Republic of Rwanda
Ministry of Education

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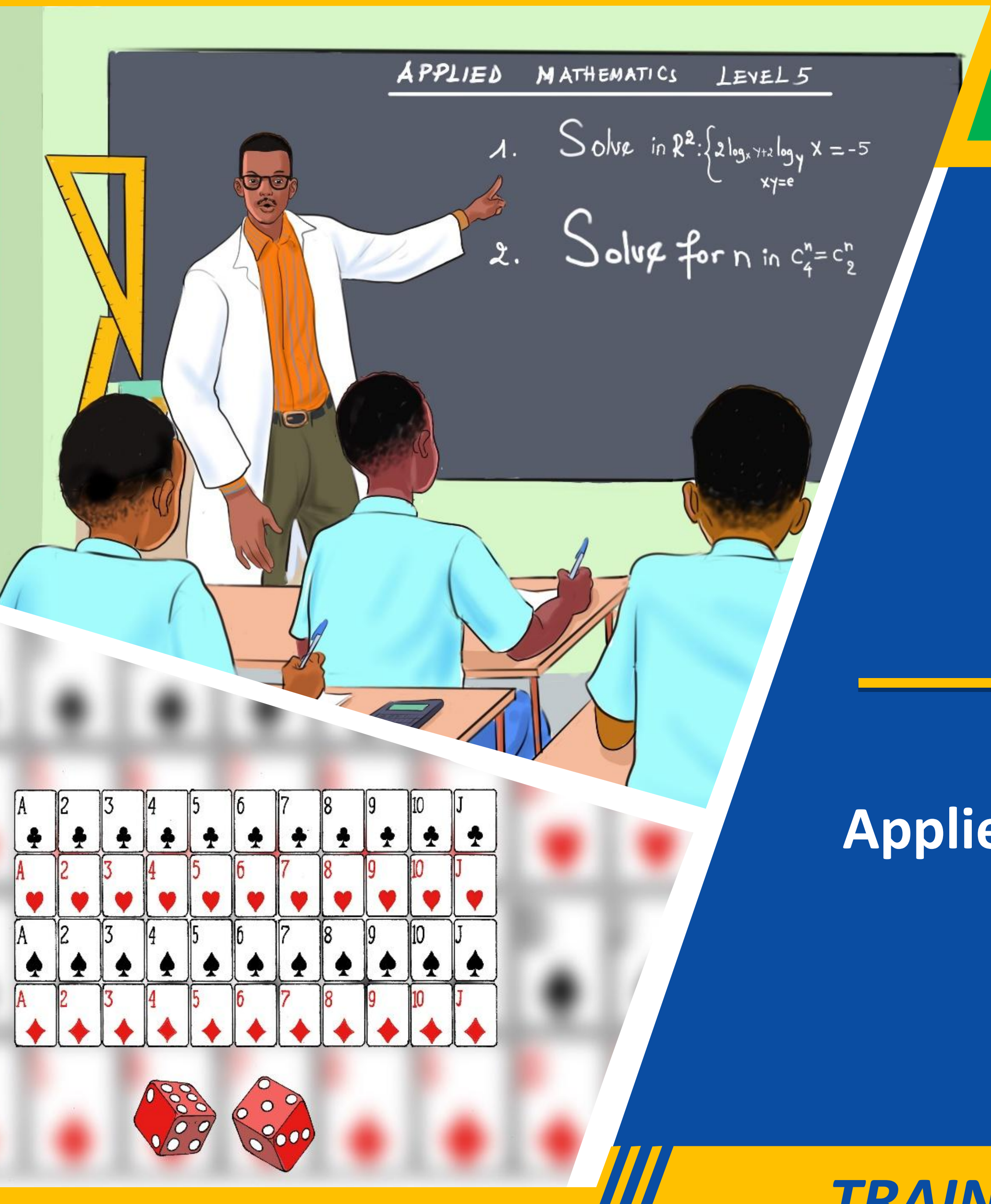

TOURISM

GENMA501
Applied Mathematics

TRAINEE'S MANUAL

April 2025

Republic of Rwanda
Ministry of Education

RTB | RWANDA TVET BOARD

APPLY LOGARITHMS, EXPONENTIAL EQUATIONS AND PROBABILITY

2025

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LIST OF ABBREVIATIONS AND ACRONYMS

CBET: Competence Base Education and Training

ln: Natural logarithm (logarithm to base e)

log: Logarithm (commonly base 10 if unspecified)

P(A): Probability of event A occurring

P(A|B): Conditional probability (probability of A given B)

RQF: Rwanda Qualification Framework

RTB: Rwanda TVET Board

TVET: Technical and Vocational Education and Training

INTRODUCTION

This trainee's manual encompasses all necessary skills, knowledge and attitudes required to **apply Logarithms, Exponential Equations and Probability**. Students undertaking this module shall be exposed to practical activities that will develop and nurture their competences. The writing process of this training manual embraced competency-based education and training (CBET) philosophy by providing practical opportunities reflecting real life situations.

The trainee's manual is subdivided into units, each unit has got various topics. You will start with a self-assessment exercise to help you rate yourself on the level of skills, knowledge and attitudes about the unit.

A discovery activity is followed to help you discover what you already know about the unit.

After these activities, you will learn more about the topics by doing different activities by reading the required knowledge, techniques, steps, procedures and other requirements under the key facts section, you may also get assistance from the trainer. The activities in this training manual are prepared such that they give opportunities to students to work individually and in groups.

After going through all activities, you shall undertake progressive assessments known as formative and finally conclude with your self-reflection to identify your strengths, weaknesses and areas for improvement.

Do not forget to read the point to remember the section which provides the overall key points and takeaways of the unit.

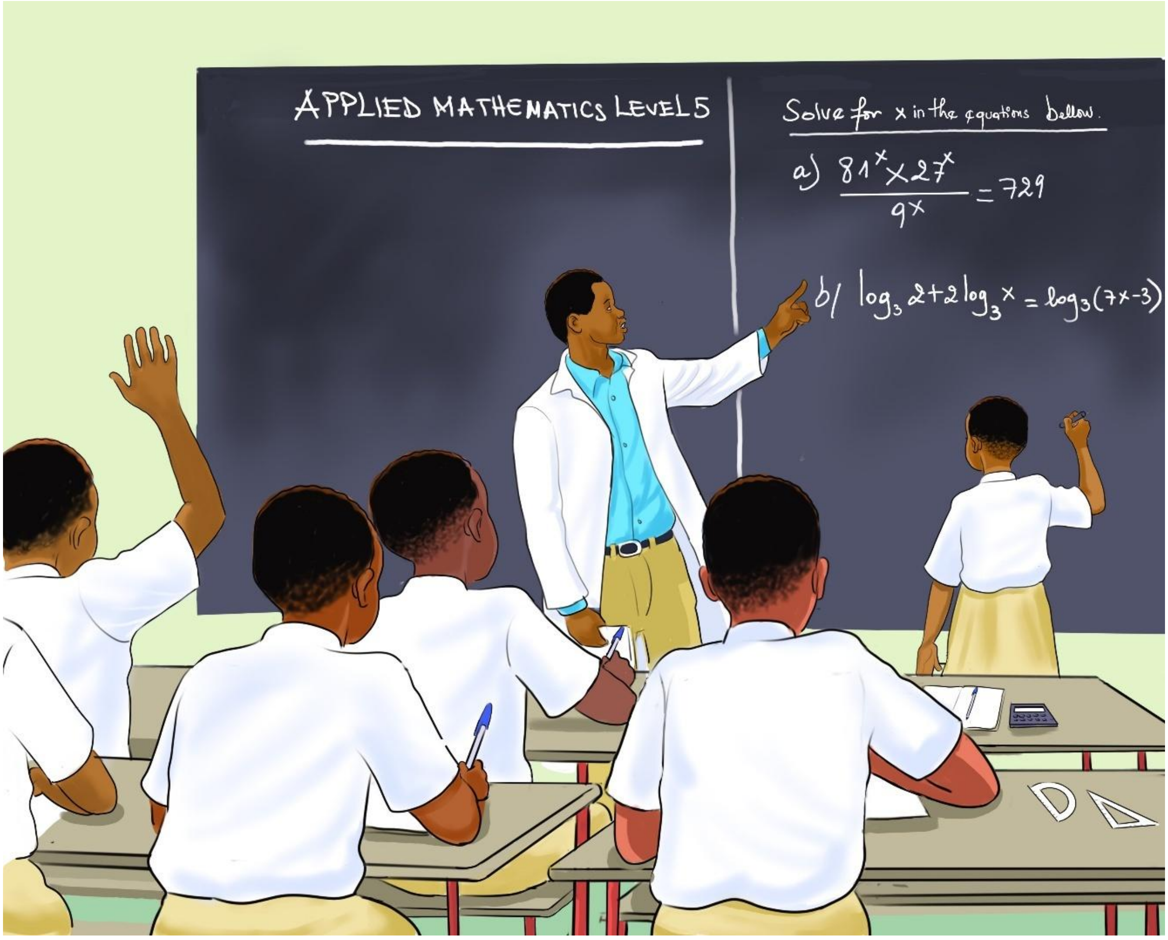
Module Units:

Unit 1: Solve exponential and logarithmic equations

Unit 2: Apply exponential and logarithmic expression

Unit 3: Apply fundamentals of probabilities

UNIT 1: SOLVE EXPONENTIAL AND LOGARITHMIC EQUATIONS



Unit summary

This unit provides you with the knowledge, skills and attitudes required to solve exponential equations and logarithmic Equations. It covers the description of properties for exponential expressions, the description of properties for logarithmic expression, solving exponential equation and solving logarithmic equations

Self-Assessment: Unit 1

1. Answer the following questions:
   a. What is exponential function?
   b. What is a logarithmic function?
   c. What is the difference between exponential and logarithmic expressions?
   d. Differentiate exponential to logarithmic expressions?
   e. Without using calculators, solve each of the following equations for X:
      i. $2^{x+1} = 8$
      ii. $3^{2x} = 81$
      iii. $5^x = \frac{1}{25}$
      iv. $e^{2x} = 20$
      v. $\ln(x-3) = 2$
      vi. $\log(x+1) + \log(x-1) = 1$
      vii. $\log(x^2-1) - \log(x+1) = 0$
2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.
   a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
   b. Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
   c. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.
3. At the end of this unit, you'll assess yourself again.

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Define properly exponential expressions					
Isolates Exponential terms correctly					
Pay attention to details while isolating exponential and logarithmic expression.					
Define properly logarithmic expression					
Isolates logarithmic terms correctly					
Think logically when applying properties of exponential expressions.					
Identify properties of exponential expressions					
Apply properties of exponential and logarithmic expression					
Identify properties of logarithmic expressions					
Apply properties of logarithmic expression					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Be patient while solving exponential expressions and logarithmic equations.					
Describe different technics to solve exponential equations					
Solve exponential equations based on their properties.					
Describe different technics to solve logarithmic equations.					
Solve exponential expressions and logarithmic equations based on their properties					

Key Competencies:

Knowledge	Skills	Attitudes
1. Define properly exponential expressions	1. Isolates Exponential terms correctly	1. Pay attention to details while isolating exponential and logarithmic expression.
2. Define properly logarithmic expression	2. Isolates logarithmic terms correctly	2. Pay attention to details while logarithmic expression.

Knowledge	Skills	Attitudes
3. Identify properties of exponential expressions	3. Apply properties of exponential and logarithmic expression	3. Think logically when applying properties of exponential expressions.
4. Identify properties of logarithmic expressions	4. Apply properties of logarithmic expression	4. Think logically when applying properties of logarithmic expressions.
5. Describe different technics to solve exponential equations	5. Solve exponential equations based on their properties.	5. Be patient while solving exponential equations.
6. Describe different technics to solve logarithmic equations.	6. Solve exponential expressions and logarithmic equations based on their properties.	6. Be patient while solving exponential expressions and logarithmic equations.
7. Demonstrates exponential and logarithmic expressions.	7. Analyse correctly exponential expressions and logarithmic forms.	

Discovery activity:

Scenario 1:

Nowadays every trending information is almost known by anyone in short time as information is being spread rapidly throught social media. Take an example of one person who shares a photo to 10 people and each of these 10 people share it to another 10, these 10 also share it to another 10 and son on. Within short time, this photo willl reach to many people.

Task 1:

1. What is the best function of calculating the number of people who received that image?
2. If each person was sharing the image to 10 people, how can we call this 10 that is keeping repeating?

3. If each time the image was shared is termed level, Calculate the number of people that received the image at level 4.
4. If each this picture was shared to 10 people in one minute, how long will it take for the picture to be shared to 900,000 people?

Topic 1.1: Exponential expressions

Activity 1: Problem Solving

Task 2:

Answer the questions below:

1. What do you understand by term exponential rate?
2. Write the following number 2x2x2x2 in exponential expression/equation.
3. From your answer in question 2, what is the appropriate term of 2 in the exponential equation?
4. State the exponent rules for solving exponential equations

Key Facts 1.1a: Description of properties for exponential expression

- **Definition of exponential expressions**

  An **exponential expression** is a mathematical expression in which a base is raised to an exponent (also called a power). The base number represents the number being multiplied repeatedly, while the exponent represents the number of times the base number is multiplied by itself.

  The general form of an exponential expression is written as $\boldsymbol{y^n}$, where "y" is the base number and "n" is the exponent.

  $y^n = y.y.y.y.y.y.y.y.y.y.y.y$ …. (n times)

- **Key Concepts in Exponential Expressions:**

  ✓ A positive exponent means you multiply the base that number of times.

Example: $2^4 = 16$

In the above example, the base is 2 and the exponent is 4, Therefore the result is 16 because 2 is multiplied by itself four times $2x2x2x2 = 16$

- ✓ A negative exponent means you take the reciprocal of the base raised to the positive exponent**.** Which is written as $y^{-x} = \frac{1}{y^x}$

  Example: $2^{-4} = \frac{1}{16}$

  In the above example, the base is 2 and the exponent is - 4, therefore the result is

  $2^{-4} = \frac{1}{2^4}$, you remember that $2^4 = 16$ therefore,

  $2^{-4} = \frac{1}{16}$

- ✓ For any non-zero base, when an exponent of 0 it means the value of the expression is always 1

  Example: $X^0 = 1$, $100^0 = 1$, $-1^0 = 1$, etc….

- **Exponent rules**

Exponent rules are those laws that are used for simplifying expressions with exponents. Many arithmetic operations like addition, subtraction, multiplication, and division can be conveniently performed in quick steps using the laws of exponents.

