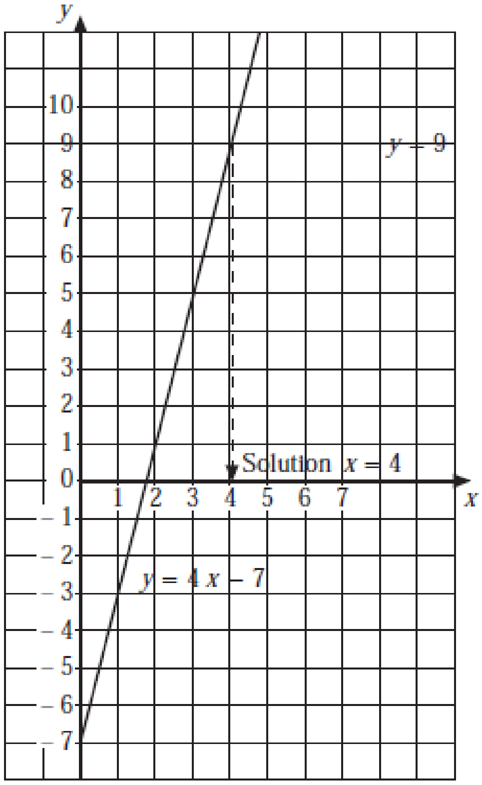
Solutions

a) $x + 6 = 14$ $\Leftrightarrow x = 14 - 6$ $\Rightarrow x = 8$ $S = \{8\}$	b) $4x + 5 = 20 + x$ $\Leftrightarrow 4x - x = 20 - 5$ $\Leftrightarrow 3x = 15$ $\Leftrightarrow x = \frac{15}{3}$ $\Rightarrow x = 5$ $S = \{5\}$	c) $x = 14 - x$ $\Leftrightarrow 2x = 14$ $\Rightarrow x = 7$ $S = \{7\}$

- **Graphical method**

$4x-7=9$

Draw the lines $y=4x-7$ and $y=9$.



The solution is given by the value on the x-axis immediately below the point where $y=4x-7$ and $y=9$ cross.

The solution is $x=4$.

✓ **Linear inequalities**

- **Algebraic method**

They are solved as linear equations except that:

a) When we multiply an inequality by a negative real number the sign will be reversed

b) When we interchange the right side and the left side, the sign will be reversed.

Examples: Algebraically solve the following inequalities in the set of real numbers

Example 1:	Example 2:	Example 3:
$-2x+5\le 0$	$x-4>0$	$x>2x-4$
Solution	***Solution***	***Solution***
$-2x+5\le 0$	$x-4>0$	$x>2x-4$
$\Leftrightarrow -2x\le -5$	$\Leftrightarrow x>4$	$\Leftrightarrow -x>-4$
$\Leftrightarrow x\ge \frac{5}{2}$	$S=\left]4,+\infty\right[$	$\Leftrightarrow x<4$
$S=\left[\frac{5}{2},+\infty\right[$		$S=\left]-\infty,4\right[$

Example 5:

$2x+5 \le 2x+4$

Solution

$2x+5 \le 2x+4 \quad 0x \le -1$

Since any real number times zero is zero and zero is not less or equal to -1 then the solution set is the empty set. $S=\varnothing$

Example 6:

$2(x+5) > 2x-8$

Solution

$2(x+5) > 2x-8$

$\Leftrightarrow 2x+10 > 2x-8$

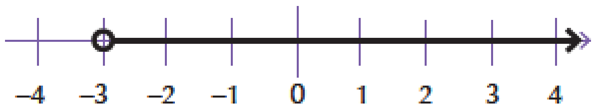
$\Leftrightarrow 0x > -18$

Since any real number times zero is zero and zero is greater than -18, then the solution set is the set of real numbers. $S = \mathbb{R} =]-\infty, +\infty[$

Graphical method

The graph of a linear inequality in one variable is a number line. We use an unshaded circle for < and > and a shaded circle for ≤ and ≥.

The graph for $x > -3$:



Example: Solve the following inequality

$2x - 6 < 2.$

Solution

Add 6 to both sides: $2x - 6 + 6 < 2 + 6$

Divide both sides by 2: $\frac{2}{2}x < \frac{8}{2}$

$x < 4$

Open circle at 4 (since x cannot equal 4) and an arrow to the left (because we want values less than 4).

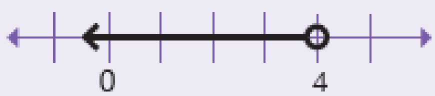


Fig 5.4

- **Discuss parameter in linear equation**

  In this case, we solve and discuss the equation (for parameters only). We can see that after finding the value of x, it follows a discussion so that we can validate the solution.

  Example:

1. Solve and discuss the equation: $(2 - 3m)x + 1 = m^2(1 - x)$

   Solution

$$(2 - 3m)x + 1 = m^2(1 - x)$$
$$2x - 3mx + 1 = m^2 - m^2x$$
$$2x - 3mx + m^2x - m^2 + 1 = 0$$
$$x(2 - 3m + m^2) - m^2 + 1 = 0$$
$$x(2 - 3m + m^2) = m^2 - 1$$
$$x = \frac{m^2 - 1}{2 - 3m + m^2}$$

$$x = \frac{(m-1)(m+1)}{(m-1)(m-2)} = \frac{m+1}{m-2}$$

   If $m = 2$, then there is no solution.

   If $m \neq 2$, then the solution is $x = \frac{m+1}{m-2}$

Activity 2: Guided Practice

Task 3:

1. Solve the following linear equations:
    a. $3(x+7)=0$
    b. $2x-1=x+2$
    c. $2x+6=14$
    d. $3x-9=-15$
2. Solve the following linear inequalities:
    a. $3(x+7)>0$
    b. $2x-1\leq x+2$
    c. $2x+6\geq 14$
    d. $3x-9<-15$
3. Refer to a table below and work out the question that follow: Copy and complete a table below, then sketch a graph of $y=2x-1$

x	-3	-2	-1	0	1	2	3
$y=2x-1$							

Activity 3: Application

Task 4:

1. Solve the following linear equations

1. $3x+11=\ -7$
2. $2(x-5)+3x=4(x-6)+1$
3. $\frac{3}{x-2}=\frac{4}{3x+4}$

2. Solve the following inequalities and write the solution set- in interval notation
    a. $6(x+4)-7-(3x+10)\geq 8(x-1)$
    b. $5x-2(x-3)<3(2x-1)$

$6t+3<3t+12$

3. solve and discuss the following linear equations:
    a. (m+2)x= m+1
    b. x- mx= 24
    c. $(2-3m)x+1=m(1-x)$

Topic 1.2: Solving two simultaneous linear equations

Activity 1: Problem Solving

Task 5:

1. What is a system of simultaneous linear equations in two variables?
2. State the general form of two simultaneous linear equations.
3. Name four methods for solving systems of linear equations in two variables.
4. Which method do you prefer for solving two simultaneous linear equations? Why?

Key Facts 1.2a: Solving Two simultaneous linear equations.

✓ **Two System of linear equations**

A linear equation in two variables x and y is an equation of the form

$ax + by = c \text{ where } a \neq 0 , b \neq 0 \text{ and } a, b, c$ are real numbers.

In general, two simultaneous linear equation are of the forms: $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

where $a_1; b_1; c_1; a_2; b_2; c_1$ are constants or simultaneous linear equations in two unknowns or a system of two linear equations in two unknowns.

We can solve such systems of linear equations by using one of the following methods:

1. substitution method
2. Elimination method
3. Comparison method
4. Graphical method

Comparison method

To find the value of unknown from simultaneous equation by equating the same variable in terms of another, we do the following steps:

i) Find out the value of one variable in first equation
ii) Find out the value of another variable in second equation
iii) Equating the obtained two same variables
iv) Solve the equation to find out the unknown variables
v) Substitute the obtained value of one unknown in one equation to get the second value.

Example

1) Algebraically, solve the simultaneous linear equation by equating the same variables.

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

Solution:

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

From equation (1) $4x+5y=2 \Rightarrow x=\frac{2-5y}{4}$, from equation (2) $x + 2y = -1 \Rightarrow x = -1 - 2y$

Equalize the values of x from equation (1) and (2)

$\frac{2-5y}{4}=-1-2y$

$2 - 5y = 4(-1 - 2y)$

$2-5y=-4-8y$

$-5y + 8y = -4 - 2$

$y = -2$

$x=\frac{2-5y}{4}$, $x=\frac{2-5(-2)}{4}=\frac{12}{4}=3$ then, $S=\{(3,-2)\}$

Elimination method

To eliminate one of the variables from either of equations to obtain an equation in just one unknown, make one pair of coefficients of the same variable in both equations negatives of one another by multiplying both sides of an equation by the same number. Upon adding the equations, that unknown will be eliminated.

Example

Solve the system of equations using elimination method.

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

Solution

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}\left|\begin{matrix} -2 \\ 1 \end{matrix}\right. \Leftrightarrow \begin{cases} -2x-2y=-2 \\ 2x+3y=2 \end{cases}$$

$$y=0$$

$x+y=1 \Leftrightarrow x=1-y=1$

$S = \{(1,0)\}$

- **Substitution method**

This method is used when one of the variables is given in terms of the other.

Example

Find the simultaneous solution of the following pair of equations: $y = 2x - 1, y = x + 3$.

Solution

Note that the system can also be written as: $\begin{cases} y = 2x - 1 \\ y = x + 3 \end{cases}$

, then $2x - 1 = x + 3$

$x = 4$

And so $y = 4 + 3$

$y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

- **Graphical Method**

One way to solve a system of linear equations is by graphing. The intersection of the graphs represents the point at which the equations have the same x-value and the same y-value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the solution to the system of equations.

The following steps can be applied in solving system of linear equation graphically:

1. Find at least two points for each equation.
2. Plot the obtained points in *XY* plane and join these points to obtain the lines. Two points for each equation give one line.
3. The point of intersection for two lines is the solution for the given system

Example

Solve the following system by graphical method

$$\begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$$

Solution

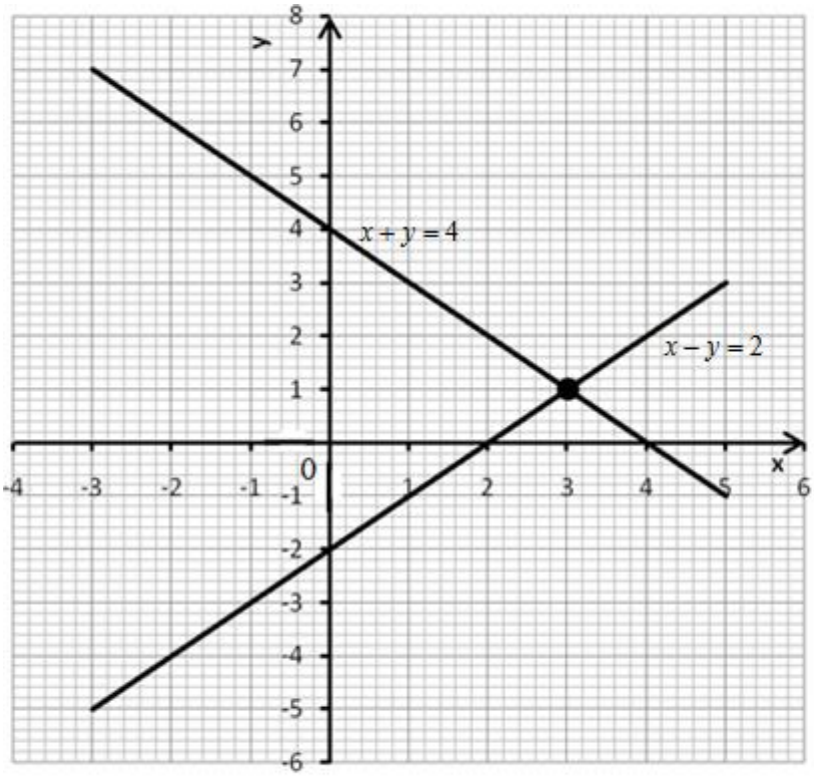
For $x + y = 4$

	-3	
	7	

For $x - y = 2$

	-3	
	-5	

Graph



The two lines intersect at point $(3,1)$. Therefore, the solution is $S = \{(3,1)\}$.

Activity 2: Guided Practice

Task 6:

1. What does it mean to solve a system of two simultaneous linear equations?
2. Solve the following simultaneous linear equations

   a. $\begin{cases} \boldsymbol{x - y = 2} \\ \boldsymbol{-3x + 2y = 6} \end{cases}$

   b. $\begin{cases} \boldsymbol{3x + 5y - 7 = 0} \\ \boldsymbol{4x - 3y - 19 = 0} \end{cases}$

3. Use a graph to solve the simultaneous linear equations

$$\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}$$

Activity 3: Application

Task 7:

1. Find the solution of the following system by using elimination method.

$$\begin{cases} 2x - 3y = 3 \\ x + 4y = 5 \end{cases}$$

2. Find the solution of the following system by using substitution.

$$\begin{cases} 2x - 3y = 3 \\ x + 4y = 5 \end{cases}$$

3. Solve the following simultaneous equations by using Comparison

$$\begin{cases} 3x - 5y = 2 \\ 2y - x + 6 = 0 \end{cases}$$

4. Solve the following simultaneous equations by using graphical method:

$$\begin{cases} 3x - y = 0 \\ x - 2(x - y) = 1 \end{cases}$$

Topic 1.3: Solving quadratic equation and inequality

Activity 1: Problem Solving

Task 8:

1. What do you mean by the term quadratic equation?
2. State the general form of quadratic equation and quadratic inequality.
3. Distinguish between quadratic equation and quadratic inequality.
4. Which method do you prefer for solving quadratic equation and quadratic inequality? Why?

Key Facts 1.3a: Solving Quadratic equation and inequality.

✓ **Quadratic equation**

The standard form of a quadratic equation looks like this: $ax^2 + bx + c = 0$

$\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are real values and $\boldsymbol{a}$ cannot be zero.

x is a variable or an unknown.

We can solve such a quadratic equation by using one of the following methods: Factorizing method, square root property, Completing the square, quadratic formula and Graphically.

Examples of quadratic equations:

a) $5x^2 + 2x + 6 = 0$, here $a = 5, b = 2$ and $c = 6$
b) $x^2 - 5x = 0$ here $a = 1, b = -5$ and $c = 0$
c) $6x - 3 = 0$ this one is not a quadratic equation because the term x^2 is missing a $i.e., a = 0$

✓ **Algebraic method**

There are four main ways of algebraic method for solving quadratic equations:

Factorizing

The method of solving quadratic equations by factorization should only be used if is readily factorized by inspection.

To factorize a quadratic equation, one can use the sum and product of its roots.

Let *x* and *y* be two real numbers such that $x + y = s$ and $xy = p$.

Here $y = s - x$ and $x(s - x) = p$. Or $sx - x^2 = p$ or $x^2 - sx + p = 0$. This equation is said to be quadratic equation and *s*, *p* are the sum and product of the two roots respectively.

Quadratic equation or equation of second degree has the form $ax^2+bx+c=0$, where the sum of two roots is $s=-\frac{b}{a}$ and their product is $p=\frac{c}{a}$.

Example 1: *Solve* $x^2+2x-24=0$	**Example 2:** *Solve* : $5x^2+7x-6=0$
Solution $x^2+2x-24=0$ $\Leftrightarrow (x+6)(x-4)=0$ So, either $x+6=0$ or $x-4=0$ giving $x=-6$ or $x=4$. $S=\{-6,4\}$	**Solution** $5x^2+7x-6=0$ $\Leftrightarrow 5x^2-3x+10x-6=0$ $\Leftrightarrow x(5x-3)+2(5x-3)=0$ $\Leftrightarrow (5x-3)(x+2)=0$ So, either $5x-3=0$ or $x+2=0$ giving $x=\frac{3}{5}$ or $x=-2$. $S=\left\{-2,\frac{3}{5}\right\}$

- **Square root property**

This property states: if A and B are algebraic expressions such that $A^2=B$, then $A=\pm\sqrt{B}$. This method is used if the form of the equation is: $x^2=k\ or(ax+b)^2=k$ where k represents a constant.