Rule	Description	Formula
Product of Powers Rule	When multiplying two exponential expressions with the same base, add the exponents.	$a^m \times a^n = a^{m+n}$
Quotient of Powers Rule	When dividing two exponential expressions with the same base, subtract the exponents.	$\frac{a^m}{a^n} = a^{m-n}$
Power of a Power Rule	When raising an exponential expression to another power, multiply the exponents	$(a^m)^n = a^{mn}$

Power of a Product Rule	When raising a product to a power, apply the exponent to both factors	$(ab)^m = a^m b^m$
Power of a Quotient	When raising a quotient to an exponent, you apply the exponent to both the numerator and the denominator	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
Negative Exponent Rule	A base raised to a negative exponent is equal to the reciprocal of the base raised to the positive exponent	$a^{-m} = \frac{1}{a^m}$
Zero Exponent Rule	Any non-zero base raised to the power of zero is equal to 1	$a^0 = 1$
Fractional Exponent Rule	A base raised to a fractional exponent can be rewritten as a root	$b^{\frac{m}{n}} = \sqrt[n]{b^m}$ is written as $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$

Key Facts 1.1b: Simplifying and solving exponential expressions using exponent rules

- The first equation, is resolved by the product of powers rule

  From this, $3^4 x 3^2 = 3^{(4+2)}$

  $= 3^6$

  $= 729$

- The second equation, is resolved by the quotient of powers rule,

  Using this rule, $\frac{5^7}{5^3} = 5^{(7-3)}$

  $= 5^4$

  $= 625$

- The third equation, is resolved by the power of powers rule,

  From this rule, $(2^3)^4 = 2^{(3x4)}$

  $= 2^{12}$

  = 4096

- The forth equation, is resolved by the negative exponential rule,

  From this rule, $5^{-2} = \frac{1}{5^2}$

  $= \frac{1}{25}$

- The fifth equation, is resolved by the fractional exponent rule,

  From this, $16^{\frac{1}{2}} = (\sqrt{16})^1$

  $= \sqrt{16}$

- The sixth equation, is resolved by more than one rule,

  First, you use the product of powers property in the numerator: $3^5 x 3^2 = 3^{(5+2)} = 3^7$

  With the above step, now we have $\frac{3^7}{3^6}$

  Second, you use quotient of powers rule on the whole fraction, $\frac{3^7}{3^6} = 3^{(7-6)}$

  Now we get $3^1 = 3$

- For simplifying this $\frac{(2x^3\, y^2)^2 * x^4}{(xy)^3}$

  Step 1: Apply the power rule for products to expression $(2x^3y^2)^2$

  $(2x^3y^2)^2 = 2^2 \times (x^3)^2 \times (y^2)^2 = 4x^6y^4$

  Step 2: Apply the power rule for quotients to $(xy)^3 = x^3y^3$

  Step 3: Substitute back into the original expression $\frac{4x^6y^4 \times x^4}{x^3y^3}$

  Step 4: Apply the product rule for exponents for the x -terms and the y -terms**:**

  $\frac{4x^{6\oplus 4}y^4}{x^3y^3} = \frac{4x^{10}y^4}{x^3y^3}$

  Step 5: Apply the quotient rule for exponents to simplify further**:** $4x^{10-3}y^{4-3} = 4x^7y$

  Final simplified expression now is $4x^7y$

Activity 2: Guided Practice

Task 3:

As seen above, exponent rules are used to simplify and manipulate expressions with exponents. These rules help in performing operations like multiplication, division, and powers of exponents more efficiently. By understanding and applying exponent rules, we can easily solve complex algebraic equations and problems. Using the exponent rules, simplify and evaluate the following exponential expressions

1. $3^4 x 3^2$=
2. $\frac{5^7}{5^3}$ =
3. $(2^3)^4$=
4. 5^{-2}=
5. $16^{\frac{1}{2}}$ =
6. $\frac{3^5 x\ 3^2}{3^6}$ =
7. $\frac{(2x^3\ y^2)^2 * x^4}{(xy)^3}$ =

Activity 3: Application

Task 4:

With the current social media, information sharing grows at an exponential rate. Let's assume that a shared trending message doubles at each 5 minutes. If the message is currently seen by 10,000 people (at zero minute) what will the number of people who will see it in 20 minutes? Using exponent rules, solve the asked question.

Topic 1.2: Logarithmic expressions

Activity 1: Problem Solving

Task 5:

Answer the questions below:

a. What is the logarithmic identity, and how is it used?

b. What is the difference between logarithmic expressions and exponential expressions?

c. What is the difference between the two kinds of logarithm?

d. State and explain the properties used for solving logarithmic equations

e. How does the change of base formula for logarithms work?

f. What is the logarithm of 1 for any base?

g. How is the logarithm of a base raised to an exponent simplified?

Key Fact 1.2a: Description of properties for logarithmic expressions

- **Definition of logarithmic expression**

  A **logarithmic expression** is a mathematical expression that represents the logarithm of a number or variable. A logarithm is defined as the power to which a number must be raised to get some other values. Briefly, a logarithm is the inverse operation of exponentiation. Fundamental properties of logarithms that simplify or manipulate logarithmic expressions are called **logarithmic identities**. These identities are based on the relationship between exponents and logarithms.

  The general form of a logarithmic expression is written as $\log_b(x)$ where **b** (the base) is a positive real number not equal to **1**, **x** (the argument) is a positive real number, $\log_b(x)$ represents the exponent to which the base b must be raised to get **x.**

- **Kinds of of logarithms: Common (or Briggian) logarithms and**

  There are two kinds of logarithms: Common/decimal (or Briggian) logarithms and Natural (or Napierian) logarithms.

A **common logarithm**, is the logarithm with base **10**. It is denoted as $\log_{10}(x)$ or simply $\log(x)$ n many contexts (especially on calculators). $\log_{10}(x) = y$ if and only if $x = 10^y$

A **Natural logarithm,** is a logarithm with the base e, where $e \approx 2.718$. It is denoted as $\ln(x)$.

If for any positive value **b** and different to zero equal to e^x ($= e^x$ *and* $\boldsymbol{b} > \mathbf{0}$) ,means that $\mathbf{ln}\ \boldsymbol{b} = \boldsymbol{x}$

- **Key concepts in Logarithmic expressions:**

1. For any base b we have $b^0 = 1,$ and $b^1 = b$. Hence $\log_b 1 = 0$ and $\log_b b = 1.$

   Example. $\log_{10} 10 = 1, \log_3 3 = 1, \log_{10} 1 = 0$

2. The logarithm of negative number and zero are not defined means doesn't exist.

- **Examples of exponential expressions vs logarithmic expressions**

Exponential expression	Logarithmic expression
$3^4 = 81$	$\log_3 81 = 4$
2^5=32	$\log_2 32 = 5$
$e^0 = 1$	$\ln 1 = 0$
$e^2 = 7.4$	$\ln 7.4 = 2$

3. **Change of bases**

The Change of Base Formula for logarithms allows you to rewrite a logarithm in terms of logs of another base. It is expressed as:

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}.$$

This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm which always has base of 10.

Example: $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

Note: Let b be a positive number. The logarithm of any positive number x to be the base written as $\log_b(x)$ represents the exponent to which b must be raised to x. That is $y = \log_b(x)$ and $b^y = x$

Example:

a. $\log_2 8 = 3$ since $2^3 = 8$

b. $\log_2 64 = 6$ since $2^6 = 64$

4. **Rules of Decimal and Natural logarithms**

These are those laws that are useful for simplifying and solving logarithmic expressions operations like addition, subtraction, Multiplication and division can be appropriately achieved in fast steps using the laws of logarithms. The logarithm of a base raised to an exponent can be simplified using the **power rule of logarithms**

Rule	Description	Formula
Product Rule	The **Product Rule of Logarithms** states that the logarithm of a product is equal to the sum of the logarithms of its factors.	$\log(ab) = \log(a) + \log(b)$ $\ln(x \bullet y) = \ln(x) + \ln(y)$
Quotient rule	The Quotient Rule of Logarithms states that the logarithm of a quotient is equal to the difference of the logarithms of the numerator and denominator.	$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$ $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
Power rule	The **Power Rule of Logarithms**_states that the logarithm of a number raised to an exponent is equal to the exponent multiplied by the logarithm of the base number.	$\log(x)^n = n\log x$ $\ln(y)^n = n\ln y$

Power of the product Rule	The logarithm of a product raised to a power can be rewritten as the power (n) multiplied by the logarithm of the product (x·y). It combines the Power Rule and the Product Rule of logarithms into a single statement.	$\log(x \bullet y)^n = n\log(x \bullet y) \Leftrightarrow n\log(x) + n\log(y)$ $\ln(x \bullet y)^n = n\ln(x \bullet y) \Leftrightarrow n[\ln(x) + \ln(y)]$
Power of a Quotient	This rule says that the logarithm of a quotient raised to a power is equal to the power (n) multiplied by the logarithm of the quotient $\left(\frac{x}{y}\right)$	$\log\left(\frac{x}{y}\right)^n = n\log\left(\frac{x}{y}\right) \Leftrightarrow n[\log(x) - \log(y)]$ $\ln\left(\frac{x}{y}\right)^n = n\ln\left(\frac{x}{y}\right) \Leftrightarrow n[\ln(x) - \ln y]$
Reciprocal rule	The logarithm of the reciprocal of x is the negative of the logarithm of x	$\log\left(\frac{1}{x}\right) = -\log(x)$ $\ln\left(\frac{1}{x}\right) = -\ln(x)$

Key Facts 1.2b: using logarithm rules to simplify and solve logarithmic Expressions

- The first equation, is resolved by the product rule

  From this, $\log_3(20 \times 5) = \log_3(20) + \log_3(5)$

  $\log_2(3 \times 5) = \log_2(3) + \log_2(5)$

- The second equation, is resolved by the quotient of powers rule, Using this rule,

  $\log\left(\frac{8}{4}\right)^3 = 3\log\left(\frac{8}{4}\right) \Leftrightarrow 3[\log 8 - \log 4]$

$$\ln\left(\frac{x}{2}\right)^4 = 4[\ln(x) - \ln(2)]$$

- The third equation, is resolved by the power of a product rule,

  From this rule,

  $$\log(50 \times 25)^5 = 5\log(50) + 5\log(25)$$

  $$\log(3 \times 4)^2 = 2\log(3 \times 4) = 2[\log(3) + \log(4)]$$

  $$\ln(5 \times 9)^5 = 5\ln(5 \times 9) \Leftrightarrow 5[\ln(5) + \ln(9)]$$

- The fourth equation, is resolved by negative logarithm rule, From this rule,

  $$\ln\left(\frac{10}{20}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$$

  $$\log\left(\frac{1}{9}\right) = -\log 9 \quad \ln\left(\frac{1}{5}\right) = -\ln 5$$

Activity 2: Guided Practice

As seen above, logarithms rules are very useful for simplifying and manipulating logarithmic expressions with exponents. These rules help in performing operations like multiplication, division, and powers of logarithms more efficiently. By understanding and applying logarithmic rules, we can easily solve complex algebraic equations and problems.

Task 6:

Using the logarithmic rules, simplify and evaluate the following logarithmic expressions

1. $\log_3(20 \times 5)$ **2.** $\log\left(\frac{8}{4}\right)^3$ **3.** $\log(50 \times 25)^5$ **4.** $\ln\left(\frac{10}{20}\right)$

Activity 3: Application

You are tour guide, and you are with a geologist studying volcanic earthquakes in Virunga National park caused by the movement of magma beneath the surface of the Earth. He explained to you **Richter scale (**measurement of the magnitude of an earthquake) is used to compare the energy released by different earthquakes. The formula for the Richter scale is:

$M = \log_{10}\left(\frac{A}{A_0}\right)$ Where: M = magnitude of the earthquake A= amplitude of the seismic waves and A0= reference amplitude (a very small constant value).

The geologist found out that the two recent earthquakes A and B had the reference amplitude of 10,000 and 1,000,000 respectively.

Task 7:

Help him to calculate the magnitude of Earthquake A and Earthquake B. Which earthquake has released much energy?

Topic 1.3: Exponential equation

Activity 1: Problem Solving

Task 8:

a. What is an exponential function? How is it written?

b. Consider functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$, complete the following table

x	**-2**	**-1**	**0**	**1**	**2**
$f(x)$					
$g(x)$					

c. Sketch the graph of function $f(x)$ and $g(x)$ from question b.

d. What is domain and range of validity for exponential functions?

e. Discuss whether $\forall x \in \Re, f(x) \in \Re$ and deduce the domain of $f(x)$

f. Discuss whether $f(x)$ can be negative or not and deduce the range of $f(x)$

g. What are the methods for solving exponential functions? What there their steps?

Key Facts 1.3a: Solving exponential equation

- **Definition**

Axponential function is written as function in the form $f(x)=a^x$, with a ≠ 0 and a ≠ 1. where a is the base and X is an exponent. An exponential function is also defined a $\exp_a:\Re\rightarrow\Re_0^+ : x\rightarrow y=\exp_a x$.

For simplicity we write $\exp_a x=a^x$.If the base a of exponential function $f(x)=a^x$ is equal to e^x or $f(x)=\exp(x)$. The exponential function $y=e^x$ has graph which is like the graph of $f(x)=a^x$ where $a\succ 1$

Example 1

Given exponential functions $f(x)=2^x$ and $g(x)=\left(\frac{1}{2}\right)^x$, the table below can be filled as:

x	**-2**	**-1**	**0**	**1**	**2**
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

These functions' graphs can be plotted as shown below:

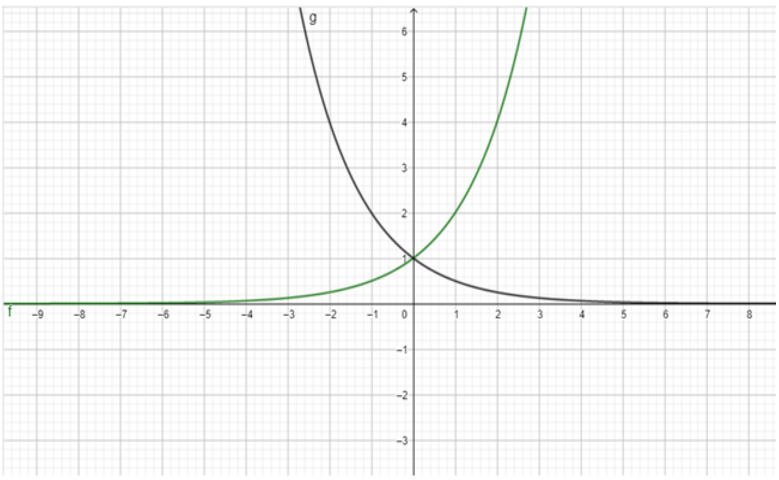


Figure 1:graph of $f(x)=2^x$ *and* $g(x)=\left(\frac{1}{2}\right)^x$

From this, graph $f(x)$ is increasing while $g(x)$ decrease and they meet at point (0,1) . It is very important to note that exponential function $f(x) = a^x$ is:

- ✓ Ever zero
- ✓ Always positive
- ✓ Only taking value 1 when x=0

- **Domain and range of exponential functions**

With the definition $y = f(x) = a^x$ and the restrictions that $a \succ 0$ and that a ≠ 1, **the domain** of an exponential function is the set of all real numbers or $]-\infty,+\infty[$. The range is the set of all positive real numbers or $]0,+\infty[$ because there is no exponent that can turn $y = f(x) = a^x$ into a zero nor into a negative result.

Note: Generally, if $u(x)$ is a defined function of x , the domain and range of f(x) depend on the values of x that make the expression valid and the resulting outputs.

Example

Determine the domain and the range of the function $f(x) = 3^{\sqrt{2x}}$

Solution

Condition for the existence of $\sqrt{2x}$ in $\Re$: x ≥ 0 . Thus, $Domf = [0,+\infty[$ and the range is [1,+∞[.

Example2

Find the domain and the range of $f(x) = 3^{\left(\frac{x+1}{x-2}\right)}$

Solution

Condition: $x-2 \neq 0 \Rightarrow x \neq 2.$ Thus $Domf = \Re - \{2\}$ and

the range is given by is $]0,3[U]3,+\infty[$.

Example3

Find the domain and the range of $f(x) = 4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2 - 4 \geq 0 \Rightarrow x \in]-\infty - 2]U[2,+\infty[.Thus$, $Domf =]-\infty - 2]U[2,+\infty[$ and the range is $[1,+\infty[$

the function $f(x) = a^{u(x)}$ will depend on $u(x)$.

- **Solving exponential equations**

  Equations that involves powers (exponents) as terms of their expressions are referred to as exponential equations. Such equations can sometimes be solved by 1) appropriately applying the properties of exponents or 2) introducing logarithms within expression.

- **Solve exponential equations using one- to- one property (Using like bases)**

  If an exponential equation can be written so that both bases are the same, the equation can be solved by comparing the exponents. To solve such exponential, follow the three steps:

  - ✓ Rewrite both sides of the equation as an exponential expression with the same base.
  - ✓ Since the bases are equal, then the exponents must be equal. Set the exponents equal to each other
  - ✓ Solve the equations and check answers

- **Solve an exponential equation by taking logarithms for both sides**

  To solve an exponential equation using logarithms, four steps are followed:

  a. Isolate the exponential expression
  b. Taking logarithms for both sides
  c. Rewriting exponential side as a linear expression
  d. Solving the obtained equation and check the answer.

Key Facts 1.3b: Solving exponential equation using like bases and logarithm

- **Using like bases**

  ✓ **Exponential equation 1:** $3^{x+2} = 27^{x}$ is solved as follows:

  For this equation, the domain of validity $x \in \Re$, therefore

$3^{x+2} = 27^x \Rightarrow 3^{x+2} = (3^3)^x$

$3^{x+2} = 3^{3x}$

$x + 2 = 3x \Leftrightarrow 2 = 2x, then, x = 1$

Therefore, **solution is** $s = \{1\}$

- ✓ **Exponential equation 2:** $4^{x-1} = 16$ is solved as follows:

Domain of validity $x \in \Re$**,**

$4^{x-1} = 16^x \Rightarrow 4^{x-1} = (4)^2$

$4^{x-1} = 4^2 \Leftrightarrow x - 1 = 2 \Rightarrow x = 3,$

Therefore, **solution** $s = \{3\}$

- ✓ **Exponential equation 3:** $e^{-x^2} = (e^x)^2 * \frac{1}{e^3}$ is solved as follows:

Domain of validity: $x \in \Re$

$e^{-x^2} = e^{2x} * e^{-3}$

$e^{-x^2} = e^{2x-3} \Rightarrow -x^2 = 2x - 3$

$then, x^2 + 2x - 3 = 0$

The solution of this equation $x^2 + 2x - 3 = 0$ gives $x = -3$ **or** $x = 1$ The solution set is $s = \{-3, 1\}$

- ✓ **Exponential equation 4:** $2^{2y} + 3(2^y) = 4$ is solved as follows:

$2^{2y} + 3(2^y) = 4 \Leftrightarrow (2^y)^2 + 3(2^y) = 4$

Let $2^y = x, then$ $x^2 + 3x = 4$

$\Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow x = -4$ **or** $x = 1$

Replacing the value of x in the equation $2^y = x$

For $x = -4$ $2^y = -4$ doesn't exist

For $x = 1, 2^y = 1 \Leftrightarrow y = 0$

$S = \{0\}$

- **Using Logarithm**
  - ✓ **Exponential equation 1:** $2e^{x+1} - 4 = 12$

Let first isolate the exponential expression, $2e^{x+1} - 4 = 12$

$2e^{x+1} = 16 \Leftarrow e^{x+1} = 8$

Now, apply logarithm for both sides to get**,** $\ln e^{x+1} = \ln 8$

$(x+1)\ln e = \ln 8 \Rightarrow x+1 = \ln 8$
$x = \ln 8 - 1, x \approx 1.079$

Therefore, solution $S = \{1.079\}$

- ✓ **Exponential equation 2:** $e^x = 5$

$Domain$ of $validity: x \in \Re$

$\ln e^x = \ln 5$
$x \ln e = \ln 5$ **where** $\ln e = 1$

$x = \ln 5$
$S = \{\ln 5\}$

- ✓ **Exponential equation 3:** $10 + e^{0.1t} = 14$

Domain of validity**:** $t \in \Re$

$e^{0.1t} = 4$
$\ln e^{0.1t} = \ln 4$
$0.1t = \ln 4$
$t = 10 \ln 4$
$S = \{10 \ln 4\}$

✓ **Exponential equation 4:** $5+3^{t-4}=7$

$$3^{t-4}=2 \Rightarrow \ln 3^{t-4}=\ln 2$$
$$\Leftrightarrow (t-4)\ln 3=\ln 2$$
$$\Rightarrow t=4+\frac{\ln 2}{\ln 4}$$

✓ **Exponential equation 5:** $3(2^{4x})-7(2^{2x})+4=0$

$$3(2^{2x})^2-7(2^{2x})+4=0$$

Let $2^{2x}=k$, **then** $3k^2-7k+4=0 \Leftrightarrow \Delta=1$

$$\Rightarrow k=\frac{4}{3}, or, k=1$$

Replacing the value of k in the equation $2^{2x}=k$, **for k=** $\frac{4}{3}$

$$2^{2x}=\frac{4}{3} \Leftrightarrow 2x\ln 2 \Leftrightarrow 2x\ln 2=\ln 4-\ln 3$$
$$\Rightarrow x=\frac{\ln 4-\ln 3}{2\ln 2}$$

$$\Rightarrow 2x\ln 2=\ln 1$$

For k=1: $x=0$

$$S=\left\{0,\frac{\ln 4-\ln 3}{2\ln 2}\right\}$$

Activity 2: Guided Practice

Task 9:

Solve the given exponential functions below using appropriate method,

Using like base	Using logarithm
1. $3^{x+2} = 27^x$ 2. $4^{x-1} = 16$ 3. $e^{-x^2} = (e^x)^2 * \frac{1}{e^3}$ 4. $2^{2y} + 3(2^y) = 4$	1. $2e^{x+1} - 4 = 12$ 2. $e^x = 5$ 3. $10 + e^{0.1} = 14$ 4. $5 + 3^{t-4} = 7$ 5. $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

Activity 3: Application

Task 10:

In a population growth model, the number of individuals in a species is given by the equation $4^{2x} - 6.2^{4x} + 6.2^{2x} - 1 = 0$,where x represents the time in years. Solve this equation to determine the critical points in time when the population reaches certain thresholds.

Topic 1.4 Logarithmic equations

Activity 1: Problem Solving

Task 11:

Answer the following questions:

1. What is a logarithmic equation?
2. How is a logarithmic equations solved?

3. For which value(s) of x, each function is defined, and use the properties for logarithm to determine the value of x in the given expressions:

   a. $\ln x = 10$
   b. $\ln x = 3$
   c. $\log x = 2$
   d. $\log(100x) = 2 + \log 4$
   e. $\log_2 x = -3$

Key Facts 1.4: Solving logarithmic equations

- **Common logarithmic equations involving one unknown**

  Logarithmic equation in $\Re$ is the equation containing the unknown within the logarithmic expression. To solve logarithmic equations, the following steps are followed:

  ✓ Set existence conditions for solution(s) of equation.

  ✓ Express logarithms in the same base

  ✓ Use logarithmic properties to obtain:

  - $\log_a u(x) = \log_a v(x) \Leftrightarrow u(x) = v(x)$ where $u(x)$ and $v(x)$ are the functions in x.

- **Make sure that the value(s) of unknown verifies the conditions above. The properties of logarithms can be used to solve logarithmic equations.**

  Example 1: Solve the following equation $\log_x 49 = 2$

  Solution: condition of validity: $x \succ 0, x \neq 1$

  From the equation $\log_x 49 = 2$ we change logarithmic equation to exponential equation to get $x^2 = 49$

  $x = \pm 7$ for $x \succ 7$ therefore ,solution set is $\{7\}$

  Example 2: Solve each equation

  a. $\log_3(x+1) = \log_3 2$

  b. $\log_{x-2} 3 = 1$

  c. $\log_2(x+14) + \log_2(x+2) = 6$

Solution

a. $\log_3(x+1)=\log_3 2$

Condition of existence : $x \succ 0 \Leftrightarrow x \succ -1$

Then, $\log_3(x+1)=\log_3 2(simplify)$ to obtain

$\Leftrightarrow x+1=2$ **Then,** $x=1, S=\{1\ \}$

b. $\log_{x-2} 3=1$

Condition : $x-2 \succ 0 \Leftrightarrow x \succ 2$ **and** $x-2 \neq 1 \Leftrightarrow x=3$ this means $x \in]2,3[\cup]3,+\infty[$

$\log_{x-2} 3=1 \Leftrightarrow \log_{x-2}(x-2)$

,Then, $\Leftrightarrow 3=x-2$

$\Leftrightarrow x=5, S=\{5\ \}$

c. $\log_2(x+14)+\log_2(x+2)=6$

Condition: $x+14 \succ 0$ **and** $x+2 \succ 0 \Leftrightarrow x \succ -14$ **and** $x \succ -2$ $\Leftrightarrow x \in]-2,+\infty[$

$\log_2(x+14)+\log_2(x+2)=6 \Leftrightarrow \log_2(x+14)(x+2)=6\log_2 2$

$\Leftrightarrow \log_2(x+14)(x+2)=6\log_2 2^6$

$\Leftrightarrow (x+14)(x+2)=64$

$x^2+16x+28-64=0 \Leftrightarrow x^2+16x-36=0$

$\Leftrightarrow (x+18)(x-2)=0$

$\Leftrightarrow x=2 or x=-18, S=\{2\ \}$

- **Systems of equations involving common logarithms**

In solving systems of equations using logarithms, like one-variable logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can used in the creation of a one-variable equation that is simple to solve.

Example: solve the following system of equations

$$\begin{cases} \log x+\log y=1 \\ \log(10x)-\log y=2 \end{cases}$$

Solution

Let us apply elimination method by eliminating the variable y,

$$\begin{cases}\log x+\log y=1\\ \log(10x)-\log y=2\end{cases} \Leftrightarrow \begin{cases}\log x+\log y=1\\ \log(10x)-\log y=2\end{cases} \Leftrightarrow \log x+\log(10x)=3$$

Use logarithmic property of addition to solve for x the new equation

$$\log x+\log(10x)=3 \Leftrightarrow \log(10x*x)=3$$
$$\log 10x^2=3 \Leftrightarrow \log 10x^2=3\log 10$$
$$\Rightarrow \log 10x^2=\log 10^3$$
$$10x^2=10^3 \Rightarrow x=\pm 10$$

Since for -10 is undefined, we reject it and keep $x=10$

- **Natural logarithmic equations**

The Natural logarithmic equation in Υ is the equation containing the unknown within the natural logarithmic expression. To solve Natural logarithmic equations, the following steps are followed:

- ✓ Set existence conditions for solution(s) of equation.
- ✓ Use logarithmic properties to obtain:

  $\ln u(x)=\ln v(x) \Leftrightarrow u(x)=v(x)$, where u(x) and v(x) are functions in x
- ✓ Make sure that the value(s) of unknown verifies the conditions above.
- ✓ $y=\ln x=\log_e x \Leftrightarrow e^y=x$ (as inverse) The equation $\ln x=1$ has a unique solution, the rational number 2.71828182845904523536...and this number is represented by letter e.The properties of natural logarithms can be used to solve natural logarithmic equations.

  Example 1:

  Solve each equation

  a. $\ln x-\ln 5=o$

  b. $3+2\ln x=7$

  c. $\ln 2x+\ln(x+2)=\ln 6$

Solution

a. $\ln x - \ln 5 = 0$

condition of validity: $x \succ 0$

Then, $\ln x = \ln 5 \Leftrightarrow x = 5$

Hence ,the solution $S = \{5\}$

b. $3 + 2\ln x = 7$

condition of validity: $x \succ 0$,By solving:

$2\ln x = 7 - 3 \Leftrightarrow 2\ln x = 4 \Leftrightarrow \ln x = 2 \Leftrightarrow x = e^2$

$S = \{e^2\}$

c. $\ln 2x + \ln(x+2) = \ln 6$

Conditions of validity: $2x \succ 0$ and $x + 2 \succ 0$

$\Rightarrow x \succ 0$ and $x \succ -2$

Domain of validity: $]-2,+\infty[\cap]0,+\infty[\Leftrightarrow]0,+\infty[$

Solving:

$$\Leftrightarrow \ln 6 = \ln[2x(x+2)]$$
$$\Leftrightarrow 2x(x+2) = 6$$
$$\Leftrightarrow 2x^2 + 4x - 6 = 0$$
$$\Leftrightarrow x^2 + 2x - 3 = 0$$

$\ln 2x + \ln(x+2) = \ln 6$ = $\Leftrightarrow (x-1)(x+3) = 0$

$x = 1$ **or** $x = -3$

As $x \in]0,+\infty[$ **,then** $S = \{1\}$

- **Systems of equations involving natural logarithms**

In solving systems of equations using natural logarithms, like one-variable natural logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the natural logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can used in the creation of a one-variable equation that is simple to solve.