Steps to solve quadratic equations by the square root property:

1. Transform the equation so that a perfect square is on one side and a constant is on the other side of the equation.

2. Use the square root property to find the square root of each side. Remember that finding the square root of a constant yield positive and negative value.

3. Solve each resulting equation. (If you are finding the square root of a negative number, there is no real solution and imaginary numbers are necessary.)

Example:

Solve the quadratic equation $(\ x+1)^2=49$

$(x+1)^2 = 49$

1. $(x+1)^2 = 49$
2. $\sqrt{(x+1)^2} = \pm\sqrt{49}$
3. $x+1 = 7$ or $x+1 = -7$

   $x = 6$ or $x = -8$

Quadratic formula

To solve this equation, first we find the discriminant (delta): $\Delta = b^2 - 4ac$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let $\Delta = b^2 - 4ac$, There are three cases:

- If $\Delta > 0$, there are two distinct real roots:

  $x_1 = \frac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b - \sqrt{\Delta}}{2a}$
- If $\Delta = 0$, there is one repeated real root (one double root):

  $x_1 = x_2 = \frac{-b}{2a}$
- If $\Delta < 0$, there is no real root.

Example 1: *Solve:*	Example 2: *Solve :* $x^2 - 7x + 5 = -5$
$x^2 + 2x + 1 = 0$ ***Solution*** $x^2 + 2x + 1 = 0$ $\Delta = 2^2 - 4(1)(1) = 0$ $x_1 = x_2 = \frac{-2}{2} = -1$; $S = \{-1, -1\}$	***Solution*** $x^2 - 7x + 5 = -5$ $\Leftrightarrow x^2 - 7x + 10 = 0$; $\Delta = (-7)^2 - 4(1)(10) = 9$ $x_1 = \frac{-(-7) + \sqrt{9}}{2} = 5,\ x_1 = \frac{-(-7) - \sqrt{9}}{2} = 2$ $S = \{2, 5\}$

Completing the square.

Before solving quadratic equations by completing the square, let us look at some examples of expanding a binomial by squaring it.

- $(x+3)^2 = x^2+6x+9$.
- $(x-5)^2 = x^2-10x+25$

Notice that the constant term (k^2) of the trinomial is the square of half of the coefficient of trinomial's x-term. Thus, to make the expression x^2+kx a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression.

When completing the square to solve quadratic equation, remember that you must preserve the equality. When you add a constant to one side of the equation, be sure to add the same constant to the other side of equation.

Example 1: Solve $x^2-4x+1=0$ by completing the square ***Solution*** $x^2-4x+1=0$ Rewrite original equation $x^2-4x=1$ Subtract 1 from both sides. $x^2-4x+(-2)^2=1+(-2)^2$ Add $(-2)^2=4$ to both sides. $(x-2)^2=3$ Binomial squared. $x-2=\pm\sqrt{3}$ Take square roots. $x=2\pm\sqrt{3}$ Solve for x . ***The equation has two solutions:*** $x=2+\sqrt{3}$ ***and*** $x=2-\sqrt{3}$ $S=\{2+\sqrt{3},2-\sqrt{3}\}$	**Example 2:** Solve $4x^2+2x-5=0$ by completing the square ***Solution*** $4x^2+2x-5=0$ Rewrite original equation. $4x^2+2x=5$ Add 5 to both sides. $x^2+\frac{1}{2}x=\frac{5}{4}$ Divide both sides by 4. $x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2=\frac{5}{4}+\frac{1}{16}$ Add $\left(\frac{1}{4}\right)^2=\frac{1}{16}$ to both sides. $\left(x+\frac{1}{4}\right)^2=\frac{21}{16}$ Binomial squared. $x+\frac{1}{4}=\pm\frac{\sqrt{21}}{2}$ Take square roots. $x=-\frac{1}{4}\pm\frac{\sqrt{21}}{2}$ ***then*** $x=-\frac{1}{4}+\frac{\sqrt{21}}{2}$ ***and*** $x=-\frac{1}{4}-\frac{\sqrt{21}}{2}$ $S=\left\{-\frac{1}{4}+\frac{\sqrt{21}}{2},-\frac{1}{4}-\frac{\sqrt{21}}{2}\right\}$

✓ **Graphical method**

Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e., a line $x = h$ where h is constant value. This line is called **axis of symmetry**.

For any quadratic function $f(x) = ax^2 + bx + c$ whose axis of symmetry is the line $x = -\frac{b}{2a}$, we can get the y-coordinate of the vertex by substituting the x-coordinate of axis of symmetry. The vertex becomes $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$. The vertex of a quadratic function is the point where the function crosses its axis of symmetry. If the coefficient of the x^2 term is positive, the vertex will be the lowest point on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the ∩-shape. The shapes are as below.

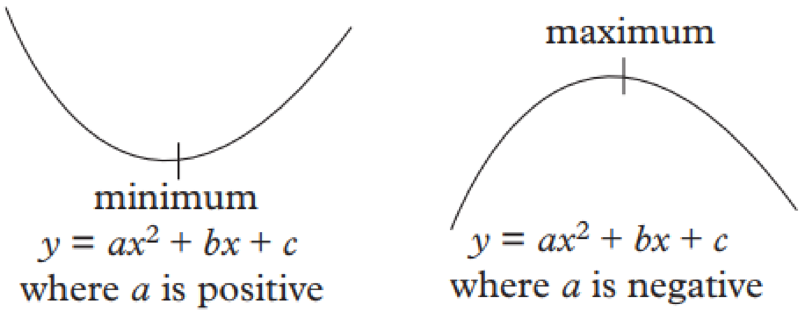


Figure 1:Maximum and minimum of quadratic function

The intercepts with axes are the points where a quadratic function cuts the axes. There are two intercepts i.e. x-intercept and y-intercept. x-intercept for any quadratic function is calculated by letting *y = 0* and y- intercept is calculated by letting *x = 0*

Graph of a quadratic function

The graph of a quadratic function can be sketched without table of values if the following are known.

- The vertex
- The x-intercepts
- The y-intercept

Example: Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

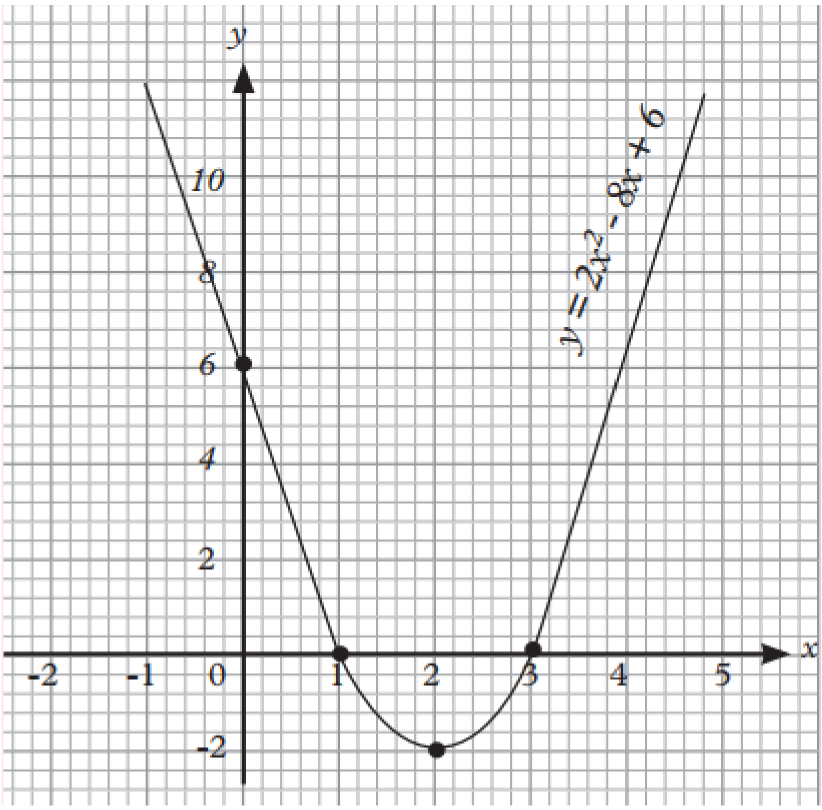
Solution

- The coefficients are a = 2, b = –8 and c = 6

- The x-coordinate of the vertex is $h = -\frac{b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2$
- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get $y = 2(2)^2 - 8(2) + 6 = -2$
- The vertex is (2, –2) and the axis of symmetry is $x = 2$.
- When x = 0, $y = 2(0)^2 - 8(0) + 6 = 6$.
- The y-intercept is (0, 6)

When y = 0, $0 = 2x^2 - 8x + 6$, we therefore solve the quadratic equation for the values of x and we find the x-intercepts are (1,0) or (3,0)

The graph is as below.

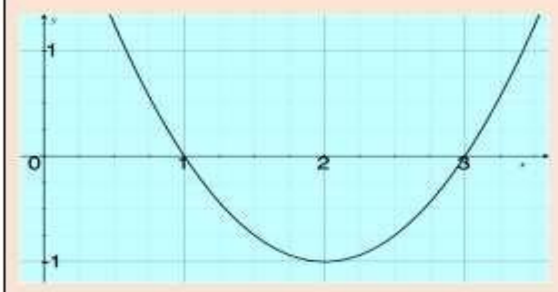


Determination of solutions

In using this method, we draw a graph of a quadratic function by creating a table of values. The solutions or roots are obtained by reading-off the x-coordinates of the point of intersection of the curve and the horizontal axis (when the equation = 0) Recall that the quadratic can have a maximum of two roots – this occurs when the graph cuts the x-axis at two distinct points. If the x-axis is a tangent to the curve, then the two roots are equal to each other and so there is just one solution.

If the curve does not cut or touch the x-axis, there are no solutions. These cases are illustrated below.

Example:



Graph of $x^2 - 4x + 3 = 0$

The roots are $x = 1$ and $x = 3$

✓ quadratic inequality

Form of a quadratic inequality:

After rearrangement, quadratic inequality has the following standard form

$$ax^2 + bx + c > 0 \quad (\geq,\ <,\ \leq)$$

We can solve such a quadratic inequality by using one of the following methods: Algebraic and graphically.

Algebraic method

We can solve such a quadratic inequality by using the following steps:

1. Factorization of the given inequality
2. Determination of roots
3. Study of sign
4. Determination of interval of solutions

Example:

Solve the inequality (x + 3)(x −2) > 0.

Solution

The critical values are x = −3, x = 2.

The required sign diagram is:

x / Factors	−∞		−3		2		+∞
x + 3		−	0	+		+	
x − 2		−		−	0	+	
(x + 3) (x − 2)		+	0	−	0	+	

Fig. 6.3

The answer is **x** < −3 **or x** > 2 ⇔ x ∈] − ∞, −3 [∪]2, + ∞[

Graphical method

We can solve such a quadratic inequality by using the following steps:

1. Plotting a parabola
2. Shading the region satisfying the given inequality
3. Determination of interval of solutions

Example

Solve graphically the following quadratic inequality

$x^2 - 4x + 3 > 0$

Solution

The standard form $y = (x-2)^2 - 1$

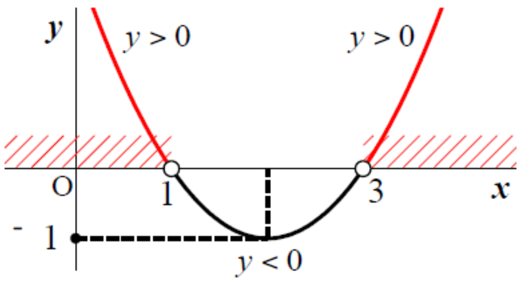
By factoring, we have $y = (x-1)(x-3)$

therefore, the roots are $x = 1,\ x = 3$

The inequality is satisfied in the shaded domain.

The solution is

$x < 1, \quad x > 3$



Activity 2: Guided Practice

Task 9:

1. Solve algebraically the following quadratic equations
   $a.\ 5x^2 + 7x - 6 = 0$
   $b.\ x^2 + 2x + 1 = 0$
2. Solve graphically a following quadratic equation.
   $y = x^2 - 3x + 2$

Activity 3: Application

Task 10:

I. Given the following quadratic inequality $x^2 + 5x + 6 > 0$. Solve by using the following methods:

a. Algebraic

b. Graphical

II. Solve the following quadratic equation in three methods $4x^2 + 5x + 1 = 0$

a. Factorization method

b. Completing square method

c. Quadratic formula

III. Sketch the curve $f(x) = -x^2 - 2x$

Formative Assessment

Formative Assessment for Unit 1

Instructions: Attempt all questions

1. Describe the linear equations and inequalities

2. From the following mathematical expression: $x + 1 = 0$; $5 < x$; $1 = x^2$ and $x^3 - 1 = 0$

   Which one of these expressions is of the form of:

   i) Linear inequality

   ii) Quadratic equation

   iii) Linear equations

3. Solve the following linear equations in the set of real numbers

a) $x+5=9$

b) $6x+5=5$

c) $x-2=3$

d) $25=2x-5$

4. Without tables of values, state the vertices, intercepts with axes, axes of symmetry, and sketch the graphs.

a. $y = 2x^2 + 5x - 1$

b. $y = 3x^2 + 8x - 6$

5. Solve the following system of linear equations

a. $\begin{cases} 4y + x = 8 \\ -x + y = 2 \end{cases}$

b. $\begin{cases} 3x + y = 1 \\ x - y = -1 \end{cases}$

Solve in set of real numbers the following equations by factorization, quadratic formula and completing the square.

a. $x^2 - 2x = 3$

b. $x^2 - 12x + 11 = 0$

Points to Remember

- Linear equation in one unknown is of the form $ax+b=0$, with a and b constant and $a \neq 0$
- Inequalities in one unknown are of the forms: $ax+b>0$, $ax+b<0$, $ax+b \geq 0$, $ax+b \leq 0$, with a and b constant and $a \neq 0$
- The two different methods such as algebraic or graphical in solving linear equations and inequalities.
- In certain coefficients of equations contain one or several letter variables, the equation is called **parametric** and the letters are called **real parameters.**
- Two simultaneous linear equation are of the forms: $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where $a_1; b_1; c_1; a_2; b_2; c_1$ are constants
- Inequality of the type $ax^2 + bx + c >, \leq, \geq, < 0$ are called quadratic inequality.
- The standard form of a quadratic equation looks like this: $ax^2 + bx + c = 0$
  $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are values and $\boldsymbol{a}$ cannot be zero.

x is a variable or an unknown.

- There are four main ways of algebraic method for solving quadratic equations:
- The method of solving quadratic equations by factorization.

  This method to factorize a quadratic equation, one can use the sum and product of its roots.

- Solving quadratic equation by Square root property

  The **Square Root Property** states:

  If $A^2 = B$, where A and B are algebraic expressions, then $A = \pm\sqrt{B}$.

  This method is applicable when the quadratic equation is in one of the following forms:

  1. $x^2 = k$
  2. $(ax + b)^2 = k$

  Here, k is a constant.

- Solving quadratic equation by Square root property

  Thus, to make the expression $x^2 + kx$ a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression.