Example1: solve the following system of equations

$$\begin{cases} \ln(xy) = 7 \\ \ln\left(\dfrac{x}{y}\right) = 1 \end{cases}$$

Solution

$$\begin{cases} \ln(xy) = 7 \\ \ln\left(\dfrac{x}{y}\right) = 1 \end{cases} \Leftrightarrow \begin{cases} \ln x + \ln y \\ \ln x - \ln y \end{cases}$$

solve the system of equations started eliminating variable y

$$\begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases} \Leftrightarrow 2\ln x = 8$$

$$\Rightarrow \ln x = 4 \Rightarrow x = e^4$$

replacing the value of x in equation (1), $\ln e^4 + \ln y = 7$ then by solving this equation

the solution for y is given by:

$$\ln e^4 + \ln y = 7$$
$$\Rightarrow 4\ln e + \ln y = 7$$
$$\Rightarrow 4 + \ln y = 7$$
$$\Rightarrow \ln y = 7 - 4 \Leftrightarrow \ln y = 3$$
$$y = e^3$$

Therefore the solution of the system is. $S = \{(e^4, e^3)\}$

Example2: Solve the following:

$$\begin{cases} 3\ln x + 2\ln y = 144 \\ 4\ln x + \ln y = 152 \end{cases}$$

solve the system of equations started eliminating variable y

$$\begin{cases} 3\ln x + 2\ln y = 144 \\ 4\ln x + \ln y = 152 \end{cases}$$ By multiplying 1 to the first equation and -2 to the second equation of the system we will get

$$\begin{cases} 3\ln x + 2\ln y = 144 \\ -8\ln x - 2\ln y = -304 \end{cases} \Leftrightarrow -5\ln x = -160$$

$$\Rightarrow \ln x = \frac{160}{5} \Leftrightarrow \ln x = 32$$

$$\log_e x = \ln x, so \Leftrightarrow \log_e{}^x = 32$$

Remember that $\Rightarrow x = e^{32}$

$$3\ln x + 2\ln y = 144$$
$$\Rightarrow 3\ln e^{32} + 2\ln y = 144$$
$$\Rightarrow 3x32\ln e + 2\ln y = 144$$
$$\Rightarrow 96\ln e + 2\ln y = 144$$
$$\Rightarrow 96 + 2\ln y = 144$$
$$\Rightarrow 2\ln y = 144 - 96$$
$$\Rightarrow 2\ln y = 48 \Leftrightarrow \ln y = \frac{48}{2}$$

replacing the value of in equation(1)

$$\ln y = 24 \Leftrightarrow y = e^{24}$$

Therefore, the solution of the system is: $S = \{(e^{32}; e^{24}\}$

Activity 2: Guided Practice

Task 12:

For each of the following function solve for x .

a. $\log(x+2) = 2$

b. $\log x + \log(x^2 + 2x - 1) - \log 2 = 0$

c. $\log(35-x^3)=3\log 5-x)$

d. $\log(3x-2)+\log(3x-1)=\log(4x-3)^2$

e. $\log(1-x)=-1$

Notes:

- For any clarifications you may contact your trainer.
- You may also refer to the **key facts 1.4** in this manual.

Activity 3: Application

Task 13:

1. In a biological study, the growth of bacteria population is modeled by the following logarithmic functions, where t represents the time in days. Determine for which value(s) of t each function is defined to understand the time periods during which the population growth can be accurately modeled.

   a. $f(t)=\ln(t+4)$

   b. $f(t)=\ln t$

   c. $f(t)=\ln(t^2-5t+16)$

2. Solve the following system: In an environmental study, the relationship between two species' populations is modeled by a system of logarithmic equations, where x and y represent the populations of the two species. Solve the system to find the equilibrium populations:

$$\begin{cases} 2\ln x+3\ln y=-2 \\ 3\ln x+5\ln y=-4 \end{cases}$$

Formative Assessment

1. Write each of the following in logarithmic form

   a. $4^3 = 64$

   b. $2^{-3} = \frac{1}{8}$

   c. $\left(\frac{1}{2}\right)^x = y$

   d. $5^{-p} = q$

2. Use the properties of logarithms to rewrite each expression as a single logarithm:

   a. $2\log_b x + \frac{1}{2}\log_b (x+4)$

   b. $4\log_b (x+2) - 3\log_b (x-5)$

   c. $\log_b \left(\frac{x\sqrt{y}}{z^5}\right)$

3. Find the numerical value

   a. $\log_2 32$

   b. $\log_4 8$

   c. $\log_6 7$

   d. $\log_5 \sqrt{125}$

   e. $\log_5 0.008$

   f. $\log_9 10$

4. Find the exact value of x ,showing your working steps:

   a. $\log_2 8 = x$

   b. $\log_x 125 = 3$

   c. $\log_x 64 = 0.5$

d. $\log_4 64 = x$

e. $\log_9 x = 3$

5. Solve the following exponential equations

a. $2^{3x} = 3^{2x-1}$

b. $2e^{2x} - e^{x} - 6 = 0$

c. $3^{x+2} = 27^{x}$

d. $e^{x} - 12 = \frac{-5}{e^{-x}}$

6. Find the domain of definition of the functions:

a. $f(x) = \ln(x+1)$

b. $f(x) = \frac{1+\ln x}{-1+\ln x}$

7.Solve the following in real number:

$$\log x + \log(x^2 + 2x - 1) - \log 2 = 0$$

8. Evaluate the value of x and y for the following system:

a. $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$

b. $\begin{cases} \ln(xy) = 7 \\ \ln y\left(\frac{x}{y}\right) = 1 \end{cases}$

Points to Remember

- Definition of exponential expression
- Definition of logarithmic functions
- Properties of exponential functions.
- Properties of logarithmic functions
- Change of base law.
- Change from logarithm into exponential and vice
- Don’t use calculators while solving exponential and logarithmic expressions
- Before solving any exponential and logarithmic expressions, think of the appropriate rule/property to use for simplification
- Exponential equations.
- Logarithmic equations.
- Domain and range of exponential functions
- Domain and range of logarithmic functions

Self-Reflection

Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Define properly exponential expressions					
Isolates Exponential terms correctly					
Pay attention to details while isolating exponential and logarithmic expression.					

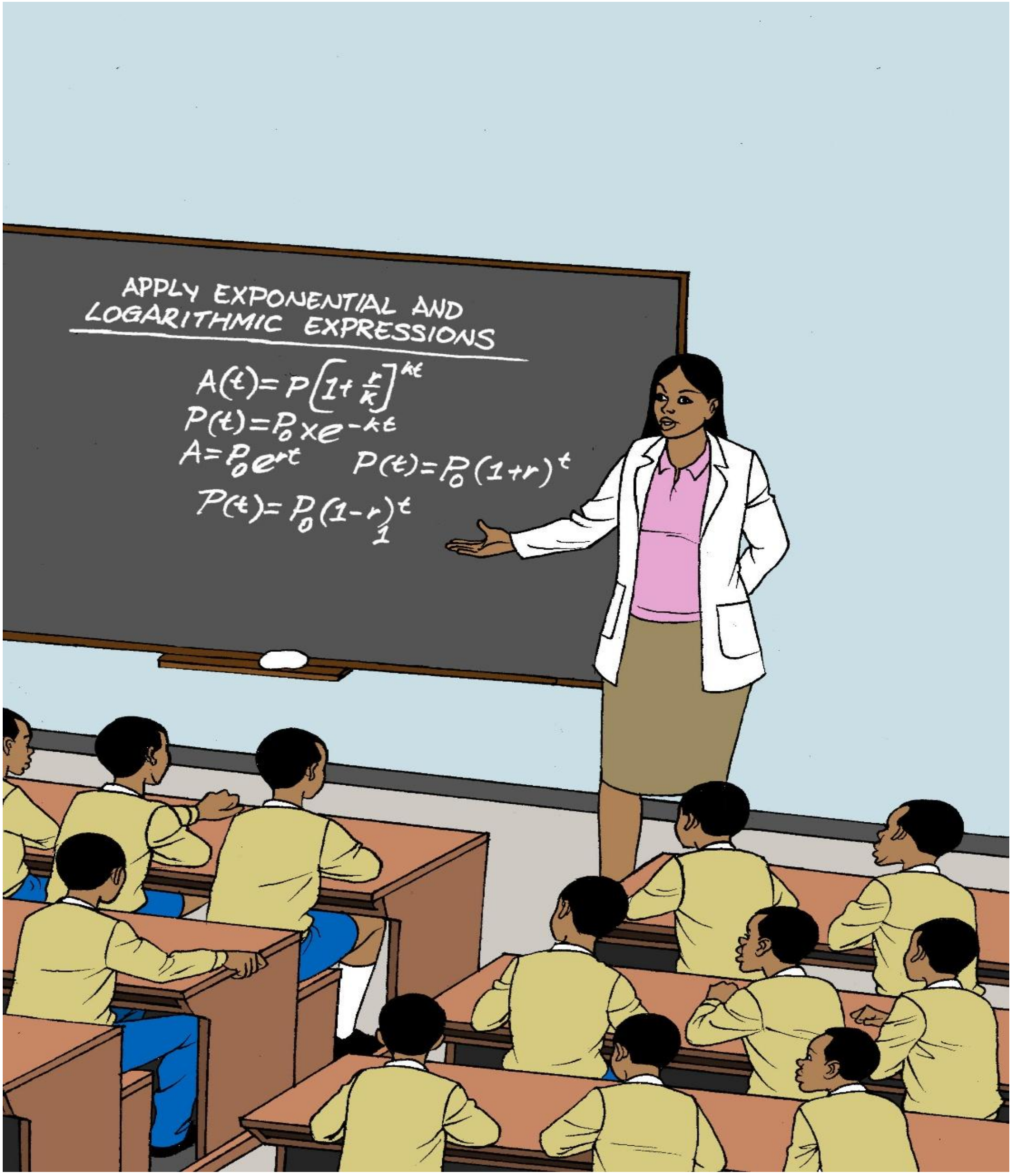
Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Define properly logarithmic expression					
Isolates logarithmic terms correctly					
Think logically when applying properties of exponential expressions.					
Identify properties of exponential expressions					
Apply properties of exponential and logarithmic expression					
Identify properties of logarithmic expressions					
Apply properties of logarithmic expression					
Be patient while solving exponential expressions and logarithmic equations.					
Describe different technics to solve exponential equations					
Solve exponential equations based on their properties.					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Describe different technics to solve logarithmic equations.					
Solve exponential expressions and logarithmic equations based on their properties					

1. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1. 2. 3.	1. 2. 3.	1. 2. 3.

UNIT 2: APPLY EXPONENTIAL AND LOGARITHMIC EXPRESSIONS.



Unit summary

This unit provides you with the knowledge, skills and attitudes required to apply exponential equations and logarithmic Expressions. It covers Calculation of Compound interest, Calculation of population growth and Calculation of population decay.

Self-Assessment: Unit2

1. Answer the following questions:
   a. What is compound interest?
   b. Given the formula for calculating the compound interest below, what does each letter stands for?

   $$A = P\left(1+\frac{r}{n}\right)^{nt}$$

   c. What is population growth and its importance?
   d. What is the formula for calculation of population growth?
   e. What is population decay?
   f. What is the formula for calculation of population decay?

2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.
   a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
   b. Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
   c. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.
   d. At the end of this unit, you'll assess yourself again.

Experience **Knowledge, skills, and attitudes**	**I do not have any experience doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Define correctly simple and compound interest.					
Calculate simple and compound					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
interest accurately using formulas.					
Pay attention to details while calculating simple and compound interest					
Recognize the meaning of each variable needed in calculation of compound interest (e.g. principal value, rate, time, and compounding frequency or periods)					
Rearrange logarithmic and exponential equations to solve for unknown variables (e.g., time t, rate r)					
Think logically when applying formula of compound interest.					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Explain exponential growth using its formula.					
Apply exponential growth formulas to real-world problems (e.g., predicting future population size).					
Be patient while solving problems related to population growth.					
Recognize the meaning of each variable needed in calculation of population growth (growth rate, initial population etc.).					
Predict future population sizes using exponential models.					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Develop a problem solving mindset when working with population growth formula.					
Explain exponential decay using its formula.					
Solve real-world problems involving population reduction using exponential decay formula.					
Think logically when applying the population decay formula to ensure accurate calculations and meaningful interpretations.					

Key Competencies:

Knowledge	Skills	Attitudes
1. Define correctly simple and compound interest.	1. Calculate simple and compound interest accurately using formulas.	1. Pay attention to details while calculating simple and compound interest
2. Recognize the meaning of each variable needed in calculation of compound interest (e.g. principal value, rate, time, and compounding frequency or periods)	2. Rearrange logarithmic and exponential equations to solve for unknown variables (e.g., time t, rate r)	2. Think logically when applying formula of compound interest.
3. Explain exponential growth using its formula.	3. Apply exponential growth formulas to real-world problems (e.g., predicting future population size).	3. Be patient while solving problems related to population growth.
4. Recognize the meaning of each variable needed in calculation of population growth (growth rate, initial population etc.).	4. Predict future population sizes using exponential models.	4. Develop a problem solving mindset when working with population growth formula.
5. Explain exponential decay using its formula.	5. Solve real-world problems involving population reduction using exponential decay formula.	5. Think logically when applying the population decay formula to ensure accurate calculations and meaningful interpretations.

Discovery activity:

Scenario 1:

Imagine you are the analyst in Tourism and you are hired to analyze how Akagera National Park will be in 10 years in three aspects, namely 1) Investment 2) A Growing Population of carnivore animals/ predators (lions, Hyenas and cheetahs), and 3) Decay in Population of herbivore animals as predators eat them. The park has an initial treasury fund of 200,000,000 frw which is invested in bank at an annual interest rate of 6%, compounded monthly. The Park has an initial population of 5,000 predators that grows at a rate of 4% per year and of 100,000 herbivore animals that decays at a rate of 7% per year.

Task 14:

1. What is the appropriate term for the initial amount of money the park has in the bank?
2. What is the formula used to calculate the final amount of money?
3. How much money will the park be having after 10 years?
4. What will be the number of carnivores in 10 years?
5. What will be the number of herbivores in 10 years?

Topic 2.1: Compound interest

Activity 1: Problem Solving

Three (3) years ago, X bank marketing team reached out to Mr. MUSINGUZI, a tour guide in Virunga National Park and they explained about a new saving account with compound interest aimed at encouraging people to save money for future. Mr. MUSINGUZI immediately opened that account and deposited 2,000,000 frw. After three years he went to the bank to withdraw the money from this bank account and was surprised when he was told that, with all interests, his money has grown up to 8,000,000 frw.

Task 15:

1. What does the compound interest in the above scenario mean?
2. Explain how Mr. MUSINGUZI can discover that power which raised his money.

3. What is the total interest earned over the 3 years?
4. How much interest does Mr. MUSINGUZI earns each year?
5. What was the growth factor (1 + r) over the 3 years?
6. What would the final amount be if Mr. MUSINGUZI left his money for 5 years instead of 3?

Key Facts 2.1a: Calculation of Compound interest

- **Description of Compound interest**

  To understand what Compound interest, let's first start with knowing what Interest is. Interest is the compensation that one gets for lending a certain asset.

  For Instance, suppose that you put some money in a bank account for a year. To reward you for that, it pays you your money plus extra amount, the extra amount is called' 'Interest''. There are two types of interest namely 1) Simple interest and 2) Compound interest

  ✓ **Simple interest**

  Simple interest is the interest calculated using the principal only and does not include compounding interest.

  ✓ **Compound interest**

  Compound interest is the interest calculated on the initial principal and also on the accumulated interest of previous periods. Essentially, it allows your money to grow at an increasing rate over time, as the interest earned in each period is added to the principal for future interest calculations. The compounding frequency, which is how often the interest is calculated, can be annual, semi-annually, quarterly, monthly, or even daily, depending on the terms of the investment or loan.

- **Terminologies used in interest calculation**

  ✓ **Principle value:**

  The Principal Value (denoted as **P**) is the initial amount of money that is invested, borrowed, or deposited, before any interest is added. It serves as the foundation upon which interest is calculated in both simple and compound interest scenarios.

  Example:

  If you invest $1,000 in a savings account, the $1,000 is the principal.
  If you borrow $5,000 from a lender, the $5,000 is the principal.

- ✓ **Interest rate.**

  Interest rate **(**denoted as **r)** is the amount of interest expressed as proportion of the principal or as percentage.

- ✓ **Time:**

  Time (denoted as **t**) refers to the duration (in years) for which the money is invested, borrowed, or left to grow. It plays a crucial role in determining how much the principal will grow over time, as interest compounds periodically.

- ✓ **Rate of change:**

  The rate of change in the context of compound interest measures how the value of an investment or loan grows over time due to compounding. It quantifies how the accumulated amount (future value) changes as time or other variables (e.g., interest rate, compounding frequency) change.

- ✓ **Future value:**

  The Future Value (FV) is the value of an investment or loan at a specific point in the future, including the accumulated interest or returns. It shows how much money your principal will grow to, given a specific rate of return and time period.

- **Formulas for calculation of interest**

  - ✓ **Simple interest**

    The formula for simple interest (SI) is straightforward: $\mathrm{SI} = P * r * t$

    Where:

    P = Principal amount (initial investment)

    r = Annual interest rate (in decimal form, e.g., 5% = 0.05)

    t = Time of investment.

  - ✓ **Compound interest**

    Calculating interest using compound interest methods consists in charging the interest on the principal plus any accumulated interest. Calculating interest using compound interest methods one may need to use the following formulas:

    - **For interest compounded annually**: $A = P \times (1 + r)^t$,

      Where

      A = Total amount after interest (principal + interest)

P = Principal amount (initial investment)

r = Annual interest rate (in decimal form, e.g., 5% = 0.05)

t = Time in years

- **If the interest is compounded more frequently than annually:**

$$A = P \times \left(1 + \frac{r}{n}\right)^{n \times t},$$

Where

A = Total amount after interest (principal + interest)

P = Principal amount (initial investment)

r = Annual interest rate (in decimal form, e.g., 5% = 0.05)

t = Time in years

n = Number of times interest is compounded per year

Key Facts 2.1b: Calculation of Compound interest

- **Solution for question 1**

  Given data:

  ✓ r= 9%=9/100=0.09

  ✓ P=10,000 Frw

  $SI = P \times R \times T$

  ✓ t= 2 years

  $SI = 10{,}000 \times 0.09 \times 2$

  $\Rightarrow SI = 1{,}800$

  Amount or Accumulated : $A = P + SI \Rightarrow 10{,}000 + 1{,}800 = 11800 Frw$

  ✓ t = $\frac{1}{2} year$ $t = 0.5$ year

  $SI = 10{,}000 \times 0.5 \times 0.09 = 450 Frw$

- **Solution for question 2**

  Given data:

$P = 500,000 Frw$

$t = 6 years$

$I = 150,000 Frw$

- ✓ **When compound interest is used.**

$$A = P(1+r)^t$$
$$A = P + I \Leftrightarrow 50,0000 + 150,000 \Rightarrow 650,000 Frw$$
$$(1+r)^t = \frac{A}{P} \Leftrightarrow (1+r)^6 = \frac{650,000}{500,000} \Leftrightarrow (1+r)^6 = 1.3$$

- ✓ **Apply log for both side to get:**

$$\log(1+r)^6 = \log 1.3$$
$$6\log(1+r) = 0.1139$$
$$\log(1+r) = \frac{0.1139}{6}$$
$$\log(1+r) = 0.0189$$
$$1 + r = 10^{(0.0189)} \Leftrightarrow 1 + r = 1.0439$$
$$\Rightarrow r = 1.0439 - 1 \Leftrightarrow r = 0.0439$$
$$\Rightarrow r = \% \Leftrightarrow 0.0439 \times 100$$
$$\Rightarrow r = 4.4\%$$

- ✓ **When simple interest is used,**

$$SI = P \times r \times t$$
$$150,000 = 500,000 \times 6 \times r \Leftrightarrow 150,000 = 3,000,000 \times r$$
$$\Rightarrow r = \frac{150,000}{3,000,000} \Leftrightarrow r = 0.05, r = 0.05 \times 100$$
$$\Leftrightarrow r = 5\%$$

- **Solution for question 3**

$$FV = 1000 \times \left(1 + \frac{0.05}{1}\right)^{1\times3}$$
$$FV = 1000 \times (1.05)^3$$
$$FV = 1000 \times 1.157625 \Leftrightarrow 1157.63$$

Hence the future value is **$1,157.63**

- **Solution for question 4**

- ✓ P=5,000

- ✓ **r**=0.06
- ✓ **n**=12 (compounded monthly)
- ✓ **t**=5 years

$$FV = 5{,}000\left(1+\frac{0.06}{12}\right)^{12\bullet 5} \Leftrightarrow 5{,}000(1+0.005)^{60}$$

$$\Rightarrow 5{,}000(1.005)^{60} \Leftrightarrow FV = 6{,}744.25Frw$$

Hence the future value **is 6,744.25 Frw.**

Activity 2: Guided Practice

Task 16:

Using the formulas in **key fact 2.1a**, answer the following questions

1. How much Interest do you get if you put 10,000 FRW in a saving account that pays simple interest at 9% per annum for two (2) years?
   a. How many interest if you invest in the account in 2 years?
   b. How many interest if you invest only the money in the account for half a year?
2. Suppose 500,000Frw principal earns 150,000Frw interest after 6 years.
   a. If compound interest is used, what was the interest rate?
   b. What if the simple interest is used?
3. A tourist saves $1,000 in a park's eco-tourism savings plan that offers an annual interest rate of 5% compounded annually. What will the amount grow to in 3 years?
4. You deposit 5,000 in a bank account earning 6% interest per year, compounded monthly. How much will you have after 5 years?

Activity 3: Application

Task 17:

A tourism company needs 1,500,000,000 frw to expand its business by increasing number of cars in its fleet as well as building a guest house. They have 300,000,000 and plan to invest it at an annual interest rate of 7.5% compounded annually. How long will it take to reach their goal?

Topic 2.2: Population growth

Activity 1: Problem Solving

Task 18:

A wildlife reserve has an initial population of 1,000 zebras. The population grows at a steady rate of 5% per year. Let's calculate the population at different time intervals and explore the relationship between the components.

a. What is the initial population of zebras in the reserve?

b. What does the growth rate of 5% per year mean?

c. How many years will we track the population to observe its growth?

d. What will the population be after 1 year? (Use the formula $P = P_O x e^{rt}$, where e≈2.718).

e. What happens to the population as time increases?

f. If the growth rate decreases to 2% per year, will the population grow faster or slower?

g. Why is it important to know the rate of growth and time of growth?

Key fact 2.2a: Calculation of population growth

- **Description of Population growth**

  Population Growth refers to the change in the number of individuals in a population over time. It is a critical aspect of demography and is influenced by factors such as birth rates, death rates, immigration, and emigration.

- **Terminologies used in population growth calculation**
  - ✓ **Initial Population**

    The initial population (denoted as **P_0)** is the starting number of individuals in a population at the beginning of a given time period.

    Example:

    If a town has 1,000 residents in the year 2000, then 1,000 is the initial population.
  - ✓ **Future Population**

    The future population (denoted **as P)** is the number of individuals in the population at a specific time in the future. The future population can be estimated based on the growth rate and the time period of interest.
  - ✓ **Rate of Growth.**

    The rate of growth (denoted as **r**) is the percentage or fraction that indicates how fast the population is increasing (or decreasing) per unit of time. A positive rate ($r>0$) indicates population growth, while a negative rate ($r<0$) suggests population decline. Growth rate is often expressed as an annual percentage, e.g., 2% per year ($r=0.02$).
  - ✓ **Time of Growth (t):**

    The time of growth (denoted as **t**) is the period over which the population changes, typically measured in years.

    Example:

    If you want to calculate the population after 10 years, $t=10$
- **Formulas for calculation of population growth**

  The formula for population growth calculation is $P = P_0 \times e^{rt}$ **or**

  $$P(t) = P_0 (1 + r)^t$$

  Where e is the base of natural logarithms (approximately 2.718)
- **Formulas for doubling time (T)**

The formula for doubling time is: $T = \frac{\ln(2)}{r}$

2.2b: Calculation of population growth

- **Solution for question1.**

  ✓ In the given example, in 1960, P_o =10,000. Thus, in 1960, we have $\mathbf{P_{10}}$=12,000 while in 1980,we have $\mathbf{P_{20}}$. So,

  $P_{10} = 12{,}000 \Leftrightarrow 12{,}000 = 10{,}000 \bullet e^{r*10}$,By applying natural logarithms both side

  $$\ln 12{,}000 = \ln 10{,}000 \times e^{10r}$$

  will get $\ln 12{,}000 = \ln 10{,}000 + \ln e^{10r}$,here we will need to apply properties of

  $$\ln 12{,}000 - \ln 10{,}000 = \ln e^{10r}$$

  logarithms ,where $\ln a^x = x \ln a$,then we will get

  $$\ln 12{,}000 - \ln 10{,}000 = 10r \ln e$$
  $$\Rightarrow \ln 12{,}000 = 9.39266$$
  $$\Rightarrow \ln 10{,}000 = 9.21034$$
  $$\Rightarrow \ln e = 1, Then$$
  $$9.39266 - 9.21034 = 10r$$
  $$\Leftrightarrow 0.18232 = 10r, \Rightarrow r = 0.0182$$

  The population in 1980 is given by $P_{20} = 10{,}000 \times e^{0.0182 \times 20}$ $\Rightarrow P_{20} = 14{,}390$.

  The population in 1980 is **14390.**

  ✓ The doubling time for the town's population means the time for which $P_n = 2P_o$

  $$\Rightarrow 2P_0 = P_0 \times e^{rt} \Leftrightarrow e^{rt} = 2$$
  $$\Rightarrow \ln e^{rt} = \ln 2, rt \ln e = \ln 2$$
  $$t = \frac{\ln 2}{r}, t \Rightarrow \frac{\ln 2}{0.0182} \Leftrightarrow t = 38 years$$

  Hence, the doubling time for the town's population is **38 years.**

- **Solution for question 2**

  ✓ We are given the formula for the population of the island: $P(t) = 4200 \times 1.04^t$ where t is the number of years after January 1, 2016. On January 1, 2016, t=0:

  $P(0) = 4200 \times 1.04^0 = 4200 \times 1 \Rightarrow 4200$. Hence**,** the population was **4200**

- ✓ We know that $P(t) = P_0 \times e^{rt}$,and we are given that $P(t) = 4200 \times 1.04^t$ by comparing these two equation we will get, $P_0 \times e^{rt} = 4200 \times 1.04^t$ this means that $P_0 = 4200 \Leftrightarrow e^{rt} = 1.04^t$ after rea-arrangement one can get $e^{rt} = 1.04^t$,apply natural logarithm for both side to get $\ln e^{rt} = \ln 1.04^t \Leftrightarrow rt \ln e = t \ln 1.04$

  $$\Rightarrow rt = t \ln 1.04 \Leftrightarrow r = \frac{t \ln 1.04}{t} \Rightarrow t = \ln 1.04 \approx 0.04 \Leftrightarrow r = 4\%$$

- ✓ On January 1, 2021, $t = 2021 - 2016, t = 5, then$ $P(5) = 4200 \times 1.04^5$,We can calculate this step by step:

  $$P(5) = 4200 \times 1.04^5 \Leftrightarrow 4200 \times 1.21665$$
  $$\Rightarrow 5{,}109.942 \approx 5110$$

  So, the population on January 1, 2021, was approximately **5110**.

- **Solution for question3**
  - ✓ **Data:**
    - Initial population=10,000
    - Final population=15,000
    - Time =5 years
    - Asked rate of growth, this is given from the following relation

  $P(t) = P_0 \times e^{rt}$,by applying natural logarithm for both side we will get

  $$\ln P(t) = \ln P_0 \times e^{rt} \Rightarrow \ln P(t) = \ln P_0 + \ln e^{rt} \Rightarrow \ln P(t) - \ln P_0 = rt \ln e \Rightarrow \ln\left(\frac{P(t)}{P_0}\right) = rt$$

  ,then $$r = \left(\frac{\ln \frac{P(t)}{P_0}}{t}\right) \Rightarrow r = \left(\frac{\ln \frac{15{,}000}{10{,}000}}{5}\right) = \frac{\ln 1.5}{5} \approx \frac{0.4055}{5} =\approx 0.0811 \approx 8.11\%$$

  - ✓ $$t = \left(\frac{\ln \frac{12{,}000}{8{,}000}}{0.04}\right) = \frac{\ln(1.5)}{0.04} \approx \frac{0.4055}{0.04} = 10 years$$

- ✓ Future number of tourists is given by

  $P(t) = P_0 \times e^{rt} \Rightarrow P(5) = 5{,}000 \times e^{0.03\times 5} \approx 5809$

  so after 5 years the number of tourist will be **5809.**

- **Solution for question 4**
  - ✓ **Given data:**
    - P0=10,000 (population in 1960),
    - P(t)=12,000 (population in 1970),
    - t=10 years (from 1960 to 1970).
  - ✓ **Estimating population in 1980**

    using formula : $P(t) = P_0 e^{rt}$,we can find the rate of growth, hence

    $12{,}000 = 10{,}000 e^{10r}$ Divide both sides by 10,000 to get:

    $1.2 = e^{10r}$ take the natural logarithm (ln) of both sides: $\ln(1.2) = \ln(e^{10r}) \Rightarrow \ln(1.2) = 10r$,solve for **r**

    $r = \frac{\ln(1.2)}{10} \approx \frac{0.1823}{10} = 0.01823(per year)$**,**so population in1980 (t=20) will be:

    r=0.01823, P_0=10,000, and t=20

    $P(20) = 10000e^{0.01823\times 20} \approx 10000 \times 1.439 \approx 14390$, The estimated population in 1980 is **14390**.