Self-Reflection

1. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

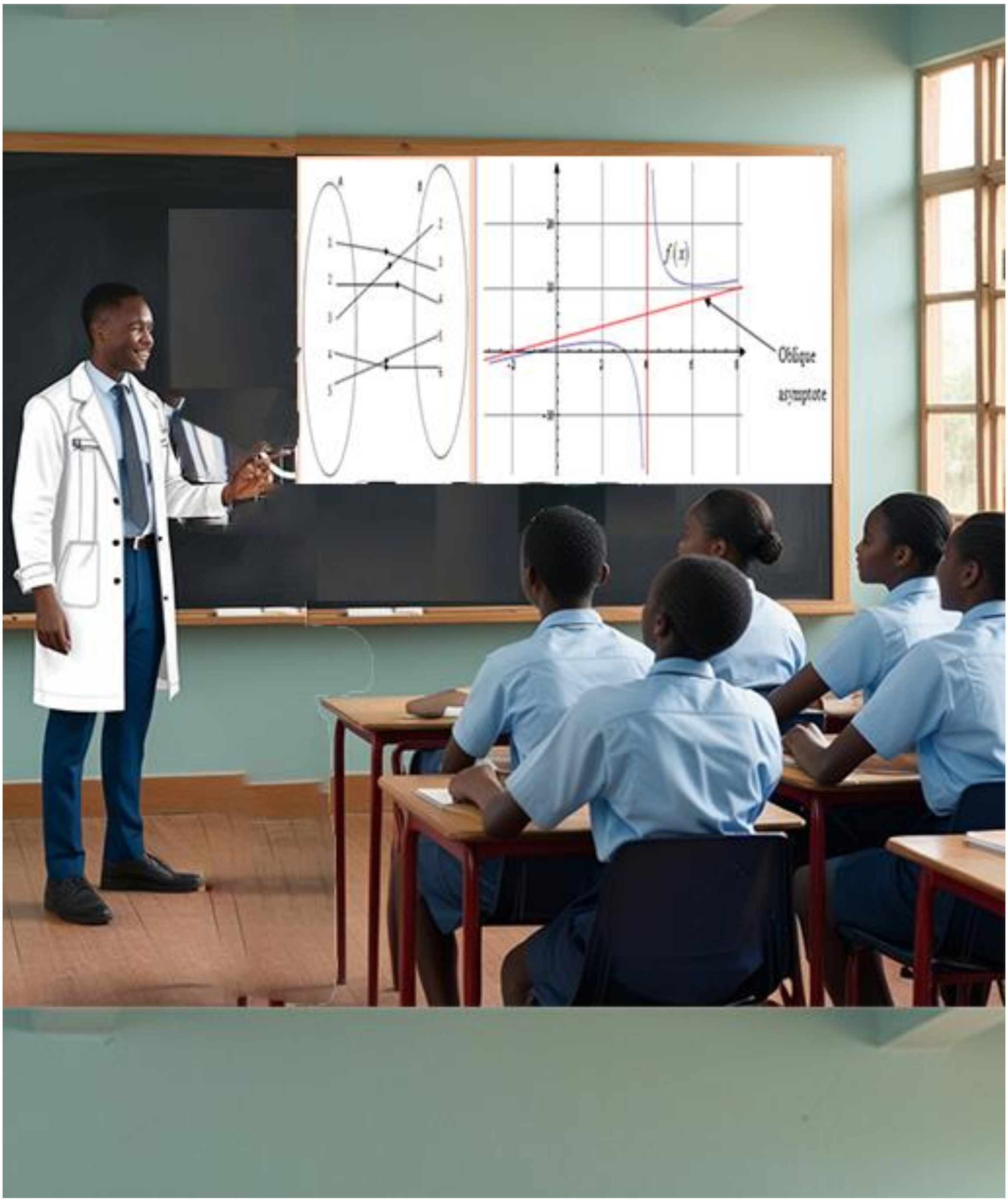
My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Describe linear equations and inequality					
Solve linear equations and inequality using algebraic and graphical method					

My experience **Knowledge, skills and attitudes**	**I don't have any experience doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Critical thinking in describing linear equations and inequality					
Analytical spirit and effectiveness in Solving linear equations and inequality.					
Describe simultaneous linear equations					
Solve the simultaneous linear equations using different methods such substitution, elimination and comparison					
Critical thinking and cooperation in describe simultaneous linear equations.					
Attention in using different methods to Solve the simultaneous linear equations					
Describe quadratic equations					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Solve the quadratic equation using various convenient methods such factoring, Square root property, completing the square, or using quadratic formula.					
Critical thinking in describing quadratic equations					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1. **2.** **3.**	**1.** **2.** **3.**	**1.** **2.** **3.**


f(x)
Oblique
asymptote

Unit summary

This unit provides you with the knowledge, skills and attitudes required to Analyze algebraic functions required to Apply Basic Mathematical Analysis. It covers the Determination of domain and range of algebraic function, Identification of symmetry of algebraic function, Determination of Limits function and Determination of asymptotes.

Self-Assessment: Unit 2

I. Observe the figure illustrated above and answer to the question below:
   a. Name the relation between subset A and subset B
   b. How do we call the elements in A?
   c. How do we call the elements in B**?**
   d. Will the relation be named the same name in 1 if one element in A is paired with more than one element in B?
   e. On the graph illustrated above name all line in that graph
   f. Examine the distance between the line and the curve, what conclusion can you take?

II. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience **Knowledge, skills and attitudes**	**I don't have any experien ce doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Explain algebraic expressions and function.					
Explain the concepts of domain, range, limits, and asymptotes.					
Factorize techniques and operations with limits.					
Describe graphical properties (symmetry, end behavior, asymptotes).					
Recognize patterns and applying the correct analysis techniques.					
Identify key features of algebraic expression and					

My experience Knowledge, skills and attitudes	I don't have any experien ce doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
function (domain, range, limits).					
Precision for careful attention to detail in calculate and interprete algebraic functions					
Open-mindedness in Being open to multiple approaches and graphical tools.					
Critical Thinking in Questioning and verifying the results.					

Key Competencies:

Knowledge	Skills	Attitudes
1. Explain algebraic expressions and function	**1.** Identify key features of algebraic expression and function (domain, range, limits, etc.)	**1.** Open-mindedness in being open to multiple approaches and graphical tools.
2. Explain the concepts of domain, range, limits, and asymptotes	**2.** Find domain, range and limits of algebraic functions, drawing asymptotes	**2.** Critical thinking in questioning and verifying the results.
3. Describe graphical properties (symmetry, end behavior, asymptotes).	**3.** Recognize patterns and apply the correct analysis techniques	**3**. Precision for Careful attention to detail in calculations and interpretations.

Discovery activity:

Task 11:

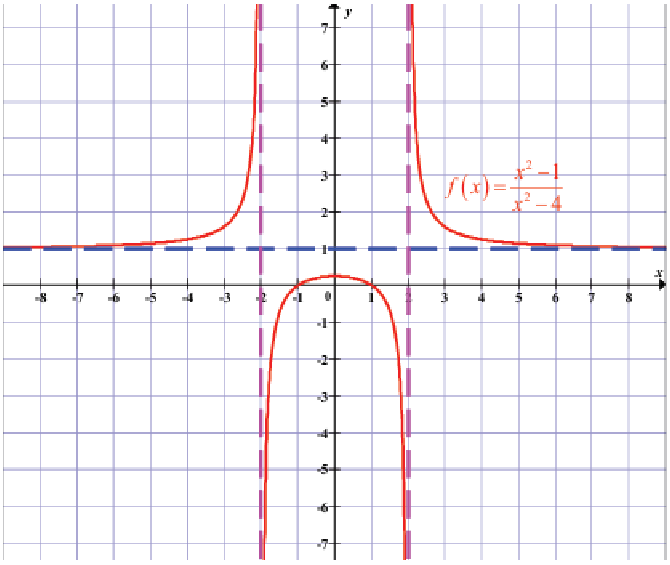
1. Consider the following sentences:
   a) The function of the heart is to pump blood
   b) Last Saturday, my sister got married; the arrangement of chairs in the main hall was in function of the number of guests.
   c) The area of a square is function of the length of its side.

   Explain what is meant by the word "Function" in each of the three sentences above.

1. Any function involves at least two variables. Identify the "independent variable" and the "dependent variable" in the following functions:

$i.\ \ y = \frac{4x - 4}{(x - 2)^2}$ $\qquad\qquad$ $ii. A = \pi r^2$

2. Observe the figure below and answer questions related to it



i.Examine the distance between the curve and lines $x = 2, x = -2, y = 1$

ii.Name the line $x = 2, x = -2, y = 1$

Topic 2.1: Determination of the domain and range of algebraic function

Activity 1: Problem Solving

Task 12:

1. Explain what is meant by the word domain of algebraic function.
2. What do you understand by a range of algebraic function?
3. What did you know about finding the domain and range of algebraic function?
4. State the domain and range of the following relations. Is the relation a function?
   {(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)
5. Given the functions $f(x) = 5x - 4$, for which real values of x does $f(x)$ has meaning?

Key Facts 2.1a: Determination of the Domain and range of algebraic function

✓ **Algebraic functions**

Two sets **A** and **B** a function from **A** into **B** is a rule which assigns a unique element of **A** exactly one element of **B**.

We write the function from A to B as **f: A → B.** Most of time we do not indicate the two sets and we simply write $\boldsymbol{Y = f(x)}$**.** The sets A and B are called the domain and range of a function, respectively.

Types of functions:

1. Polynomial function
2. Rational function
3. n^{th} roots functions

✓ Domain of a function

Domain of a function is the set of all real numbers for which the expression of the function is defined as a real number. In other words, it is all the real numbers for which the expression makes sense. The domain of f is denoted by $\boldsymbol{dom(f)}$.

Existence conditions

When determining domain of definition, we have to take into account the following conditions of existence**:**

- You cannot have a zero in the denominator.
- You cannot have a negative number under an even root.

Key Facts 2.1b: Determination of the Domain and range of algebraic function

Polynomial functions

Polynomial function is any function that can be written in the form:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ where $a_n, a_{n-1}, \ a_1, a_0$ are real numbers then $dom(f)$ = $\mathbb{R}$ = $]-\infty, \ +\infty[$.

Example: Find the domain of the following functions:

a. $f(x) = x + 5.$

b. $f(x) = 2x^2 + 4x - 6$

Solution :

a. $dom(f)$ = $\mathbb{R}$ = $]-\infty,\ +\infty[$.

b. $dom(f)$ = $\mathbb{R}$ = $]-\infty,\ +\infty[$.

Rational function

Rational function is any function that can be written as :$f(x) = \frac{h(x)}{g(x)}$ where $h(x) and\ g(x)$ are polynomial , then $dom(f)$ = $\{x \in \mathbb{R}\ :\ g(x)\ \neq 0\ \}$

Example: Find the domain of the function of the following: $f(x) = \frac{2x+3}{x-3}$

Solution

The function $f(x) = \frac{2x+3}{x-3}$ is defined for all real numbers except the number that makes the denominator to be zero. We set $x - 3\ \neq 0$ which gives us $x \neq\ 3$.

Therefore, the domain of f is $\{x \in \mathbb{R}: x \neq\ 3\} \leftrightarrow \boldsymbol{domf} = (-\infty, \mathbf{3}) \cup (\mathbf{3}, +\infty)$

2. $\boldsymbol{n^{th}}$ roots functions

A functions $f(x) = \sqrt[n]{x}$ where n is an integers.

We have two cases : when n is even number $dom(f)$ = $\{x \in \mathbb{R}\ :\ x \geq\ 0\ \}$ and when n is odd number then $dom(f)$ = $\mathbb{R}$ = $]-\infty,\ +\infty[$.

Example: Find the domain of the function of each of the following:

1. $f(x) = \sqrt{6 - 3x}$
2. $f(x) = \sqrt[3]{x + 3}$

Solutions

1. We know that the square root of the negative number does not exist. The function $f(x) = \sqrt{6 - 3x}$ is defined if and only if:

$$6 - 3x \geq 0$$

$$-3x \geq -6$$

$$3x \leq 6$$

$$x \leq 2$$

Therefore, the domain of the function$f(x)$ is $\{x \in \mathbb{R}: x \leq 2 = (-\infty. 2)\}$

3. Since n =3 is odd number then $dom(f)$ = $\mathbb{R}$ = $]-\infty,\ +\infty[$.

- **Composition of functions**

The composition of $\boldsymbol{g}$ and $\boldsymbol{f}$, denoted gof is defined by the rule $(\boldsymbol{gof})(\boldsymbol{x})$ = $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$ provided that $f(x)$ is in the domain of g.

Example:

Given $f(x) = 4x$ and $g(x) = x^2 + 3$

a) $gof(x)$

b) $gof(2)$

we have $gof(x) = g[f(x)]$

$= g[f(x)] = (4x)^2 + 3$

$= 16x^2 + 3$

$gof(x) = 16x^2 + 3$

b) $gof(2) = 16(2^2) + 3 = 67$

- **The inverse of algebraic function**

For a one-to-one function defined by $Y = f(x)$, the equation of the inverse can be found as follows:

- Replace $f(x)$ by Y.
- Interchange x and y.
- Solve for y.
- Replace y by $f^{-1}(x)$

The composition of $\boldsymbol{g}$ and $\boldsymbol{f}$, denoted gof is defined by the rule $(\boldsymbol{gof})(\boldsymbol{x})$ = $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$ provided that $f(x)$ is in the domain of g.

Example Determine the inverse of the following functions

a) $\boldsymbol{f(x) = 3x + 12}$

b) $f(x) = x^2 - 1$

Solutions

$f(x) = 3x + 12$

$y = 3x + 12$

$x = 3y + 12$

$x - 12 = 3y$

$y = \frac{x-12}{3}$

$f^{-1}(x) = \frac{x-12}{3}$

b) $f(x) = x^2 - 1$

$y = x^2 - 1$

$x = y^2 - 1$

$x + 1 = y^2$

$y = \sqrt{x+1}$

f^{-1}(x) $= \sqrt{x+1}$

✓ Range of a function

Let f: A → B be a function. The range of f, denoted by Im(f) is the image of A under f, that is, Im (f) = f [A]. The range consists of all possible values the function f can have. The range of f is denoted by $\boldsymbol{im(f)}$**.**

Steps to finding range of function

To find the range of function f described by formula where the domain is taken to be the natural domain:

1. Put $y = f(x)$.
2. Solve x in terms of y
3. The range of f is the set of all real numbers y such that x can be solved.

Example: For each of the following functions, find the range.

1.$\boldsymbol{f(x) = x + 5}$

2. $\boldsymbol{f(x) = \frac{8}{2x-4}}$

3. $\boldsymbol{f(x)}$**=**$\boldsymbol{\sqrt{6-3x}}$

ANSWERS:

1. Put $y = f(x) = x + 5$

   Solve for x; $x = y - 5$

   Note that x can be solved for any value of y. Therefore, the range is $\mathbb{R}$

2. Put $y = \frac{8}{2x-4}$

   Solve for x ; $(2x - 4)y = 8$

   $2xy = 8 + 4y$

   Thus ; $x = \frac{8+4y}{2y}$

   Note that x can be solved if and only if $2y \neq 0$ therefore, the range of the function $f(x)$ is $\{y \in \mathbb{R}: y \neq 0\} = \mathbb{R}/\{0\}$

3. Put $y = \sqrt{6 - 3x}$; we see that $y \geq 0$

   $y^2 = 6 - 3x$ With $y \geq 0$ then$x = \frac{6-y^2}{3}$. Here x can be solved whenever $y \geq 0$

   Therefore, the range of the function $f(x)$ is$\{y \in \mathbb{R}: y \geq 0\} = [0, \infty)$.