The formula for doubling time T in exponential growth is: $T = \frac{\ln(2)}{r} = \frac{\ln(2)}{0.01823} \approx \frac{0.693}{0.01823} \approx 38.02$, The doubling time for the town's population is approximately **38 years**.

Activity 2: Guided Practice

Task 19:

Using the formulas in key fact 2.2a, answer the following questions

1. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.
   a. Assuming an exponential growth model, estimate the population in 1980.
   b. What is the doubling time for the town's population?
2. The population, P, of an island t years after January 1st 2016 is given by this formula $P = 4200 \times (1.04)^t$
   a. What was the population of the island on January 1st 2016?
   b. What is the constant rate?
   c. Work out the population of the island on January 1st 2021.
3. If the tourist population grew from 10,000 to 15,000 in 5 years,
   a. Calculate the rate of this growth.
   b. If the current population is 8,000, growing at 4% annually, how long to reach 12,000?
   c. If the tourist population is 5,000, growing at 3% annually, calculate the number of tourist population after 5 years?
4. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.
   a. Assuming an exponential growth model, estimate the population in 1980.
   b. What is the doubling time for the town's population?

Activity 3: Application

Task 20: Answer the following questions

1. The population of a city is growing at a rate of 5% per year. If the current population is 10,000, what will the population be in 10 years?
2. A tourist destination has a current annual visitor count of 5,000, and the number of tourists is increasing at a rate of 7% per year. How long will it take for the number of tourists to double?
3. The number of tourists visiting a park increased from 2,000 to 3,000 in 5 years. What is the annual growth rate?

4. According to United Nation data, the world population at the beginning of 1975 was approximately 4 billion and growing at rate of about 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2030.
5. The population of a country grows according to the law $P = Ae^{0.06t}$ where P million is the population at time t years and A is a constant. Given that at time t=0, the population is 27.3 million, calculate the population when
   a. t=10 b) t=15 c) t=25
6. The population of a country grows according to the law $P = 12e^{kt}$ where p million is the population at time t years and k is the constant .Given that when t=7, P=15. Find the time for which the population will be a) 20 million b) 30 million c) 35 million
7. A city in Texas had a population of 75,000 in 1970 and a population of 200,000 in 1995. The growth between the years 1970 and 1995 followed an exponential pattern of the form $f(t) = A \times e^{\alpha t}$
   a. Find the values of A and α
   b. Using the given model, estimate the population for the year 2010

Topic 2.3: Population decay

Activity 1: Problem Solving

Task 21:

During an experiment, a scientist notices that the number of bacteria halves every second. If there were 2.3×10^{30} bacteria at the start of the experiment, how many bacteria were left after 5 seconds? Give your answer in standard form correct to two significant figures.

Key Facts 2.3a: Calculation of population decay

- **Description of population decay**

  Population decay refers to a process where the size of a population decreases over time. This phenomenon is commonly modeled using **exponential decay,** which assumes that the rate of change of the population is proportional to its current size.

This can be due to various factors, such as resource depletion, disease, natural disasters, or environmental changes.

- **Terminologies used in calculation of Population Decay**
  - ✓ **Decay Rate:**

  The decay constant (denoted as **k**) determines how quickly the population decreases. A larger k corresponds to a faster rate of decay.

  - ✓ **Half-Life:**

    The half-life (denoted as $\left(T_{\frac{1}{2}}\right)$) is the time it takes for the population to reduce to half its original size.

  - ✓ **Doubling time**

    Doubling Time (denoted as T**)** refers to the period it takes for a population (or any other quantity) to double in size at a constant growth rate. It is a useful concept in demography, economics, and environmental studies to understand how quickly a population or resource is growing.

  - ✓ **Exponential Behavior:**

    The decrease is rapid initially and slows over time, never truly reaching zero but approaching it asymptotically.

- **Formulas for calculation of population decay**
  - ✓ **Decay Rate:**

    A population whose rate of decrease is proportional to the size of the population at any time obeys a law of the forms $P(t) = Ae^{-kt}$, $P(t) = P_0e^{-kt}$ and $P(t) = P_0(1-r)^t$. The negative sign on exponent indicates that the population is decreasing. This is known as exponential decay. If a quantity has an exponential growth model, then the time required for it to double in size is called the doubling time. Similarly, if a quantity has an exponential decay model, then the time required for it to reduce in value by half is called the halving time. For radioactive elements, halving time is called half-life.

✓ **Half-Life**

The formula of half-life is : $T_{\frac{1}{2}} = \frac{\ln 2}{k}$

✓ **Doubling time**

The formula of half-life is: $T = \frac{\ln 2}{k}$

Key Facts 2.3b: Calculation of population decay

- **Solution for question 1**

Use the exponential decay formula: $P = P_0 e^{-kt}$

Where:

✓ P0=10,000 (initial population)

✓ k=0.08 (decay rate)

✓ t=5 years

Substitute into the formula: $P(5) = 10{,}000 \times e^{-0.08\times5} \Leftrightarrow P(5) = 10{,}000 \times e^{-0.4}$ Using $e^{-0.4} \approx 0.67032$

$P(5) = 10{,}000 \times 0.67032 = 6{,}703.2$, so after 5 years, the population will be approximately **6,703 individuals**.

- **Solution for question 2**

Use the exponential decay formula: $P(t) = P_0 e^{-rt}$ Rearrange to solve for t let apply natural logarithm in both side to get $\ln P(t) = \ln P_0 e^{-rt}$ with the help of basic rules of logarithms we will get $\ln P(t) = \ln P_0 + \ln e^{-rt} \Leftrightarrow \ln P(t) - \ln P_0 = -rt \ln e$ remember that natural logarithm of e is equal to 1,so the time will be given by :

$$t = \frac{\ln\left(\frac{P(t)}{P_0}\right)}{-r} =$$

Where:

- ✓ P0=50,000 (initial population)
- ✓ P(t)=20,000P (final population)
- ✓ r=0.06 (decay rate)

Substitute values in formula to get $t = \frac{\ln\left(\frac{20,000}{50,000}\right)}{-0.06} \Leftrightarrow \frac{\ln(0.4)}{-0.06}, t \approx 15.27$. It will take approximately **15.27 years** for the population to decay to 20,000.

- **Solution for question3**

To solve this problem, we use the **exponential decay formula**: $N(t) = N_0 e^{-kt}$

Where:

- ✓ N(t) is the number of cells after time t (in minutes),
- ✓ N0=5,000,000 is the initial number of cells,
- ✓ k is the decay constant,
- ✓ t is the time in minutes.

The problem states that **45% of the cells are dying every minute**, meaning 55% of the cells remain after each minute. This corresponds to a decay factor of 0.55 per minute.

The decay factor is related to the decay constant k: $e^{-k} = 0.55$ Take the natural logarithm in both side to get $-k = \ln(0.55) \Leftrightarrow -\ln 0.55 \approx 0.5978$

Substitute N0=5,000,000 and k≈0.5978 in exponential decay formula.

$1,000 = 5,000,000 \times e^{-0.5978t}$. Divide both sides by 5,000,000 to get

$$\frac{1,000}{5,000,000} = e^{-0.5978t} \Rightarrow 0.0002 = e^{-0.5978t}$$

Take the natural logarithm of both sides: ln(0.0002) =−0.5978t.

$$t = \frac{\ln(0.0002)}{-0.5978} \approx 14.25$$

.

So it will take approximately **14.3 minutes** for the cell population to drop below 1,000.

Activity 2: Guided Practice

Using the formulas in key fact 2.3a, answer the following questions

Task 22:

1. A population of bacteria starts with 10,000 individuals and decreases at a decay rate of 8% per year. What will the population be after 5 years?
2. If a population of 50,000 decreases by 6% per year, how long will it take for the population to decay to 20,000?
3. If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 1,000 cells?

Activity 3: Application

Task 23: Answer the following questions.

1. A city's population is decreasing exponentially by 4% per year. If the current population is 100,000, how long will it take for the population to halve?
2. In a certain experiment, the number of bacteria reduces by a quarter each second. If the number of bacteria initially was X, write a formula that can be used to calculate the number of bacteria, V, remaining after t seconds
3. The population of a particular town on July 1, 2011 was 20,000. If the population decreases at an average annual rate of 1.4%, how long will it take for the population to reach 15,300?
4. A certain town has an initial population of 40,000 people. If the population is decreasing by 2% per year, how long will it take for the population to fall below 30,000?
5. A population of a certain species of fish in a lake is initially 5,000. If the population decreases at a rate of 3% per year, how many years will it take for the population to reduce to 2,000?
6. A certain species of plant in a controlled environment has a population of 500. If the population decreases at a rate of 4% per month, how many plants will remain after 6 months?

Formative Assessment

1. Multiple Choice questions

   a. The formula for compound interest is given by $A = P\left(1+\frac{r}{n}\right)^{nt}$. What does the variable **t** represent?

      i. Time in years

      ii. Interest rate

      iii. Principal amount

      iv. Number of times interest is compounded

   b. Population grows according to the formula $P(t) = P_0 e^{kt}$. If $k \succ 0$, what does it indicate?

      i. Population decay

      ii. Population stability

      iii. Population growth

      iv. None of the above

   c. If the half-life of a substance is 5 years, what is the value of the decay constant k (rounded to 3 decimal places)?

      i. 0.693

      ii. −0.693

      iii. 0.138

      iv. −0.138

   d. Which of the following equations is used to calculate continuous compound interest?

      i. $A = P(1+r)^t$

      ii. $A = P\left(1+\frac{r}{n}\right)^{nt}$

      iii. $A = Pe^{rt}$

      iv. $A = P + \text{Pr}t$

2. Differentiate Initial population to Future population

3. Differentiate simple to Compound Interest.

4. A \$1000 deposit is made at a bank that pays 12% compounded weekly. How much will you have on your account at the end of 10 years?
5. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.
   a. Assuming an exponential growth model, estimate the population in 1980.
   b. What is the doubling time for the town's population?
6. A principal amount of \$10,000 is invested at an annual interest rate of 5%, compounded quarterly. Calculate the amount after 3 years. (Use $A = P\left(1+\frac{r}{n}\right)^{nt}$)
7. A radioactive substance has a half-life of 10 years. How much of a 100g sample remains after 30 years? Use $P(t) = P_0 e^{-kt}$ and calculate $k = \frac{\ln(2)}{half - life}$
8. A population of bacteria decreases from 100,000 to 25,000 in 8 hours. Calculate the decay constant k and the time it takes for the population to reduce to 12,500.
9. A loan of \$50,000 is taken out with an annual interest rate of 6%, compounded continuously. Calculate the total amount to be repaid after 10 years. (Use $A = Pe^{rt}$)
10. The population of a country is 1 million, and it is observed to double every 35 years. Derive the exponential growth equation for this population and predict the population after 70 years.
11. Solve for t in the compound interest equation: $2000 = 1000(1+0.05)^t$
12. A population decreases by 3% annually. If the current population is 12,000, how long will it take for the population to reduce to 8,000? Use the formula ($P(t) = P_0 e^{-rt}$

Points to Remember

- Exponential and logarithmic functions are used in population growth, half-life, and decay in radioactivity. Logarithmic functions are also used to find interest rate problems. A quantity is said to have an exponential growth (decay) model if at each instant of time its rate of increase (decrease) is proportional to the amount of the quantity present.
- Formula for compound interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$ or $A = Pe^{rt}$ for continuous compounding. Where:
  - ✓ A: Final amount
  - ✓ P: Principal amount
  - ✓ r: Annual interest rate (decimal form)
  - ✓ n: Number of compounding periods per year
  - ✓ t: Time in years
- Exponential growth is given by $P(t) = P_0 e^{rt}$ .Where :
  - ✓ P(t): Population at time t
  - ✓ P0: Initial population.
  - ✓ k: Growth rate constant (positive for growth).
  - ✓ t: Time elapsed.
  - ✓ Use $k = \ln(1 + r)$ if the growth rate is given in percentage.
  - ✓ The population double when $t = \frac{\ln(2)}{k}$ (known as doubling time)
- Exponential decay is given by $P(t) = P_0 e^{-rt}$ .Where:
  - ✓ P(t): Population or amount remaining at time t.
  - ✓ P0: Initial Population
  - ✓ k: Decay rate
  - ✓ t: Time elapsed.
  - ✓ The decay constant k is calculated using the half-life: $k = \frac{\ln(2)}{half - life}$

- For exponential growth model, the time required for it to double in size is called the doubling time. Similarly, for exponential decay model, the time required for it to reduce in value by half is called the halving time. For radioactive elements, halving time is called half-life.
- Remember the distinction between growth $(k \succ 0)$ and decay $(k \prec 0)$
- Logarithms can help determine how long it takes for a substance to decay to a specific amount.

Self-Reflection

1. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

My experience **Knowledge, skills and attitudes**	**I don't have any experience doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Define correctly simple and compound interest.					
Calculate simple and compound interest accurately using formulas.					
Pay attention to details while calculating simple and compound interest					
Recognize the meaning of each variable needed in calculation of compound interest (e.g. principal value, rate, time, and					

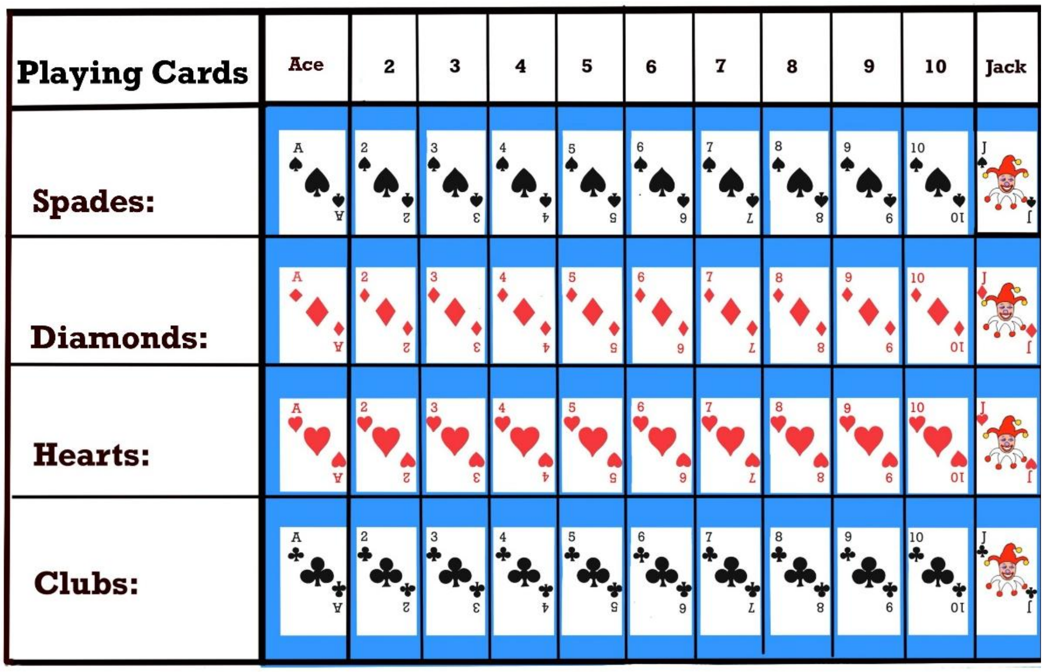
My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
compounding frequency or periods)					
Rearrange logarithmic and exponential equations to solve for unknown variables (e.g., time t, rate r)					
Think logically when applying formula of compound interest.					
Explain exponential growth using its formula.					
Apply exponential growth formulas to real-world problems (e.g., predicting future population size).					
Be patient while solving problems related to population growth.					
Recognize the meaning of each variable needed in calculation of population growth (growth rate, initial population etc.).					
Predict future population sizes using exponential models.					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Develop a problem solving mindset when working with population growth formula.					
Explain exponential decay using its formula.					
Solve real-world problems involving population reduction using exponential decay formula.					
Think logically when applying the population decay formula to ensure accurate calculations and meaningful interpretations.					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1. 2. 3.	1. 2. 3.	1. 2. 3.

UNIT 3: APPLY FUNDAMENTALS OF PROBABILITY



Unit summary

This unit provides you with the knowledge, skills and attitudes required to apply fundamentals of probability. It covers application of counting techniques, computation of probabilities and calculation of the conditional probability.

Self-Assessment: Unit3

1. Answer the following questions:
   a. Define probability, events, sample spaces, and outcomes.
   b. What are some real life examples of probability?
   c. How to calculate probability?
   d. Ivan rolls a fair dice, with sides labelled A,B,C,D,E and F. What is the probability that the dice lands on a vowel?
   e. Max tested a coin to see whether it was fair. The table shows the results of his coin toss experiment

Head	Tail
26	**41**

   What is the relative frequency of the coin landing on heads?
2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.
   a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
   b. Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
   c. Read the statements across the top, put a check-in a curriculum that best represents your level of knowledge, skills, and attitudes.
3. At the end of this unit, you'll assess yourself again.

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Define correctly probability and its associated terms					
Determine the number of permutations and combinations of “n” items, “r” taken at a time.					
Pay attention to details while calculating probability					
Describe the application of probability to different fields including tourism, social sciences etc…					
Use counting techniques to solve related problems.					
Approach problems systematically and logically.					
Distinguish properly between combination and permutation					
Use properties of combinations					
Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios.					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Define the combinatorial Analysis.					
Analyse problems to determine relevant probabilities.					
Be patient while solving complex probability problems.					
Describe different formulas used to solve problems related to calculation of probability					
Apply concepts to solve real-world problems and scenarios.					
Describe uses of Unions, Intersections and Complements in probability contexts.					
Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment.					
Interpret data to make probabilistic inferences					

Key Competencies:

Knowledge	Skills	Attitudes
1. Define correctly probability and its associated terms	1. Determine the number of permutations and combinations of “n” items, “r” taken at a time.	1. Pay attention to details while calculating probability
2. Describe the application of probability to different fields including tourism, social sciences etc...	2. Use counting techniques to solve related problems.	2. Approach problems systematically and logically.
3. Distinguish properly between combination and permutation	3. Use properties of combinations	3. Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios.
4. Define the combinatorial Analysis.	4. Analyse problems to determine relevant probabilities.	4. Be patient while solving complex probability problems.
5. Describe different formulas used to solve problems related to calculation of probability	5. Apply concepts to solve real-world problems and scenarios.	
6. Describe uses of Unions, Intersections and Complements in probability contexts.	6. Interpret data to make probabilistic inferences	
7. Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment.		

Discovery activity:

Scenario 1:

Nyungwe National Park is home to a variety of primates, including chimpanzees. Based on data: the probability of spotting a chimpanzee during a morning tour is **70%**, the probability of spotting colobus monkeys during the same tour is **50%**, the probability of spotting both species is **40%.**

Task 24: Read the following scenario and respond the corresponding questions

1. What is the meaning of term probability?
2. What is the probability of spotting at least one species (either chimpanzees or colobus monkeys)?
3. What is the probability of not spotting either species during the tour?
4. If 3 tours are conducted on different days, what is the probability of spotting chimpanzees on all 3 tours?

Topic 3.1: Counting techniques in probability

Activity 1: Problem Solving

Task 25:

Answer the following questions:

1. What is probability?
2. What is the formula for probability calculation?
3. What are the branches of probability?
4. What are the Techniques for counting the number of elements in a sample space?
5. Provide formula for each technique.

Key Facts 3.1a: Counting techniques in probability

- **Introduction to Probability**

  ✓ **Definition**

  Probability is a branch of mathematics that deals with the likelihood or chance of different outcomes occurring in a random experiment. It provides a framework for analyzing situations where outcomes are uncertain, allowing us to make predictions and informed decisions based on quantitative reasoning. It is the chance that something will happen or how likely it is that some event will happen.

  For example, when the weather forecast says there' a 60% chance of rain, that's a probability.

  ✓ **Basic formula of probability**

  The probability of an event **A,** denoted P**(A),** is a measure of how likely the event is to occur.

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes in sample (S)}}$$

  ✓ **Branches of Probability**

  There are 4 main branches of probability:

- **Theoretical probability**: Also known as classical probability. This is the probability of an event occurring in an experiment where all outcomes are equally likely**.**

  Example: When rolling a fair die, what is the probability of rolling each number?

  Solution

  The probability is 1/6. This is because that a die has six sides with numbers from 1 to 6. As each number is represented only once, the probability is 1/6.

- **Empirical probability**: Also known as experimental probability. This is refers to a probability that is based on historical data of a conducted experiment. In other words empirical probability is an event occurring based on historical data.

$$\text{Empirical probability} = \frac{\text{frequency of event}}{\text{total frequency}}$$

Example: A bag contains 5 red marbles, 3 blue marbles, and 2 green marbles. You draw one marble from the bag, record its color, and then replace it back into the bag. This process is repeated 50 times, and the results are as follows: Red: 28 times, Blue: 15 times and Green: 7 times. What is the experimental probability of drawing each color?

Solution

✓ What is the experimental probability of drawing a red marble? Experimental Probability = (Number of times red was drawn) ÷ (Total number of trials)

$$\Rightarrow \frac{28}{50} = 0.56 \approx 56\%$$

✓ What is the experimental probability of drawing a blue marble? Experimental Probability = (Number of times blue was drawn) ÷ (Total number of trials)

✓ What is the experimental probability of drawing a green marble? Experimental Probability = (Number of times green was drawn) ÷ (Total number of trials)

$$\Rightarrow \frac{7}{50} = 0.14 \approx 14\%$$

In the above example, the theoretical probability of drawing each color is:

- Red: Theoretical probability: $\frac{5}{10} = 0.5 \approx 50\%$
- Blue: Theoretical probability: $\frac{3}{10} = 0.3 \approx 30\%$
- Green: Theoretical probability: $\frac{2}{10} = 0.2 \approx 20\%$

- **Subjective probability**: Is likelihood event based on their personal judgment or experience, rather than on calculations or observations. This is also called a personal probability.

  Example: Anna is planning to go hiking tomorrow. She checks the weather forecast, which says there's a 40% chance of rain. However, Anna notices that the sky is clear today and believes the chance of rain tomorrow is much lower. Based on her experience, she estimates there's only a 10% chance of rain. What is Anna's subjective probability of rain tomorrow, and how does it differ from objective probability?

  Solution

  Anna's subjective probability is **10%**. This is based on her personal judgment and experience, rather than statistical or objective data. And The objective probability **is 40%**, as provided by the weather forecast, which is based on meteorological data and statistical analysis.

- **Axiomatic probability**: Is a mathematical theory that describes the likelihood of an event by establishing a set of axioms that apply to all probability approaches.

  Probability is a set function P(E) that assigns to every event E a number called the probability of E such that:

  ✓ The probability of an event is greater than or equal to zero. $P(E) \geq 0$

  ✓ The probability of the sample space is one**.** $P(U) = 1$

- **Techniques for counting the number of elements in a sample space.**

  There are five (5) techniques for Counting the number of elements in a sample space namely:

  ✓ Venn diagram,

  ✓ Tree diagram,

  ✓ Multiplication principle,

  ✓ Permutations and

  ✓ Combination

- **Venn diagram**

This is a way of representing sets in a closed curve. The universal set, written U or ∈ is a set containing all the sub-sets considered in the problem.

In probability theory, the universal set is called the **sample space** and its subsets are called the **events**.

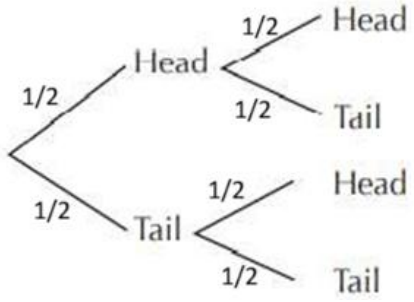
- ✓ **The complement** of a set say A is the set written A′ or $\bar{A}$ of an element in the universal set but not in the set A.
- ✓ **Intersection of sets** A and B is denoted by A ∩ B. This represents the set of all elements common to both A and B.
- ✓ **Union of set** A and B is denoted by A ∪ B. This represents the set of all elements belonging to A or B or both A and B.
- ✓ **Disjoint sets** are sets with no elements in common. If A and B are disjoint sets, then A ∩ B = ∅. Also A and B are said to be **mutually exclusive**.

- **Tree diagram**

A tree diagram is a diagram with a structure of branching connecting lines representing a relationship. It can be used to find the number of possible outcomes of experiments where each experiment occurs in a finite number of ways in succession.
The outcome is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.
For each trial, the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, A and B, of each trial. For example, when you toss a coin, the outcome is either head or tail.

 Head	 Tail

A second toss would also give head or tail. We represent this as:



- **Multiplication rule**

This is also known as product rule or product principle or multiplication principle. Suppose that an experiment (operation) is to be performed in two successive ways. If the operation is composed of k successive steps which may be performed in n_1 , n_2 , ..., n_k distinct ways, respectively, then the operation may be performed in $n_1 \times n_2 \times \ldots \times n_k$ distinct ways.

- **Permutations**

Permutation is an ordered arrangement of the items in a set.

The number of arrangements of **r** objects, taken from **n** unlike objects, can be considered as the number of ways of filling **r** places in order by the **n** given objects and is given by:

$$P_r^n = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots..(n-r+1)$$

Notes:

- ✓ **!** is factorial notation
- ✓ A different notation can be used of **permutation with no repetition**: $P_r{}^n$ = P (n, r) Condition n ≥ r and the number of arrangements of r objects, taken from n unlike objects where each of which may be repeated any number of times is $\boldsymbol{P_r{}^n}$ **= n^{r}**.
- ✓ The number of permutations of n object with n1 identical objects of type 1,n2 identical objects of type 2,…………, and nk identical objects of k type is **P=**$\frac{\boldsymbol{n}!}{\boldsymbol{n1}!\boldsymbol{n2}!\ldots\ldots\boldsymbol{nk}!}$
- ✓ Number of permutations of n objects, repetition not allowed is **P=n!**

- **Combination**

Combination is a selection of objects where the order is not taken into account. In other words, the order in which the objects are arranged is not important.

When considering the number of combinations of **r** objects, the order in which they are placed is not important.

In general, the number of combinations of **r** objects from **n** unlike objects is

$\binom{n}{r}$ or C $(n, r) = \frac{n!}{r!(n-r)!}$

Condition n ≥ r

Note:

Other notations that can be used are ^nCr , $C^n r$ or $_n Cr$.

Some properties are:

- ✓ $\binom{n}{r} = \binom{n}{n-r}$
- ✓ $\binom{n}{n} = \binom{n}{0}$
- ✓ $\binom{n}{1} = n$

Sometimes we are given a condition that must be taken into account in a combination.

Note:

$C_r^n = \frac{P_r^n}{r!}$, where P_r^n is the permutation of n objects taken r at a time.

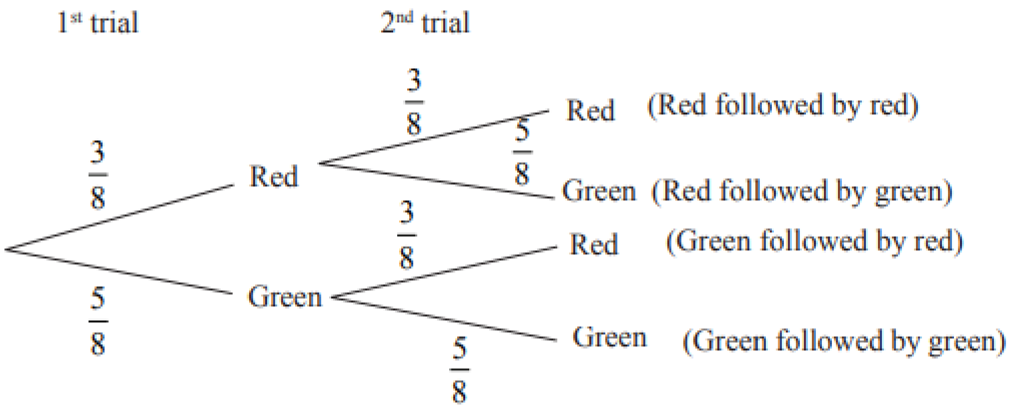
Key Facts3.1b: Application of techniques for Counting the number of elements in a sample space.

- **Solution for question 1**
  - ✓ $A \cup B = \{1,3,4,5,6,7\}$
  - ✓ $A \cap C = \{3,5,7\}$
  - ✓ $B - C = B \setminus C = \{4,6\}$

- ✓ $A \cup B' = A \cup (U \setminus B) = A \cup (1,2,3) \Rightarrow \{1,2,3,5,7\}$
- ✓ $C' - A'$,Let us first find $C' = U \setminus C = \{1,4,6\}$ and $A' = U \setminus A = \{2,4,6\}$ so $C' - A' = C' \setminus A' = \{1\}$
- ✓ $A' \cap C \{2,4,6\} \cap \{2,3,5,7\} = \{2\}$

- **Solution for question 2**

  Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is 3 ÷ 8 and probability of choosing a green ball is5 ÷ 8 and since after the 1st trial, the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial. Draw a tree diagram showing the probabilities of each outcome of the two trials.