Activity 2: Guided Practice

Task 13:

1. **Find the domain of the function of each of the following:**
   a. $f(x) = 3x^3 - 2x^2 + 4x - 6$
   b. $f(x) = \frac{2x+3}{x-3}$
   c. $\sqrt{6 - 3x}$
2. **Find the range of the function of each of the following:**
   a. $f(x) = 4x - 6$
   b. $f(x) = \frac{x+3}{x-3}$
   c. $\sqrt{x - 3}$

Activity 3: Application

Task 14:

1. Determine the domain and range of the given function

a. $\boldsymbol{f(x) = -x^4 + 4}$

b. $\boldsymbol{f(x) = \frac{x^2+x-2}{x^2-x+2}}$

2. Find the domain of definition for

a. $f(x) = \sqrt{4x - 8}$

b. $f(x) = \frac{x-2}{x^2-25}$

c. $f(x) = x^2 + 5x\text{-}6$

d. $\boldsymbol{f(x) = -x^5 - 3x^4 + x^2 - 5}$

e. $f(x) = \frac{7x^2+9}{x+6}$

Topic 2.2: Identification of symmetry of algebraic function

Activity 1: Problem Solving

Task 15

1. Explain what is meant by the word even of function.
2. What do you understand by study the parity of algebraic function?
3. What do you know about odd function?
4. List any four examples of odd functions and any four examples of even function.

Key Facts 2.2a: Identification of Symmetry of algebraic function

✓ **Even function**

Let f be a function of $\mathbb{R}$ in $\mathbb{R}$ we say that f is even if $\forall$ x $\in$ Dom (f), (–x) $\in$ Dom(f);

f(–x) = f(x).

The graph of an even function is symmetric with respect to the y-axis.

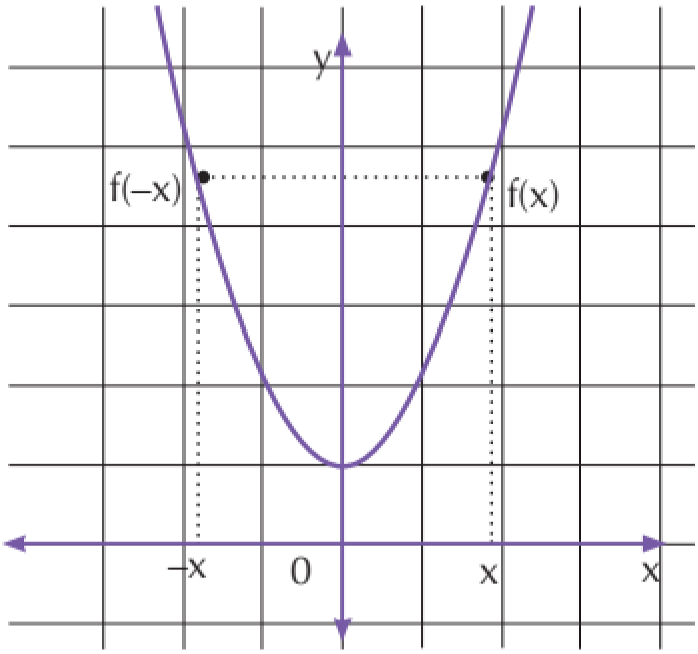


Figure 2: curve of even function

<u>Figure:</u> This is the curve $f(x) = x^2+1$

Example: Determine whether the function (f) are even in each of the following cases:

$f(x) = 2x^2 + 1$

Solution

$f(x) = 2x^2 + 1$

$f(-x) = 2(-x)^2 + 1$

$= 2x^2 + 1$

$= f(x)$

Therefore $f(x)$ is an even function

✓ **Odd function**

We say that a function f is odd if ∀x ∈ Dom (f), (–x) ∈Dom(f);

f(–x) = –f(x).

Note that an odd function has symmetry at origin.

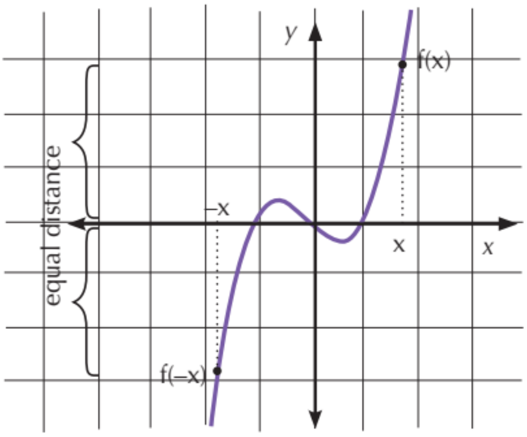


Figure 3: graph of odd function

<u>Figure:</u> This is the curve f(x) = x^3 – x

Example: Determine whether the function (f) are even in each of the following cases:

$f(x) = x^3 - 3x$

Solution

$f(-x) = (-x)^3 - (-x)$

$= -x^3 + x$

$= -f(x)$ thus $f(x)$ is odd function because $f(-x) = -f(x)$

Activity 2: Guided Practice

Task 16

1. Determine whether the function (f) are even and odd or neither in each of the following cases:
   a. $f(x) = 2x^2 + 1$
   b. $f(x) = 2x^3 - x$
   c. $f(x) = \frac{1}{2}x^3 - 2x^2 + \frac{3}{2}$

Activity 3: Application

Task 17

1. State whether the following function is even, odd or neither $\boldsymbol{f(x) = \frac{x^3+x}{5}}$
2. Study the parity of the following functions
   a) $f(x) = x^4 - 3x^2$
   b) $f(x) = x^2 + 4$
   c) $\boldsymbol{f(x) = x^3 + x}$

Topic 2.3: Determination of limits function

Activity 1: Problem Solving

Task 18:

1. You have heard about the term 'neighborhood' in everyday life. What does it mean?
2. Give the function $f(x) = 2x + 2$, complete the following table

x	0.8	0.9	0.95	0.99	0.999	1.2	1.1	1.01	1.001	1.0001
$f(x)$										

What do you observe as x approaching the value 1

Key Facts 2.3.a. Determination of limits function.

✓ **Finite limits**

Suppose the function y = f(x) is a numerical function with independent variable x and dependent variable y where the value y depends on the variable of x. If the values of x can be made as closer to a value a as we please, then this **can be written x** →**a**. This is **read as** '**x** tends to **a'** or '**x** approaches **a** Since the variable y depends on the variable **x**, we can say that as **x** tends to **a,** y will also tend to a certain value that we have to determine. The obtained value of y as **x** tends to **a** is what we call **the limiting value of f(x).**

If f(x) can be made as close as we like to some number a by making x sufficiently close to a, then we say that f(x) has a limit of L as x approaches a, and we write:

$$\lim_{x \to a} f(x) = L$$

Example: 1. Evaluate the following limit

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} 2x + 1 = 2(3) + 1 = 7 = f(3)$$

Example 2. Evaluate the following limit

$\lim_{x \to 2} x^3 - 3x^2 + 4x + 5$ **= 8- 12+8+5 = 9**

✓ **Infinity limits**

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

Example: Evaluate the following limit:

$$\lim_{x \to 2} \frac{x^3 - x^2}{x-2} = \frac{(2)^3 - (2)^2}{2-2} = \frac{4}{0} = \infty$$

✓ **Limits at infinity**

Let f be a function defined on some interval, (a, ∞) . Then

$$\lim_{x \to \infty} f(x) = L$$

Note: To compute the limit of a function as $x \to \pm\infty$, we use the following principles

$\forall a \in \mathbb{R}$

- $a(\pm\infty) = \begin{cases} \pm\infty \text{ if } a > 0 \\ \mp\infty \text{ if } a < 0 \end{cases}$,
- $-\infty \pm a = -\infty$ and $+\infty \pm a = +\infty$,
- $\frac{a}{\pm\infty} = 0$,
- $\frac{a}{0} = +\infty$ or $-\infty$.

Example:

Evaluate the following limit $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Solution

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{x^2\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} = \frac{\lim_{x \to \infty} x^2\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$\frac{\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - \lim_{x \to \infty} \frac{2}{x^2}}{\lim_{x \to \infty} 5 + \lim_{x \to \infty} \frac{4}{x} + \lim_{x \to \infty} \frac{1}{x^2}} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

✓ **Indeterminate cases (I F)**

Computation of limits may results in indeterminate form (I.F) such as $\frac{0}{0}, \frac{\infty}{\infty}$, $\infty - \infty$, $0.\infty$, that must be taken away.

Indeterminate form $\frac{0}{0}$

It may be that the limit of that Indeterminate form $\frac{0}{0}$ can be found by some methods such as factor method, rationalization method.

a. Factor method

Example: Evaluate the following limits

$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{(2)^2-4}{2-2}=\frac{0}{0}$ IF, Then to remove the IF we have to use Factor Method

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2=2+2=4$$

b. Rationalization method

In functions which involve square roots, rationalization of either numerator or denominator and simplifications will facilitate the work.

Example: find $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{0}{0}\ \text{(indeterminate form)}$$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1}$$

$$=\lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}=\frac{0}{0}\ \text{(indeterminate form)}$$

$$\lim_{x\to 0}\frac{1}{(\sqrt{1+x}+1)}=\frac{1}{(\sqrt{1+0}+1)}=\frac{1}{2}$$

c. Indeterminate form $\frac{\infty}{\infty}$

Computing the following limits, allows us to conclude what is given after:

Examples:

Compute the following:

a) $\lim\limits_{x\to\infty} \dfrac{3x-4}{2x^2-3x+5} = \lim\limits_{x\to\infty} \dfrac{x^2\left(\frac{3}{x}-\frac{4}{x^2}\right)}{x^2\left(2-\frac{3}{x}+\frac{5}{x^2}\right)} = \lim\limits_{x\to\infty} \dfrac{\frac{3}{x}-\frac{4}{x^2}}{2-\frac{3}{x}+\frac{5}{x^2}} = \dfrac{0-0}{2-0+0} = \dfrac{0}{2} = 0$

b) $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{5x^2+4x-5} = \lim\limits_{x\to\infty} \dfrac{x^2\left(2-\frac{3}{x^2}\right)}{x^2\left(5-\frac{4}{x}-\frac{6}{x^2}\right)} = \lim\limits_{x\to\infty} \dfrac{2-\frac{3}{x^2}}{5+\frac{4}{x}-\frac{6}{x^2}} = \dfrac{2-0}{5+0+0} = \dfrac{2}{5}$

c) $\lim\limits_{x\to\infty} \dfrac{3x^3-2x+1}{-x^2+3x-5} = \lim\limits_{x\to\infty} \dfrac{x^3\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{x^2\left(-1+\frac{3}{x}-\frac{5}{x^2}\right)} = \lim\limits_{x\to\infty} \dfrac{x\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{-1+\frac{3}{x}-\frac{5}{x^2}} = \dfrac{3(+\infty)}{-1} = \dfrac{+\infty}{-1} = -\infty$

d) $\lim\limits_{x\to-\infty} \dfrac{3x^3-2x+1}{-x^2+3x-5} = \lim\limits_{x\to-\infty} \dfrac{x^3\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{x^2\left(-1+\frac{3}{x}-\frac{5}{x^2}\right)} = \lim\limits_{x\to-\infty} \dfrac{x\left(3-\frac{2}{x^2}+\frac{1}{x^2}\right)}{-1+\frac{3}{x}-\frac{5}{x^2}} = \dfrac{3(-\infty)}{-1} = \dfrac{-\infty}{-1} = +\infty$

d. Indeterminate form $\infty - \infty$

Compute $\lim\limits_{x\to\infty} \left(\sqrt{4x^2+3x+1}-2x\right)$

Solution

$\lim\limits_{x\to+\infty} \left(\sqrt{4x^2+3x+1}-2x\right) = +\infty - \infty$ (indeterminate form)

$\lim\limits_{x\to+\infty} \left(\sqrt{4x^2+3x+1}-2x\right) = \lim\limits_{x\to+\infty} \dfrac{\left(\sqrt{4x^2+3x+1}-2x\right)\left(\sqrt{4x^2+3x+1}+2x\right)}{\sqrt{4x^2+3x+1}+2x}$

$= \lim\limits_{x\to+\infty} \dfrac{4x^2+3x+1-4x^2}{\sqrt{4x^2+3x+1}+2x} = \lim\limits_{x\to+\infty} \dfrac{3x+1}{\sqrt{4x^2+3x+1}+2x}$

$= \lim\limits_{x\to+\infty} \dfrac{x\left(3+\frac{1}{x}\right)}{x\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}+2} = \lim\limits_{x\to+\infty} \dfrac{3+\frac{1}{x}}{\sqrt{4+\frac{3}{x}+\frac{1}{x^2}}+2} = \dfrac{3}{\sqrt{4}+2} = \dfrac{3}{4}$

Activity 2: Guided Practice

Task 19:

Evaluate the following limit

1. $\lim\limits_{x\to3} 2x+1$
2. $\lim\limits_{x\to2} x^3-3x^2+5$
3. $\lim\limits_{x\to2} \dfrac{x^3-x^2}{x+1}$

Activity 3: Application

Task 20:

1. Find the limits of the following functions

a. $\lim_{x \to 2} x^3 - 3x^2 + 4x + 5$

b. $\lim_{x \to -3} \frac{x^2-9}{x+3}$

c. $\lim_{x \to +\infty} \frac{3x^2+3x+4}{5x+4}$

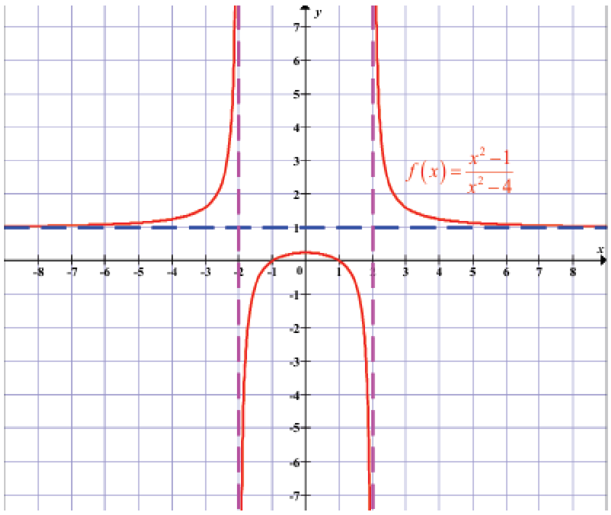
d. $\lim_{x \to \infty} \sqrt{x^2 + x - 2} - x$

Topic 2.4: Determination of asymptotes

Activity 1: Problem Solving

Task 21

1. Observe the figure below and answer questions related to it



i. Examine the distance between the curve and lines $x = 2, x = -2, y = 1$

ii. Name the line $x = 2, x = -2, y = 1$

Key Facts 2.4a: Determination of asymptotes

✓ **Vertical asymptote**

A line with equation $x = x_0$ is called a **vertical asymptote** for the graph of a function $f(x)$ if $\lim_{x \to x_0} f(x) = \pm\infty$

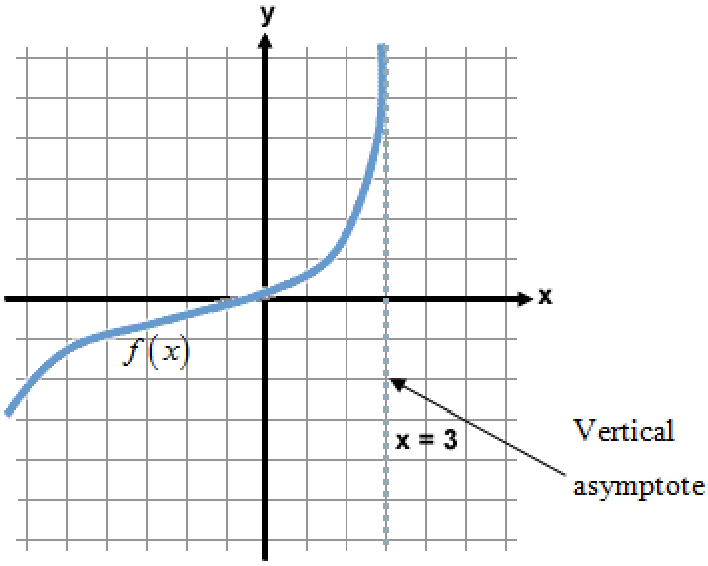


Figure 4: vertical asymptote

V.A≡ $x = a$, where $\lim_{x \to a} f(x) = \pm\infty$

Example: *Let* $f(x) = \frac{2x^2+7x-1}{x+1}$; $\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{2x^2+7x-1}{x+1}$

$= \infty$

Hence, $VA \equiv x = -1$is vertical asymptote for $f(x) = \frac{2x^2+7x-1}{x+1}$.

- ✓ **Horizontal asymptote**

A line with equation $y = L$ is called a **horizontal asymptote** for the graph of a function $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$

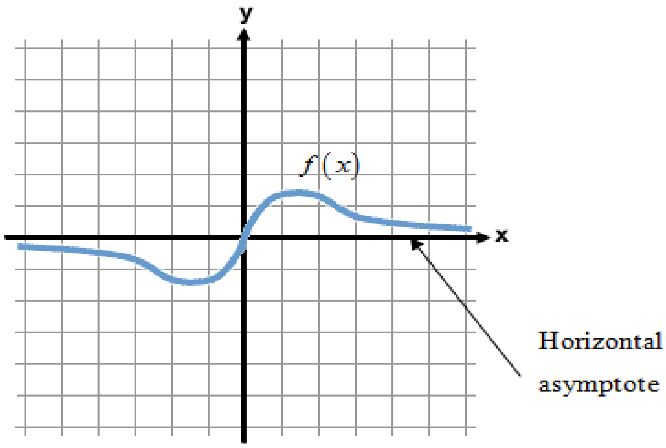


Figure 5: Horizontal asymptotes

$H.A \equiv Y = b$, where $b = \lim_{x\to\infty} f(x)$

Example:

Let $f(x) = \frac{3x^2+4x-9}{2x^2+1}$,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3x^2}{2x^2}$$
$$= \frac{3}{2}$$

Thus, $y = \frac{3}{2}$ is a horizontal asymptote for $f(x) = \frac{3x^2+4x-9}{2x^2+1}$.

✓ **Oblique asymptote**

If a rational function, $\frac{P(x)}{Q(x)}$, is such that the degree of the numerator exceeds the degree of the denominator by one, then the graph of $\frac{P(x)}{Q(x)}$ will have an **oblique asymptote (or a slant asymptote)**; that is, an asymptote which is neither vertical nor horizontal.

Notice

Horizontal asymptote and oblique asymptote do not exist simultaneously on the same side, that is, at infinity of the same sign

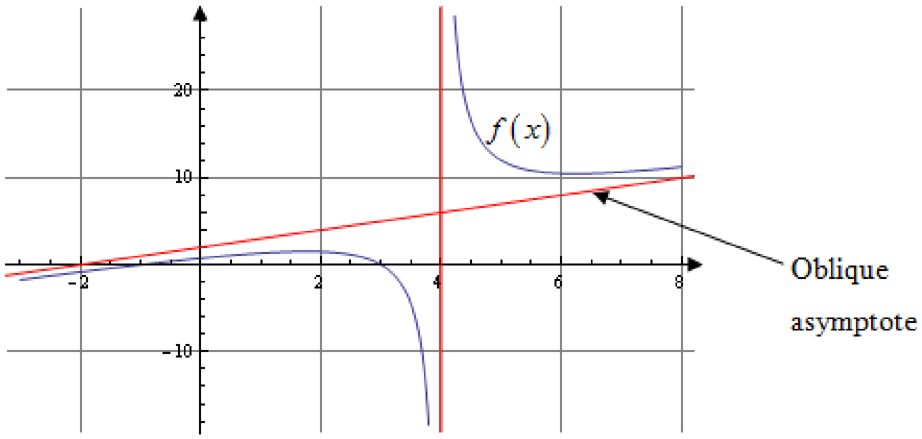


Figure 6: Oblique asymptote

We perform the division of $P(x)$ by $Q(x)$ to obtain $\frac{P(x)}{Q(x)} = (ax+b) + \frac{R(x)}{Q(x)}$, where, $ax+b$ are the quotient and $R(x)$ is the remainder. Another way to find the values of constants a and b are $a = \lim_{x\to\pm\infty} \frac{f(x)}{x}$ $a \neq 0$, and

$b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$. Therefore, **we can write that** $O.A\equiv y=ax+b$**, where** $a=\lim_{x\to\pm\infty}\frac{f(x)}{x}$, $a\neq 0$, **and** $b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$

Example1:

Find the equation of the oblique asymptote of function $f(x)=\frac{3x^3+4x-5}{x^2+1}$

Solution:

Let $y=ax+b$ be the oblique asymptote. Then:

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{3x^3+4x-5}{x^3+x}$$
$$=3$$

$$b=\lim_{x\to\pm\infty}\left[f(x)-3x\right]$$
$$=\lim_{x\to\pm\infty}\frac{3x^3+4x-5-3x^3-3x}{x^2+1}$$
$$=\lim_{x\to\pm\infty}\frac{x-5}{x^2+1}$$
$$=0$$

Thus, $y=3x$ is the oblique asymptote for $f(x)=\frac{3x^3+4x-5}{x^2+1}$

Example2: Let $f(x)=\frac{x^2+2x-3}{x}$. Find all possible asymptotes

Solution:

$\lim_{x\to 0^+}f(x)=-\infty$ and $\lim_{x\to 0^-}f(x)=+\infty$. Thus, there is a vertical asymptote $\text{V.A}\equiv x=0$

$\lim_{x\to\pm\infty}f(x)=\pm\infty$. Thus, the horizontal asymptote does not exist.

To find oblique asymptote, let $y=ax+b$ be the oblique asymptote.

$$a=\lim_{x\to\pm\infty}\frac{f(x)}{x}$$
$$=\lim_{x\to\pm\infty}\frac{x^2+2x-3}{x^2}$$
$$=1$$

Since $1\neq 0$, let us find b.

$$b=\lim_{x\to\pm\infty}\left[f(x)-ax\right]$$
$$=\lim_{x\to\pm\infty}\left[\frac{x^2+2x-3}{x}-x\right]$$
$$=\lim_{x\to\pm\infty}\frac{2x-3}{x}$$
$$=2$$

Then, $O.A \equiv y = x + 2$.

As the degree of the numerator exceeds the degree of the denominator by one, we could find oblique asymptote after performing long division

	$x+2$
x	x^2+2x-3 $-(x^2)$ $2x-3$ $-(2x)$ -3

$f(x) = x + 2 - \frac{3}{x}$. Thus, there exists an oblique asymptote $O.A \equiv y = x + 2$

Activity 2: Guided Practice

Task22

1. Find the asymptotes of the following functions
   a. $f(x) = \frac{x^2+2x+1}{x+3}$
   b. $f(x) = \frac{x-8}{x+4}$

Activity 3: Application

Task23

1. Find the equation of horizontal asymptote of the equation:

$$f(x) = \frac{x^2 - 1}{x^2 + 2}$$

2. Find the equation of the oblique asymptote to the curve representing $f(x) = \frac{x^2+3x-1}{2x+1}$

3. Find the equation of the vertical asymptote to the curve representing $f(x) = \frac{x^2+3x-1}{2x+1}$

Formative Assessment

Formative Assessment for Unit2

Instructions: Attempt all questions

1. Analyze the graph of a function to determine its parity: Based on the graph provided, is the function odd, even, or neither? Explain your reasoning in terms of symmetry."
2. Find the domain of definition of $f(x)$

a. $f(x) = \frac{x+2}{x+1}$

b. $f(x) = \frac{x^2-1}{x^2-4}$

c. $f(x) = \sqrt{x^2 + x - 2}$

3. Find the range of $f(x)$

a. $f(x) = \frac{1}{x+2}$ b. $f(x) = 2x + 5$

4. Create your own example of an even function and an odd function.
5. Ensure symmetry is visually clear for odd or even functions.
6. Study the parity of the functions

a. $f(x) = \frac{x+1}{x-4}$ b. $f(x) = \frac{1-3x^2}{1-x^2}$

c. $f(x) = x^2 - 1$

7. Find the asymptotes of the following functions

c. $f(x) = \frac{x^2+2x+1}{x+3}$

d. $f(x) = \frac{x-8}{x+4}$

8. **F**ind the limits of the following functions

a. $\lim_{x\to 2} x^3 - 3x^2 + 5$

b. $\lim_{x\to 3} \frac{x^2-9}{x-3}$

c. $\lim_{x\to +\infty} \frac{3x^2+3x+4}{5x+4}$

Points to Remember

- **Domain of a function** is the set of all real numbers for which the expression of the function is defined as a real number.
  The range of function f, denoted by $Im(f)$ is the image of A under f
- **Even function**
  we say that a function f is even if $\forall\, x \in Dom\,(f), (-x) \in Dom(f)$;

$$f(-x) = f(x)$$

- **Odd function**
  We say that a function f is odd if $\forall x \in Dom\,(f), (-x) \in Dom(f)$;

  $f(-x) = -f(x)$

- we say that $f(x)$ has a limit of L as x approaches a, and we write:
  $\lim_{x \to a} f(x) = L$
- Let f be a function defined on both sides of a, except possibly at a itself. Then
  $\lim_{x \to a} f(x) = \infty$, it's called **infinity limit**
- Let f be a function defined on some interval, (a, ∞) . Then
  $\lim_{x \to \infty} f(x) = L$, it's called **limit at infinity**
- There exist 3 kind of asymptotes:
- **Vertical asymptotes**
  $V.A \equiv x = a$, where $\lim_{x \to a} f(x) = \infty$
- **Horizontal asymptotes**
  $H.A \equiv Y = b$, where $b = \lim_{x \to \infty} f(x)$
- **Oblique asymptotes**
  $O.A \equiv y = ax + b$, where $a= \lim_{x \to \infty} \frac{f(x)}{x}$

  $b= \lim_{x \to \infty} f(x) - ax$

Self-Reflection

I. Observe the figure illustrated above and answer to the question below:

a. Name the relation between subset A and subset B

b. How do we call the elements in A?

c. How do we call the elements in **B?**

d. Will the relation be named the same name in 1 if one element in A is paired with more than one element in B?

e. On the graph illustrated above name all line in that graph

f. Examine the distance between the line and the curve, what conclusion can you take?

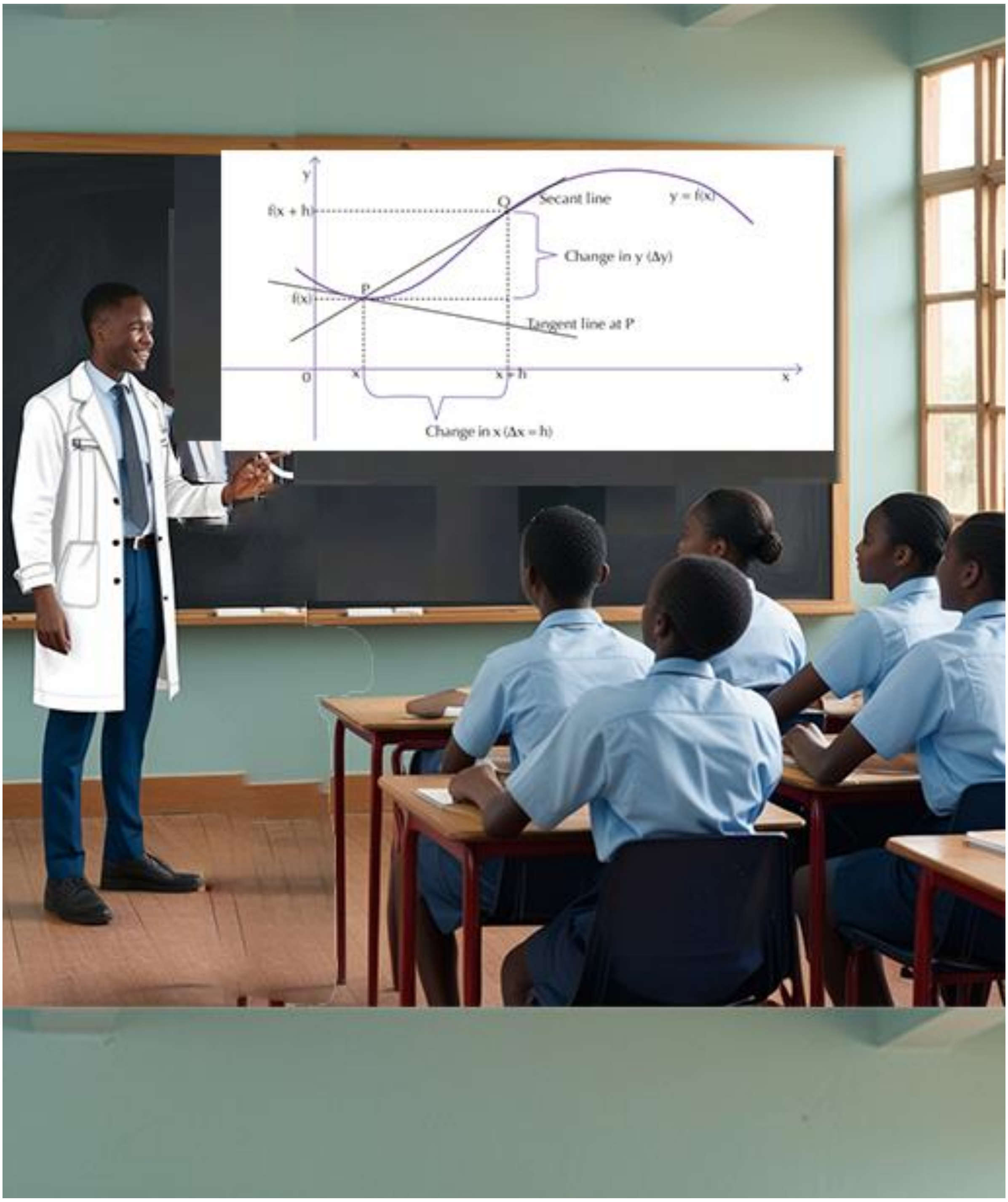
II. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Explain algebraic expressions and function.					
Explain the concepts of domain, range, limits, and asymptotes.					
Factorize techniques and operations with limits.					
Describe graphical properties (symmetry, end behavior, asymptotes).					
Recognize patterns and apply the correct analysis techniques.					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Identify key features of algebraic expression and function (domain, range, limits).					
Precision for careful attention to detail in calculations and interpretations for algebraic functions					
Open-mindedness in Being open to multiple approaches and graphical tools.					
Critical Thinking in Questioning and verifying the results.					

III. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	**1.**	**1.**
2.	**2.**	**2.**
3.	**3.**	**3.**


y
f(x + h)
Q
Secant line
y = f(x)
Change in y (Δy)
P
f(x)
Tangent line at P
0
x
x + h
x
Change in x (Δx = h)

Unit summary

This unit provides you with the knowledge, skills and attitudes required to apply fundamentals of differentiation required to apply basic mathematical analysis. It covers determination ofderivatives, application of derivatives and sketching Curve of algebraic function.

Self-Assessment: Unit 3

1. Observe the figure illustrated above on the beginning of unit 3 and answer to the questions :
    a. Evaluate the rate change of y with respect to x
    b. How do we call $\frac{\Delta y}{\Delta x}$?
    c. Evaluate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
    d. Comment your result in (c)
2. Relate the vertex of quadratic function to maximum and minimum of function.

3. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Explain the concept of limits.					
Familiarity with basic differentiation formulas.					
Grasp of the meaning of derivatives (rate of change, slope of tangents).					
Knowledge of higher-order derivatives.					
Explain applications (e.g., optimization, motion, graphing).					
Apply differentiation rules (power, product, chain).					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Solve real-world problems using differentiation.					
Simplify complex expressions before differentiation.					
Interpret derivative results in practical contexts.					
Identify points of maxima, minima, or inflection.					
Curiosity to explore mathematical concepts.					
Perseverance to solve complex problems.					
Confidence in apply mathematical reasoning.					
Open-mindedness to learn new methods.					
Willingness to analyze and verify solutions.					

Key Competencies:

Knowledge	Skills	Attitudes
1. Explain the concept of limits.	**1.** Apply differentiation rules (power, product, chain).	**1.** Curiosity to explore mathematical concepts.
2. Familiarity with basic differentiation formulas.	**2.** Solve real-world problems using differentiation.	**2.** Perseverance when solving complex problems.
3. Explain the meaning of derivatives (rate of change, slope of tangents).	**3.** Simplify complex expressions before differentiation	**3.** Confidence in applying mathematical reasoning.

Discovery activity:

Task 24

1. What does it mean a term derivatives of functions?
2. Distinguish between tangent line and normal line.
3. Explain what is meant by the word slope of function.
4. What do you understand by finding the derivative of algebraic function?
5. What do you know about sketching of the function?

Topic 3.1. Determination of derivatives

Activity 1: Problem Solving

Task 25

I. Consider the following function $f\,(x) = x^2 + 1$

   a. Evaluate the limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

   b. Name the limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

II. Why is the method of finding the derivative using the limit definition referred to as "differentiation from first principles"?

III. Explain why for non-linear functions, the slope of the tangent line remains constant at every point.

Key Facts 3.1a: Determination of Derivatives

✓ **Derivative of function .**

For a non-linear function with equation y = f(x), slopes of tangents at various points continually change.

Consider the following figure:

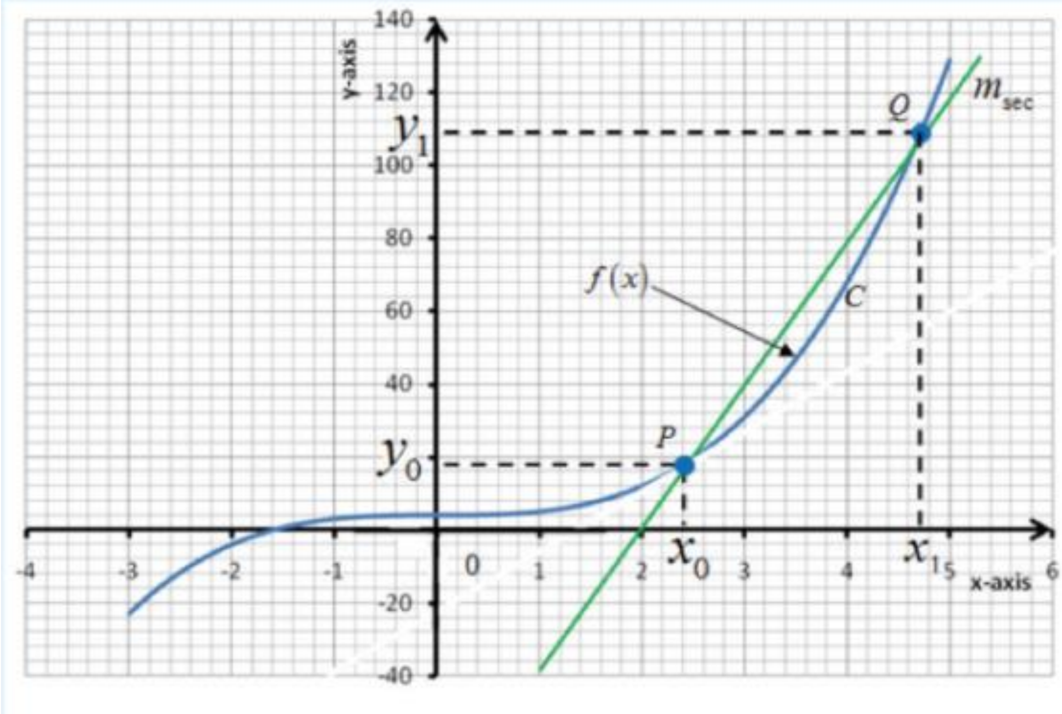


Figure 7: derivative of function

The derivative of a function *f*(*x*) with respect to *x* is denoted by ***f* '(*x*)** and provided that the limit exists.

The derivative of a function, also known as slope of a function, or derived function or simply the derivative, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note:

- The slope (gradient) of the tangent to a curve of f(x) is defined as the slope of the curve f(x), and is the instantaneous rate of change in y with respect to x.
- Finding the slope using the limit method is said to be using first principles.
- A chord (secant) of curve is a straight line segment which joins any two points on the curve.
- A tangent is straight line which touches curve at point.

✓ **Derivative of function at a given point.**

The derivative of a function f(x) at x = a is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Alternative formula for finding f′(a) is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Thus, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ is the slope of the tangent at x and is called the derivative at x = a .

The slope of the tangent at the point x = a is defined as the slope of the curve at the point where x= a, and is the instantaneous rate of change in y with respect to x at that point.

Example1:

Let $f(x) = x^2 + 1$, find $f'(x)$

Solution:

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

Thus, $f'(x) = 2x$

Remark:

The process of finding derivative of a function is called **differentiation** of that function.

- ✓ **Derivative of a polynomial, a rational and irrational function.**

Differentiation is the process of finding the derivative function. If we are given a function f(x) then f′(x) represents the derivative function. However, if we are given y in terms of x then y′ or $\frac{dy}{dx}$ are usually used to represent the derivative function.

Note: $\frac{dy}{dx}$ reads 'dee y dee x', or 'the derivative of y with respect to x'. $\frac{dy}{dx}$ is not a fraction.

We write **dx** instead of **"Δx heads towards 0"**, so "the derivative of" is commonly written as $\frac{d}{dx}$, so $\frac{d}{dx}(x^2) = 2x$.

Below, there is basic rules which can be used to differentiate more complicated functions without using the differentiation from the first principles.

- ✓ **Derivative of a polynomial, a rational and irrational function.**

<u>Rule of differentiation</u>

Differentiation is the process of finding the derivative function. If we are given a function f(x) then f'(x) represents the derivative function. However, if we are given y in terms of x then y' or $\frac{dy}{dx}$ are usually used to represent the derivative function.

1. <u>Derivative of a constant function</u>

If f is a constant function, say $f(x)=c$, for all x, then

$$\frac{df}{dx}=\frac{d}{dx}(c)=0$$

2. <u>Derivative of a power</u>

If $\boldsymbol{n}$ is any real number, then

$\frac{d}{dx}x^n = nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.

for all x where the powers x^n and x^{n-1} are defined.

This holds for any function with power. Thus, if $f \in (D,I)$ for positive and negative, and fractional value of n,

$$(f^n)' = nf^{n-1}f'.$$

<u>Particular case:</u>

Let $f(x)=\sqrt{g(x)}$.

Here $n=\frac{1}{2}$ because $\sqrt{g(x)}=[g(x)]^{\frac{1}{2}}$

The derivative of $f(x)$ is as follows

$$f'(x)=\frac{1}{2}[g(x)]^{\frac{1}{2}-1}g'(x)$$
$$=\frac{1}{2}[g(x)]^{-\frac{1}{2}}g'(x)$$
$$=\frac{1}{2}\frac{g'(x)}{[g(x)]^{\frac{1}{2}}}$$
$$=\frac{g'(x)}{2\sqrt{g(x)}}$$

Thus, if $f(x)=\sqrt{g(x)}$ then $f'(x)=\frac{g'(x)}{2\sqrt{g(x)}}$

4. Multiplication by a scalar

If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cf(x))=c\frac{d}{dx}f(x)$$

5. Derivative of a product

If f and g are differentiable at x, then their product is $f\cdot g$, hence

$$\frac{d}{dx}(f\cdot g)=g\frac{df}{dx}+f\frac{dg}{dx}$$

6. Sum (difference)of functions

Let $D(I,\mathbb{R})$ be the set of functions differentiable on I.

If $f\in D(I,\mathbb{R})$ and $g\in D(I,\mathbb{R})$, then $f\pm g\in D(I,\mathbb{R})$.

In addition $\frac{d}{dx}(f\pm g)=\frac{df}{dx}\pm\frac{dg}{dx}$

7. **Derivative of the rational function**

If f and g are differentiable at x and if $g(x) \neq 0$, then

the quotient $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

8. **Derivative of a composite function: Chain rule**

Composite functions are functions like $(3x^2 + 4x)^5$, $\sqrt{2x-1}$ or $\frac{1}{x^2+2x+3}$. These functions are made up of two simpler functions.

(a) $y = (3x^2 + 4x)^5$ is $y = u^5$ where $u = 3x^2 + 4x$

(b) $y = \sqrt{2x-1}$ is $y = \sqrt{u}$ where $u = 2x - 1$

(c) $y = \frac{1}{x^2+2x+3}$ is $y = \frac{1}{u}$ where $u = x^2 + 2x + 3$

Notice that in the example $(3x^2 + 4x)^5$, if $f(x) = x^5$ and $g(x) = 3x^2 + 4x$ then $f(g(x)) = f(3x^2 + 4x) = (3x^2 + 4x)^5$

All of these functions can be made up in this way where we compose a function of a function. Thus, these functions are called **composite functions**.

Consider the function $y = (2x + 1)^3$ which is really $y = u^3$ where $u = 2x + 1$.

- We see that $\frac{dy}{du} = 3u^2 = 3(2x + 1)^2$ and $\frac{du}{dx} = 2$
- When we expand, we obtain

  $y = (2x + 1)^3 = (2x)^3 + 3(2x)^2 1 + 3(2x)1^2 + 1^3 = 8x^3 + 12x^2 + 6x + 1$

  $\frac{dy}{du} = 24x^2 + 24x + 6 = 6(4x^2 + 4x + 1) = 3(2x + 1)^2 \times 2 = \frac{dy}{du}\frac{du}{dx}$

From the above example we derive the formula of the chain rule:

If $y = f(u)$ where $u = u(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

In general, If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

✓ **Successive derivatives**

We have seen that the derivative of $y = f(x)$ is in general also a function of *x*. This new function can be also differentiable, in which case the derivative of the first derivative is called the **second derivative** of the original function. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2\left(f(x)\right) = D^2 f(x)\cdot$$

The symbol d^2means the operation of differentiation is performed twice.

Similarly, the derivative of the second derivative is called the **third derivative** and so on.

Thus, if for example $y = 3x^4$ then,

$$\frac{dy}{dx} = 12x^3, \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = 36x^2, \quad \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 72x \text{ And so on.}$$

The successive derivatives of a function *f* are higher order derivatives of the same function.

We denote higher order derivatives of the same function as follows:

The second derivative is:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x) = y''$$

The third derivative is:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x) = y'''$$

And the n^{th} derivative is:

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = f^{(n)}(x) = y^{(n)}$$

- ✓ **Interpretation of derivative of a function**

If the function $y = f(x)$ is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of y with respect to x. Since $f''(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called convexity (bending) of a curve.

If $x = t$ is time and if $y = s(t)$ is displacement function of moving object, then $s'(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity i.e. the second derivative of the displacement function is $s''(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity function, which is, the acceleration function.

Consider a body moving along the x-axis such that its displacement, x metres on the right of the origin 0 after a time t seconds ($t \geq 0$), is given by $x = f(x)$.

The average velocity of the body in the time interval [t, t+h] is given by

$$\vec{v} = \frac{\text{total displacement}}{\text{total time taken}} = \frac{f(t+h) - f(t)}{h} \quad (h \neq 0).$$

In order to find the instantaneous velocity of the body at time t seconds, we find the average velocity in the time interval [t, t+h] and let h take smaller and smaller values. In fact the instantaneous velocity of the body at time t seconds is defined to be

$$v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

This is the first derivative of the function $x = f(t)$ and so $v(t) = \frac{dx}{dt} = f'(t)$.

Then differentiating with respect to time the 'dot' notation is often used.

Thus, if $x = f(t)$, then $v(t) = \frac{dx}{dt} = \dot{x} = f'(x)$

We define the **velocity** to be the rate of change of displacement.

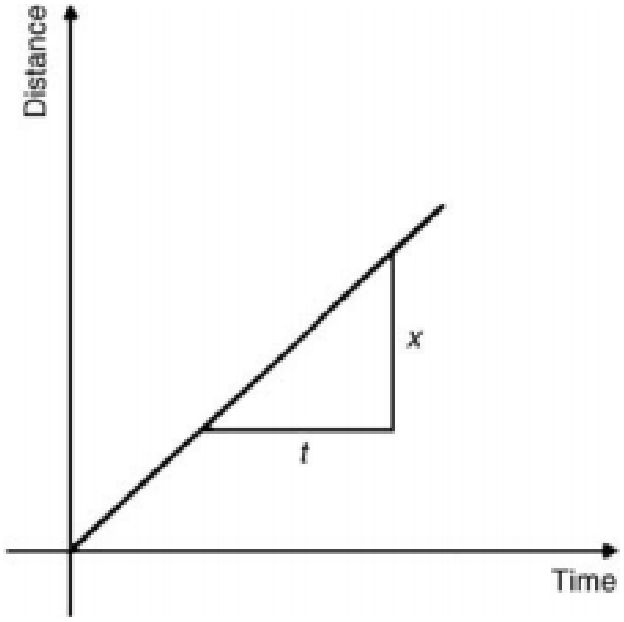
Acceleration is defined as the rate of change of velocity. Thus the acceleration of the body moving along a straight line with displacement $x(t)$ at time t and is given by

$$a(t) = \frac{d}{dx}(v(t)) = v'(t) = \frac{d^2}{dt^2}(x(t)) = x''(t) = \ddot{x}(t).$$

Velocity is a vector quantity and so the direction is critical. If the body is moving towards the right (the positive direction of the x-axis), its velocity is positive and if it is moving towards the left, its velocity is negative. Therefore, the body changes motion when velocity changes sign. A sign diagram of the velocity provides a deal with information regarding the motion of the body

- **Kinematical meaning of a derivative**

When a car moves a distance x metres in a time t seconds along a straight road, if the velocity v is constant then $v = x / t$ m/s



Summarising, if a body moves a distance x metres in a time t seconds then:

(i) **distance** $x = f(t)$.

(ii) **velocity** $v = f'(t)$ **or** $\frac{dx}{dt}$, which is the gradient of the distance/time graph.

(iii) **acceleration** $a = \frac{dv}{dt} = f''(t)$ **or** $\frac{d^2x}{dt^2}$, which is the gradient of the velocity/time graph.

Activity 2: Guided Practice

Task 26

1. Find from the first principal, the slope of the tangent of the following functions at a given value of x

   a. $f(x) = 2x^2 + 3$ at $x = 2$

   b. $f(x) = \frac{2x-1}{x+3}, at\ x = -1$

   c. $f(x = \sqrt{x}\ at\ x = 9$

Activity 3: Application

Task 27

1. Determine from first principles the derivative of the following functions:

(a) $\boldsymbol{f(x) = 2}$ (b) $f(x) = x + x^3$ (c) $f(x) = x^3 + 2x + 3$ (d) $f(x) = x^4 - \frac{1}{4}x$

2. If $f(x) = 4x + 2x^2$, find:

(a) $\boldsymbol{f'(x)}$ using the definition of derivative (b) $\boldsymbol{f'(2)}$

(c) $\boldsymbol{f'(-2)}$

3. For each of the following, find, from first principles, the derivative:

(a)$f(x) = \frac{1}{x+2}$ (b) $f(x) = \frac{1}{2x-1}$ (c) $f(x) = \frac{1}{x^2}$ (d) $f(x) = \frac{1}{x^3}$

4. Find, from first principles, the derivative of the following functions:

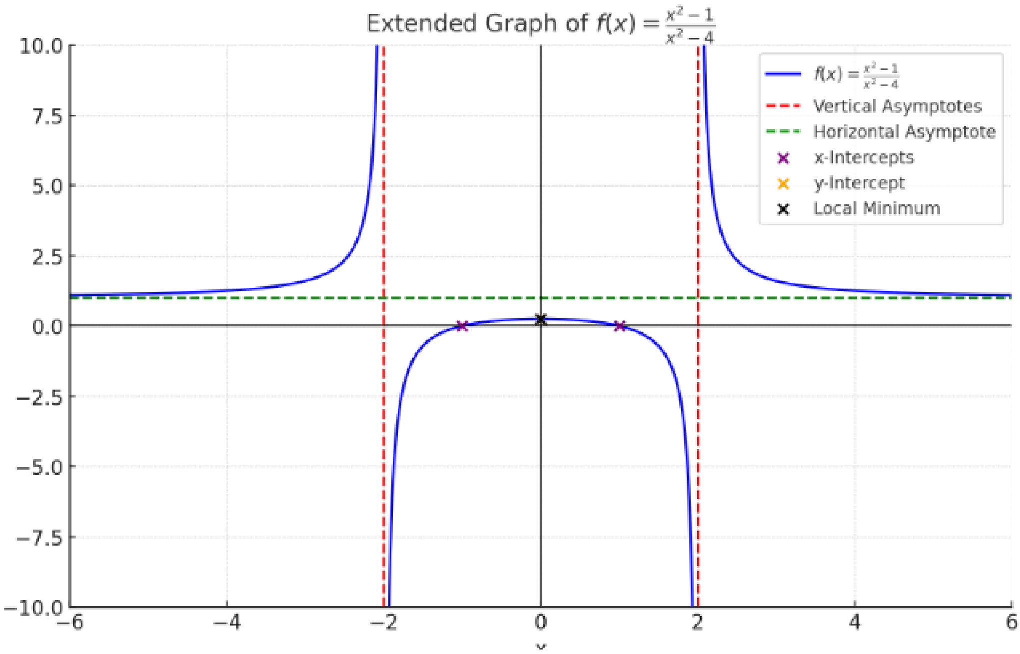
(a) $\sqrt{x+2}$ (b)$\frac{1}{\sqrt{x}}$ (c)$\sqrt{2x+1}$

Topic 3.2: Application of derivatives

Activity 1: Problem Solving

Task 28

I. **O**bserve the graph above and answer to the questions:



1. The interval where the curve turns up
2. The interval where the function is increasing
3. The interval where the function is decreasing
4. The intercept point
5. The maximum point on the graph
6. Find $f\acute{}(x)$ and study its sign
7. Find $f''(x)$ and study its sign
8. Find the interval where $f'(x) > 0$
9. Find the interval where $f''(x) > 0$
10. Compare results in 8 and 2; compare also your results in 9 and 1
11. What can conclude about 10

Key Facts 3.2a: Application of derivatives

✓ **Tangent line to a curve of function**

Consider a curve y = f(x). If P is the point with x-coordinate at point $\boldsymbol{a}$, then the slope of the tangent at this point is $f'(a)$. The equation of the tangent is obtained by equating slopes :

$$\frac{y - f(a)}{x - a} = f'(a) \text{ or}$$

$$y - f(a) = f'(a)(x - a)$$

<u>Or</u>

The slope of the tangent line of $y = f(x)$ at $(x_0, f(x_0) = y_0)$ is given by $f'(x_0) = \dfrac{y - y_0}{x - x_0}$.

Then, the equation of the tangent line is

$$T \equiv y - y_0 = f'(x_0)(x - x_0)$$

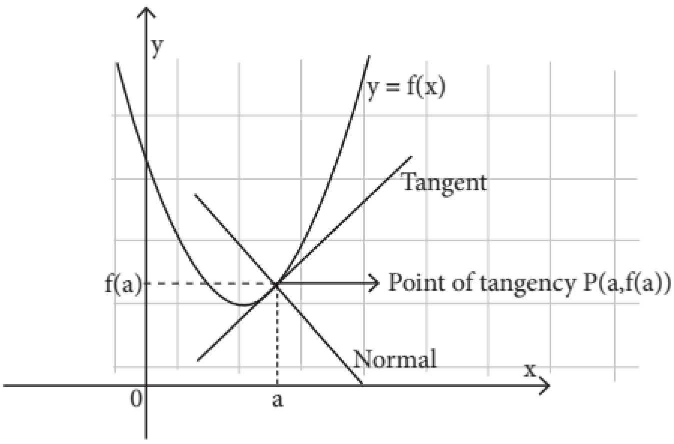


Figure 8: Equation of tangent and normal line

✓ **Normal line to a curve of function**

A normal to a curve is a line which is perpendicular to the tangent at the point of contact. The slope of a normal at x = a is $\frac{-1}{f'(a)}$. This comes from the fact that the product of gradients of two perpendicular line is $-\mathbf{1}$

Note: If a tangent touches y = f(x) at (a,b) then it has equation

$$N \equiv y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$$

- ✓ **Maximum and minimum points of a function**
- **Stationary point (extreme point, turning point)**

This is a point on the graph $y = f(x)$ at which f is differentiable and $f'(x) = 0$.

The graph of a function $y = f(x)$ at which f is differentiable and $f'(x) = 0$. Have a maximum, if $f'(x) > 0$ and minimum, if $f'(x) < 0$.

Note: Maximum and minimum values are termed as extreme values.

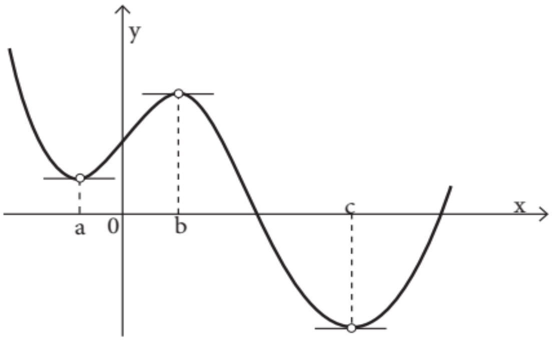


Figure 9: Maximum and Minimum of function

The points a, b and c are stationary points.

We can see the stationary points on the example below

Find the stationary point of the function defined by $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$

Solution

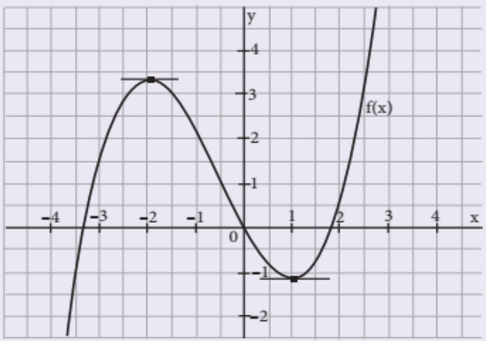
$f'(x) = x^2 + x - 2$

$f'(x) = 0 \Rightarrow x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$

$x = 1$ or $x = -2$

The stationary points of the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$ are found at $x = 1$ and $x = -2$ and are (1, –1.17) and (–2, 3.33).



- ✓ **Concavity and inflection point**

A point of inflexion is a point on a graph y = f(x) at point $\boldsymbol{x = a}$, it is necessary that $f''(a) = 0$, and so this is the usual method of finding possible points of inflexion.

A curve is said to be concave downwards (or concave) in an interval ,If $f''(x) < 0$ and curve is said to be concave upwards (or convex) in an interval ,if $f''(x) > 0$.

At a point of graph $y = f(x)$, it may be possible to specify the concavity by describing the curve as either concave **up** or concave **down** at that point, as follows:

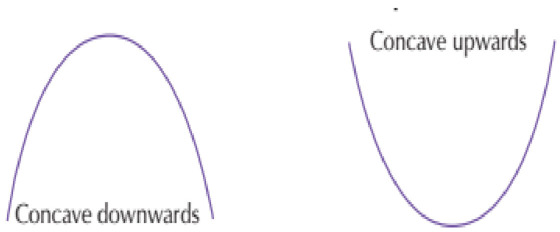


Figure 10: concavity of function

1. A curve is said to be concave downwards (or concave) in an interval]a, b[If f''(x) < 0 for all x∈]a,b[.
2. A curve is said to be concave upwards (or convex) in an interval]a, b[if f''(x) > 0 for all x ∈]a,b[.

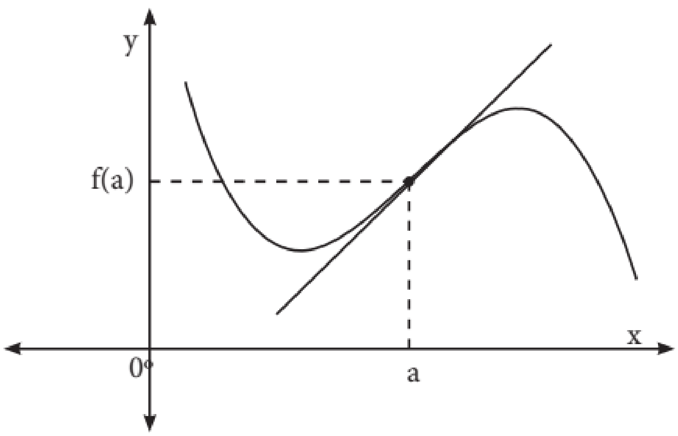


Figure 11: inflection point of function

Let study the concavity of the function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$ and State the value of x for which the curve of f is concave upwards then we Find the x coordinate of each point of inflection. $f'(x) = (\frac{x^3}{3} + \frac{x^2}{2})' = x^2 + x \leftrightarrow f''(x) = 2x + 1 \leftrightarrow f''(x) = 0 \leftrightarrow x = \frac{-1}{2}$

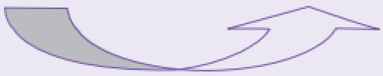
Solution

(a) $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

$f'(x) = x^2 + x = x(x + 1)$

The sign of f″(x)

$f''(x) = 2x + 1$

x \ Function	$-\infty$		$-\frac{1}{2}$		$+\infty$
$f''(x)$		$-$	0	$+$	
$f(x)$					

The curve is concave upwards for **x**$> -\frac{\mathbf{1}}{\mathbf{2}}$ **,**Dawn word for **x**$< -\frac{\mathbf{1}}{\mathbf{2}}$

d x=$-\frac{\mathbf{1}}{\mathbf{2}}$ is a point of inflection.

Activity 2: Guided Practice

Task 29

1. Find the equation of the tangent $to\ f(x) = x^2 + 2\ at\ the\ point\ where\ x = 1$

2. Find the equation of the normal $to\ f(x) = \frac{8}{\sqrt{x}}\ at\ the\ point\ where\ x = 8$**.**

3. Given function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$**,** Find the interval where the function is increasing and where it is decreasing and Find all stationary points.

Activity 3: Application

Task 30

1. In each of the following questions, find, at the given point, the equation of the tangent and the equation of the normal.
   a. $y = x^2 - 4 \; at \; x = 1$
   b. $y = x^2 + 4x - 2 \; at \; x = 0$
   c. $y = \frac{1}{x} \; at \; x = -1$
   d. $y = x^2 + 5 \; at \; x = 0$
   e. $y = x^2 + 5 \; at \; x = 0$
   f. $y = (x - 2)(x^2 - 1) \; at \; x = -2$
2. Find the equation of the tangent to each of the following functions at an indicated point:
   a. $y = x - 2x^2 + 3 \; at \; x = 2$
   b. $y = \sqrt{x} + 1 \; at \; x = 4$
   c. $y = x^3 - 5x \; at \; x = 1$
   d. $y = \frac{4}{\sqrt{x}} \; at \; (1, 4)$
3. Find the equation of the normal to each of the following functions at an indicated point:
   a. $y = x^2$ at the point (3, 9)
   b. $y = x^2 - 5x + 2 \; at \; x = -2$
   c. $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at the point (1, 4)
   d. $y = 8\sqrt{x} - \frac{1}{x^2} \; at \; x = 1$

Topic 3.3: Sketching curve of algebraic function

Activity 1: Problem Solving

Task 31

Given the function $f(x) = \frac{x^2-1}{x^2-4}$

1. Write the Domain of $f(x)$
2. Find all equations of asymptotes
3. Show the sign of first derivative and second derivative
4. Show the intercept points with axes
5. On the same plane draw y and x axes, include in in your drawing all asymptotes you egt in 2,
6. Show all stationary points on your graph and use curves to join all supplementary points of your choice.

Key Facts 3.2 a: Sketching Curve of algebraic function

✓ **Variation table and Additional points.**

The variation table is a table typically consists of:

1. **Rows** that represent:
   - The values of x (domain of the function or intervals of all stationary point).
   - The sign and value of the first derivative $f'(x)$ (positive, negative, or zero).
   - The sign and value of the second derivative $f''(x)$ (positive, negative, or zero).
   - The behavior of the function $f(x)$ (increasing, decreasing, extrema, asymptotes).
2. **Columns** that divide the domain of the function into significant intervals (e.g., based on critical points, asymptotes, etc.)

   Note that the table of variation is a summary in 4 rows and its shows the behaviour of the function

✓ **Graph of polynomial and rational function.**

To sketch a graph you have to take the following considerations:

Domain of the Function

- Determine where the function is defined.
  - Identify any restrictions (e.g., division by zero, square roots of negative numbers, or logarithms of non-positive numbers).
  - Write the domain as an interval or union of intervals.

Intercept

- **X-intercepts**: Solve $f(x) = 0$ to find where the graph crosses the x-axis.

y-intercept: Find$f(0)$ f 0is in the domain) to determine where the graph crosses the y-axis.

Symmetry

- Check for symmetry to simplify sketching:
  - **Even function**: $f(x) = f(-x)$ The graph is symmetric about the y-axis.
  - **Odd function**: $f(x) = -f(-x)$ The graph is symmetric about the origin.

Asymptotes

- **Vertical asymptotes**: Occur when $f(x) \rightarrow \pm\infty$ as x approaches certain values (e.g., denominators equal to zero).
- **Horizontal asymptotes**: Check the limits of $f(x)$ as $x \rightarrow \pm\infty$ is a real value
- **Oblique asymptotes**: If $f(x)$ approaches a slanted line $as\ x \rightarrow \pm\infty$ calculate it using long division of polynomials (if applicable).

First Derivative f′(x)**for Increasing/Decreasing Behavior**

- Find $f'(x)$ and solve $f'(x) = 0$ to determine critical points.
- Test the sign of $f'(x)$ in each interval to identify:
  - **Increasing**: $f'(x) > 0$
  - **Decreasing**: $f'(x) < 0$
- Identify **local maxima** and **local minima**:
  - **Local maximum**: $f'(x)$ changes from positive to negative.
  - **Local minimum**: $f'(x)$ changes from negative to positive.

Second Derivative (f′′(x)f''(x)f′′(x)) for Concavity and Inflection Points

- Find $f''(x)$ and solve f $f''(x)$ =0 to identify potential **inflection points**.
- Test the sign of $f''(x)$ in each interval to determine:

- **Concave up**: $f''(x)$ 0 (curve opens upward).
- **Concave down**: $f''(x)$)<0 (curve opens downward).

- An **inflection point** occurs where $f''(x)$ changes sign.

Behavior at Infinity and Near Asymptotes

- Analyze the **end behavior** of the function as $x \to \pm\infty$ Determine the limits of $f(x)$
- Examine the function's behavior near vertical asymptotes (approaches $+\infty$ or $-\infty-$).

Key Facts 3.2 b: Sketching Curve of algebraic function

Variation table and Additional points.

Steps to draw the table of variation

- Identify the **domain** of $f(x)$ by determining where the function is defined (e.g., check for values that make the denominator zero, or square roots of negative numbers).

- Calculate $f'(x)$ using the rules of differentiation.
- Set $\boldsymbol{f'(x) = 0}$ and solve for x The solutions are the critical points (extrema : maximum or minimum) where the slope of the tangent is zero.
- Determine where $f'(x)$ is undefined (these might indicate vertical asymptotes or other discontinuities).
- Show the sign of first derivative
- Calculate the second derivative and its sign
- Construct the table and remember to show concave up/down, increasing/decreasing function.

- An inflection point occurs where $f''(x)$ changes sign.

For example to find the variation table of $f(x) = \frac{x+1}{x-2}$, we fistrly compute the

fist and second derivative

$$f'(x) = \frac{(x+1)'(x-2)-(x+1)(x-2)'}{(x-2)^2}$$

$$f'(x) = \frac{x-2-x-1}{(x-2)^2}$$

$$= \frac{-3}{(x-2)^2}$$

$$f''(x) = \frac{(-3)'(x-2)^2-(-3)[(x-2)^2]'}{[(x-2)^2]^2}$$

$$f''(x) = \frac{6(x-2)}{(x-2)^4} \qquad f''(x) = \frac{6}{(x-2)^3}$$

Then construct the table of variation

x	$-\infty$	2	$+\infty$
$f'(x)$	- - - - - - - -	- - - - - -	
$f''(x)$	- - - - - - - - -	+ + + + + +	
$f(x)$			

Graph of polynomial and rational function.

Behavior at Infinity and Near Asymptotes

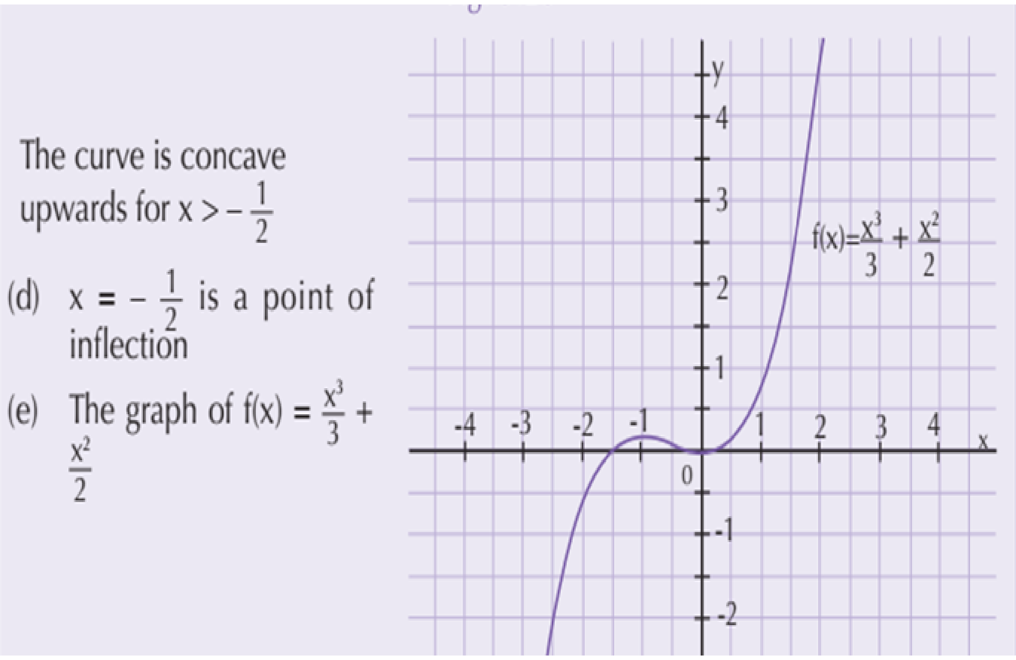
- Analyze the end behavior of the function as $x \to \pm\infty$ Determine the limits of $f(x)$
- Examine the function's behavior near vertical asymptotes (approaches $+\infty$ or $-\infty-$).

For example

For the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$:

1. **Domain:** $x \in \mathbb{R}$.
2. **Intercepts:** $x = 0$ is both the x- and y-intercept.
3. **Symmetry:** None.
4. **Asymptotes:** None (polynomial functions have no asymptotes).
5. **Critical points:**
   - $f'(x) = x^2 + x = x(x+1)$: Critical points are $x = 0$ and $x = -1$.
   - $f(x)$ increases on $(-1, \infty)$ and decreases on $(-\infty, -1)$.
6. **Second derivative:**
   - $f''(x) = 2x + 1$: Inflection point at $x = -0.5$.
   - Concave down on $(-\infty, -0.5)$ and concave up on $(-0.5, \infty)$.

Its graph is represented by



✓ Graph of rational function

Sketch the function $f(x) = \frac{x^2-1}{x^2-4}$

Step 1: Domain of the function

The denominator cannot be zero. Solve $x^2 - 4 = 0$:

$$x = \pm 2$$

So, the domain of $f(x)$ is $x \in \mathbb{R} \setminus \{-2, 2\}$.

- **Vertical asymptotes** occur at $x = -2$ and $x = 2$, since the function approaches infinity near these points.

Step 2: Horizontal asymptote

For large values of x ($x \to \pm\infty$), compare the degrees of the numerator and denominator:

- The degree of the numerator and denominator is 2.
- Divide the leading coefficients:

$$\lim_{x \to \pm\infty} f(x) = \frac{1}{1} = 1$$

So, the **horizontal asymptote** is $y = 1$.

Step 3: Intercepts

- **x-intercepts**: Solve $f(x) = 0$:

$$\frac{x^2 - 1}{x^2 - 4} = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$$

Thus, the x-intercepts are $x = -1$ and $x = 1$.

- **y-intercept**: Evaluate $f(0)$:

$$f(0) = \frac{0^2 - 1}{0^2 - 4} = \frac{-1}{-4} = \frac{1}{4}$$

Thus, the y-intercept is $(0, \frac{1}{4})$.

Step 4: Behavior near asymptotes

- Near $x = -2$ and $x = 2$, the function approaches $\pm\infty$.
  - For $x \to 2^+$: $f(x) \to +\infty$
  - For $x \to 2^-$: $f(x) \to -\infty$
  - Similar behavior occurs for $x = -2$.

Step 5: Critical points and increasing/decreasing intervals

1. **Derivative of $f(x)$:**

$$f'(x) = \frac{(x^2-4)(2x)-(x^2-1)(2x)}{(x^2-4)^2} = \frac{6x}{(x^2-4)^2}$$

2. **Critical points:** Solve $f'(x) = 0$:

$$6x = 0 \implies x = 0$$

Thus, the critical point is at $x = 0$.

3. **Sign of $f'(x)$:**

- For $x < 0$: $f'(x) < 0$ (decreasing).
- For $x > 0$: $f'(x) > 0$ (increasing).
  Thus, the function has a **local minimum** at $x = 0$.

Step 6 : Summary and

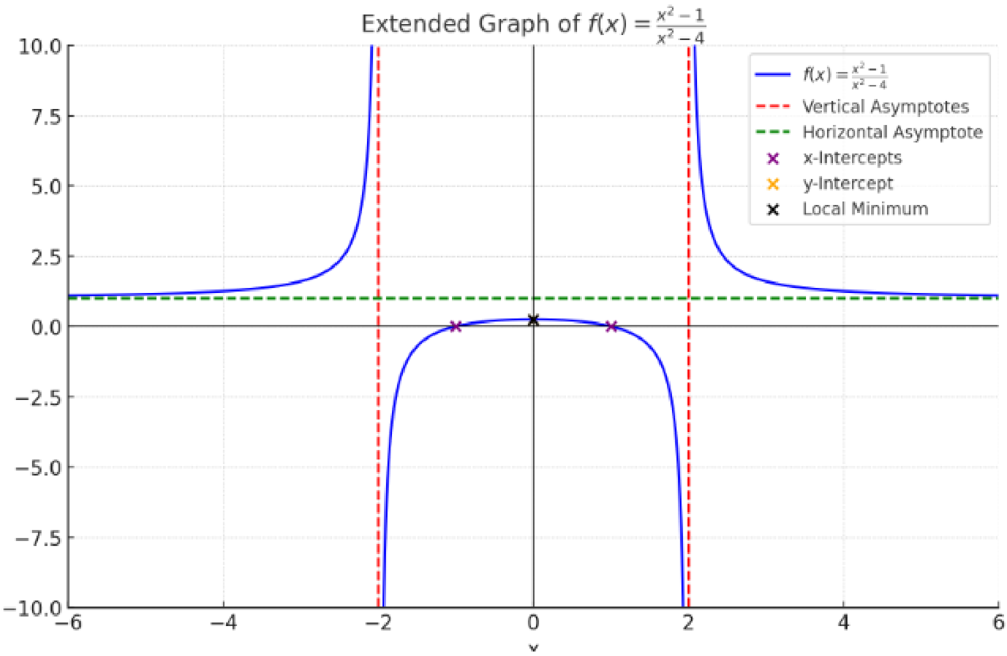
. Vertical asymptotes: x = -2, x = 2

. Horizontal asymptote: y = 1

. Intercepts: x = -1, x = 1, y =

. Local minimum: x = 0

Step 7. Graph



Activity 2: Guided Practice

Task 32

I. Find the variation table of

1. $f(x) = \frac{x+1}{x-2}$
2. $f(x) = \frac{x^2-1}{x^2-4}$
3. $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

II. Sketch the function

1. $f(x) = \frac{x^2-1}{x^2-4}$

Activity 3: Application

Task 33

1. Given function $f(x) = \frac{x+1}{x-2}$

a) Find domain of definition and boundary limits

b) Find asymptotes to the curve

c) Compute the first derivative and study its sign

d) Compute the second derivatives and study its sign.

e) Find intercept points if any

f) Sketch the graph

Formative Assessment

Formative Assessment for Unit 3

Instructions: Attempt all questions

1. List any five examples of polynomial function, for each example find the first and second derivative, stationary points, the interval where the function are increasing, the interval where they are decreasing, Draw the table of variation and graph your function.

2. List any five examples of rational function, for each example find the first and second derivative, stationary points, the interval where the function are increasing, the interval where they are decreasing, Draw the table of variation and graph your function.
3. Given the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

    a) State the value of x for which $f(x)$ is increasing

    b) Find the x-coordinate of each extreme point of $f(x)$

    c) State the values of x for which the curve of function is concave upwards

    d) Find the x-coordinate of each point of inflection

Points to Remember

- Finding the slope or derivative using the limit method is said to be using **first principles**, that is the derivative of a function $f(x)$ at $x = a$ is defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$
- **Derivative of a polynomial function**
- **If $f(x)$ and $g(x)$ are differentiable at x and c is any constant, then.**

$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$ **.**

- **Derivative of rational function**
- Suppose that $f(x)$ and $g(x)$ are differentiable functions at x and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

- The equation of tangent line is $y - f(a) = f'(a)(x - a)$.
- The equation of Normal line is $y - f(a) = \frac{-1}{f'(a)}(x - a)$.

- **The key points to remember when sketching the graph of a function:**

1. Domain:

- Determine the set of all possible input values (x-values) for which the function is defined. This will establish the boundaries of your graph.
- **Exclude any points where the function is undefined, such as:**
  - Division by zero
  - Taking the square root of a negative number
  - Taking the logarithm of a non-positive number

2. Intercepts:

- Find the x-intercept(s) by setting y = 0 and solving for x. These are the points where the graph crosses the x-axis.
- Find the y-intercept(s) by setting x = 0 and solving for y. These are the points where the graph crosses the y-axis.

3. Symmetry:

- Check for symmetry to simplify the graphing process:
  - **Even functions:** f(-x) = f(x) have symmetry about the y-axis.
  - **Odd functions:** f(-x) = -f(x) have symmetry about the origin.

- **Periodic functions:** $f(x + p) = f(x)$ for a constant p have repeating patterns.

4. Asymptotes:

- Identify any vertical asymptotes, where the function approaches infinity or negative infinity as x approaches a certain value. These often occur when there's a division by zero in the function's expression.
- Identify any horizontal or slant asymptotes, which represent lines that the graph approaches as x goes to positive or negative infinity.

5. Increasing/Decreasing Intervals:

- Analyze the function's first derivative ($f'(x)$) to determine where the function is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$). This helps you understand the overall shape of the graph.

6. Local Extrema (Maximums and Minimums):

- Find critical points by setting $f'(x) = 0$ or where $f'(x)$ is undefined.
- Use the first derivative test or second derivative test to classify these critical points as local maximums, local minimums, or neither.

7. Concavity and Inflection Points:

- Analyze the function's second derivative ($f''(x)$) to determine where the graph is concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$).
- Find inflection points where the concavity changes, often by setting $f''(x) = 0$ or where $f''(x)$ is undefined.

8. Sketch the Graph:

- Plot the intercepts, asymptotes, and any other key points identified in the previous steps

Self-Reflection

1) Observe the figure illustrated above on the beginning of unit and answer to the questions:
   i. Evaluate the rate change of y with respect to x
   ii. How do we call $\frac{\Delta y}{\Delta x}$?
   iii. Evaluate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
   iv. Comment your result in (c)

2) Relate the vertex of quadratic function to maximum and minimum of function

3) Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience **Knowledge, skills and attitudes**	**I don't have any experience doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Explain the concept of limits.					
Familiarity with basic differentiation formulas.					
Explain the meaning of derivatives (rate of change, slope of tangents).					
Explain higher-order derivatives.					
Explain applications (e.g., optimization, motion, graphing).					
Apply differentiation rules (power, product, chain).					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Solve real-world problems using differentiation.					
Simplify complex expressions before differentiation.					
Interpret derivative results in practical contexts.					
Identify points of maxima, minima, or inflection.					
Curiosity to explore mathematical concepts.					
Perseverance when solving complex problems.					
Confidence in applying mathematical reasoning.					
Open-mindedness to learning new methods.					
Willingness to analyze and verify solutions.					

4. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1. **2.** **3.**	**1.** **2.** **3.**	**1.** **2.** **3.**

References:

Arthur Adam, Freddy Goossens and Francis Lousberg (1991). Mathematisons 65, DeBoeck, 3rd edition.

David Rayner (2000). Higher GCSE Mathematics, Oxford University Press.

DPES- RWANDA (1990). Complexes 5th, Livre de l'élève. IMPRISCO-Kigali.

Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen (1990), Mathematics Principles & Process, Nelson Canada A Division of International Thomson Limited.

George B. Thomas, Maurice D. Weir & Joel R. Hass (2010), Thomas' Calculus Twelfth Edition, Pearson Education.

J CRAWSHAW, J CHAMBERS (1984). A concise course in A-Level statistics with worked examples, Stanley Thornes (Publishers) LTD.

J. Sadler, D. W. S. Thorning (1987). Understanding Pure Mathematics, Oxford University Press.

Ngezahayo, E. P. (2016). Advanced Mathematics for Rwanda Secondary Schools. Learners' Book Senior Four. Kigali: Fountain.

Ngezahayo, E. P. (2017). Advanced Mathematics for Rwanda Secondary Schools. Learners' Book Senior Six. Kigali: Fountain.

Ngezahayo, E. P. (2017). Advanced Mathematics for Rwanda Secondary Schools. Learners' Book Senior Five. Kigali: Fountain.

Peter Smythe (2005). Mathematics HL & SL with HL options, Revised Edition, Mathematics Publishing Pty. Limited.

Shampiona, A. (2005). Mathématiques 6. Kigali: Rwanda Education Board

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