- ✓ P (Red followed by green) = $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$
- ✓ P (Red and green in any order) = $\frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{15}{32}$
- ✓ P (both of the same colors) = $\frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{5}{8} = \frac{17}{32}$

- **Solution for question 3**

  The number of all possible different clothing for Habimana is 4 × 3 × 2 = 24. Therefore, The maximum number of days Habimana does not need to repeat his clothing is 24.

- **Solution for question 4**

  The first place can be filled in 5 ways, since any contestant can come first. When the first place has been filled, there are 4 more contestants to choose from for the second place. Hence the second place can be filled in 5 × 4 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 3 contestants, and the total number of ways is 5 × 4 × 3 = 60

- **Solution for question 5**

  There are $P_4^{15} = \frac{15!}{(15-4)!} = \frac{15\times14\times13\times12\times11!}{11!} = 15 \times 14 \times 13 \times 12 = 32760$ different line ups

- **Solution for question 6**

  Total number n=5 , number of digits taken each time r=3

  If there is repletion of element $P_r^n = n^r = 5^3 = 125$

- **Solution for question 7**

  Since there are three **m**, two **a** and one **l** in the word mammal. Total number of letters is **6.** We have for the number of ways we can arrange the letters in the following ways:

  P=$\frac{n!}{n1!n2!n3!} = \frac{6!}{3!\times2!\times1!}$=$\frac{6\times5\times4\times3!}{3!\times2!\times1!} = \frac{6\times5\times4}{2\times1} = 60$

- **Solution for question 8**

  The number of committees is $\binom{12}{3}$ = $\frac{12!}{3!(12-3)!}$ = $\frac{12x11x10x9!}{3!x9!}$ = 220

- **Solution for question 9**

  The number of ways of choosing the men is $\binom{12}{3}$ = $\frac{12!}{3!(12-3)!}$ = $\frac{12x11x10x9!}{3!x9!}$ = 220

  The number of ways of choosing the women is

  $\binom{8}{2}$ = $\frac{8!}{2!(8-2)!}$ = $\frac{8x7x6!}{2!x6!}$ = 28

  Therefore, the total number of ways of choosing the committee (given by the product rule) is

  220 × 28 = 6160

Activity 2: Guided Practice

Task 26:

1. Using the techniques for counting the number of elements in a sample space, answer the following questions:

   a. Let $U = \{1,2,3,4,5,6,7\}$, $A = \{1,3,5,7\}, B = \{4,5,6,7\}$ and $C = \{2,3,5,7\}$. List the elements in each of the following sets:

$$\begin{array}{ll} a. A \cup B & d. A \cup B^{'} \\ b. A \cap C & e. C^{'} - A^{'} \\ c. B - C & f. A^{'} \cap C \end{array}$$

2. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Using three diagram, find the probability that the ball drawn will be:

   a. red followed by green,

   b. red and green in any order,

   c. of the same color.

3. Habimana has 4 shirts, 3 pair of trousers and 2 pairs of shoes. He chooses a shirt, a pair of trousers and a pair of shoes to wear every day. Using multiplication rule, find the maximum number of days he does not need to repeat his clothing.

4. Using multiplication rule, find how many ways the first three places (with no ties) can be filled in a race with 5 contestants.

5. Using permutation, how many ways can 4 students from a group of 15 be lined up for a photograph?

6. Using permutation, How many three digit numbers can be formed with the digits: 1,2,3,4.5 with repletion allowed?

7. In how many ways can six letters of the word “mammal” be arranged in a row using permutation?

8. Using combination technique, how many different committees of 3 people can be chosen from a group of 12 people?

9. A committee of 5 is to be chosen from 12 men and 8 women. In how many ways can this be done if there are to be 3 men and 2 women on the committee using conditional combination?

Activity 3: Application

Suppose your company has 8 qualified tour guides, and you need to select a team of 3 guides for a safari tour.

Task 27:

From the scenario above, perform the following tasks:

1. How many different teams of 3 guides can you form?
2. If the roles of team leader, assistant, and driver need to be assigned, how many arrangements are possible?

Topic 3.2: Probabilities for an event

Activity 1: Problem Solving

Task 28:

Answer the following questions:

1. What do you understand by term event in probability?
2. What are the types of events in probability?
3. How many tourists like at least one of the activities?
4. What is the probability that a randomly selected tourist likes exactly two activities?

Key fact 3.2a: Description of formulas for computation of probabilities for an event

- **Definition of key terms**
  - ✓ **Sample space** is a set of all possible outcomes that may occur in a particular experiment is denoted by **S.**

For example:

a. When a coin is tossed: S = {H, T}.

b. When two coins are tossed: S = {HT, TH, HH, TT}.

c. When a die is thrown: S = {1, 2, 3, 4, 5, 6}

- ✓ **Possible outcomes** are all likely results of an experiment.
- ✓ **Event**: It is a set consisting of possible outcomes of an experiment with the desired qualities.

- **Types of event**

  - ✓ **Simple events**

    It is one that can only happen in one way in other words, it has a single outcome. If we consider for example of tossing a coin: we get one outcome that is a head or a tail.

  - ✓ **Certain event** is an event that will definitely take place.

    The following are examples of certain (sure) events.

    - If you put a shirt in water, it will get wet.
    - If you put your exercise book in fire, it will burn.

  - ✓ **Impossible event** is an event that cannot take place.

    The following are examples of impossible events.

    - A dice cannot give a 9.
    - A cassava plant will produce potatoes.

  - ✓ **Uncertain event** is an event that may or may not take place.

    The following are examples of uncertain events.

    - It will rain tomorrow.
    - The national football team will win their next match.
    - Mala will win the elections.
    - I will eat meat tomorrow

- **Complementary event**

  If E is an event, then E' is the event which occurs when E does not occur. Event E and E' are said to be complementary events. Theorem P(E') = 1 – P(E) or P(E) = 1 – P (E')

- **Probability of an event under equally likely events**

  The probability of an event E, denoted by P(E), is a measure of the possibility of the event occurring as the result of an experiment.

  The probability of an event E, denoted by P(E), is a measure of the possibility of the event occurring as the result of an experiment. If the sample space S is finite and the possible outcomes E are equally likely, then the probability of the event E is equal to $\frac{|E|}{|S|} = \frac{n(E)}{n(s)}$

  where |E| and |S| denote the number of elements in E and S respectively.

  n(E)=Number of favourable outcomes and n(S)= Number of total outcomes in sample space.

  Suppose that an experiment has only a finite number of equally likely outcomes. If E is an event, then $0 \leq P(E) \leq 1$.

- **Inclusive events**

  Inclusive events are events that can happen at the same time. To find the probability of an inclusive event we first add the probabilities of the individual events and then subtract the probability of the two events happening at the same time. Inclusive events are called also non- mutually exclusive.

  If A and B are events from a sample space S, then

  $P(A \cup B) = P(A) + P(B) – P(A \cap B)$.

- **Mutually exclusive events**

  A and B are said to be mutually exclusive if the events A and B are disjoint i.e. A and B cannot occur at the same time.

  For mutually exclusive events, $A \cap B = \emptyset$. $P(A \cap B) = P(\emptyset) = 0$; and the addition law reduces to $P(A \cup B) = P(A) + P(B)$.

Key fact 3.2b: Calculation of probabilities

- **Question 1**

  The first question is a Complementary event , with this
  The sample space of possible outcomes is $S = \{1,2,3,4,5,6\}$

  ✓ $P(5) = \frac{1}{6}$ b) P (not 5)=$1 - \frac{1}{6} = \frac{5}{6}$

  ✓ P(3 or 4)=$\frac{2}{6} = \frac{1}{3}$ d) P(not 3 or 4)=$1 - \frac{1}{3} = \frac{2}{3}$

- **Question 2**

  This question has the equally likely event. Since two of the eleven letters are ''A'', the probability of choosing a letter ''A'' is $\frac{2}{11}$.

- **Question 3**

  Question 3 is and an inclusive i=event, thus

  P(A) = P(an ace) = $\frac{4}{52} = \frac{1}{13}$

  P(B) = P(an spade) = $\frac{13}{52} = \frac{1}{4}$

  P(A∩ B) = P(the ace of spades) = $\frac{1}{52}$

  $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

- **Question 4**

  This is the Mutually exclusive events. Therefore

  P$(A \cup B)$ or $P(red\ or\ blue) = P(A) + P(B) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$

- **Question 5**

  Let the first machine be **M1** and the second machine be **M2** , then P M(1) = 80% =0.8 , P M(2) = = 60% =0.6 and $P(M_1 \cup M_2) = 92\% = 0.92$

  Now,

  $$P(M_1 \cup M_2 = P(M_1) + P(M_2) - P(M_1 \cap M_2)$$
  $$\Leftrightarrow P(M_1 \cap M_2) = P(M_1) + P(M_2) - P(M_1 \cup M_2)$$
  $$\Rightarrow 0.8 + 0.6 - 0.92 = 0.48 = P(M_1) \times P(M_2)$$

  Thus, the two machines operate independently

Activity 2: Guided Practice

Task 29:

Answer the following questions:

1. An ordinary die of 6 sides is rolled once. With the aid of definition of complement event, answer the following questions.

   Determine the probability of:

   a. Obtaining 5

   b. Not obtaining 5

   c. Obtaining 3 or 4

   d. Not obtaining 3 or 4

2. A letter is chosen from the letters of the word ''MATHEMATICS''. What is the probability that the letter chosen is an ''A''?
3. A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P (A), P(B), P(A ∩ B) and P(A ∪ B).
4. A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find P(A), P(B) and P(A ∪ B).
5. A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Activity 3: Application

A survey among 100 tourists reveals their preferences for three types of activities:

- **A:** Wildlife Safari (60 tourists)
- **B:** Lake Cruise (50 tourists)
- **C:** Cultural Village (40 tourists)

Overlap data:

- 20 tourists like **both A and B**
- 15 tourists like **both A and C**
- 10 tourists like **both B and C**
- 5 tourists like **all three activities**

Task 30:

Read the scenario and answer the following questions:

1. How many tourists like at least one of the activities?
2. What is the probability that a randomly selected tourist likes exactly two activities?

Topic 3.3: The conditional probability

Activity 1: Problem Solving

Task 31: Read carefully the following statement and answer the asked question

A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event "the first pen is red" and B be the event "the second pen is blue". Is the occurrence of event B affected by the occurrence of event A? Explain.

Key Facts 3.3a: Calculation of the conditional probability

- **Definition of key terms**
  - ✓ **Conditional probability**

    Conditional probability is the probability of an event occurring given that another event has already occurred. It quantifies how the occurrence of one event affects the likelihood of another event.

    - **Formula of conditional probability**
      - Mathematically, the conditional probability of event A given event B (denoted as P(A|B) is equal to $P(A \setminus B) = \dfrac{P(A \cap B)}{P(B)}$,where $P(B) \succ 0.$
      - The probability of an event B given event A (denoted as) $P(B/A)$ is equal to $P(B/A) = \dfrac{P(B \cap A)}{P(A)}$.
      - Here:

        $P(A \cap B)$: Is the probability that both events A and B occur.

        $P(B \cap A)$: Is the probability that both events Band A occur

        P(A): Is the probability that event A occurs.

P(B): Is the probability that event B occurs

- The following figure explain well the conditional probability

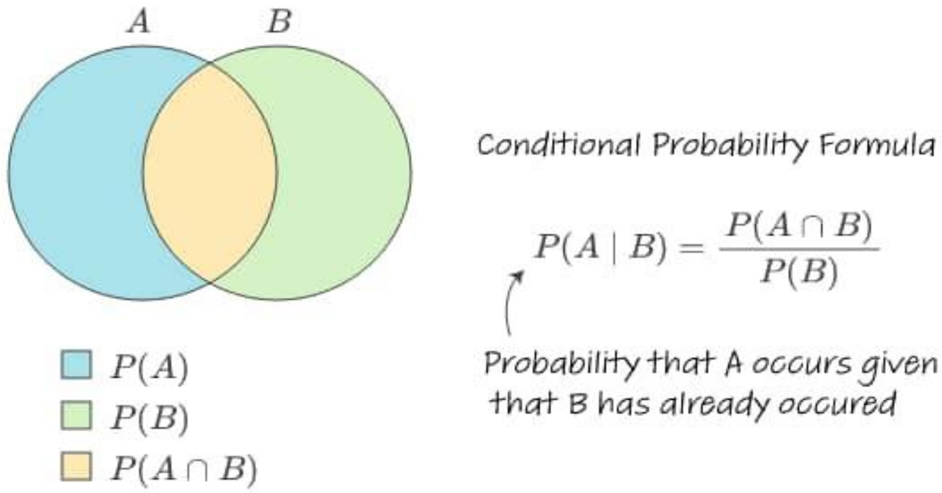


Figure 2:Conditional probability

- **Independent events**

  Independent events are two or more events whose occurrences do not influence each other. In other words, the occurrence of one event has no effect on the probability of the other event occurring. In other words, in probability, we say two events are independent if knowing one event occurred doesn't change the probability of the other event**.**

  ✓ **Multiplication rule1:** Two events A and B are independent if:

$$\boldsymbol{P(A\ and\ B\ )\ or\ P(A \cap B) = P(A).P(B)}$$

- **Dependent events**

  Dependent events are events where the occurrence of one event affects the probability of the other event occurring. In other words, the outcome of one event influences or changes the likelihood of another event.

  ✓ **Multiplication rule2:** For dependent events A and B:

  P(**A** and **B**) or $P(A \cap B = P(A) \times P(B \setminus A)$,where:

  - P(A∩B): The probability that both AAA and BBB occur.
  - P(B|A): The conditional probability of B occurring given A has occurred.
  - If P(B|A) ≠P(B), A and B are dependent events.

- **Bayes' Theorem**

  Bayes' Theorem is a mathematical formula used to calculate conditional probabilities. It states that the probability of an event A occurring, given that another event B has occurred, is proportional to the probability of B occurring given A, multiplied by the prior probability of A. This is expressed mathematically as: $P(A \setminus B) = \frac{P(B \setminus A) \times P(A)}{P(B)}$

  ,where

  - ✓ P(A|B): The conditional probability of A Given B.
  - ✓ P(B|A): The conditional probability of B given A.
  - ✓ P(A): The probability of A before considering B.
  - ✓ P(B): The total probability of B.

Key Facts 3.3b: Calculation of the conditional probability

- **Solution for question1:**

  Let A be the event: "a 4 is obtained on the first throw,

  B be the event: "an odd number is obtained on the second throw". That is $B = \{1,3,5\}$

  .Both A and B are independent events, Hence $P(A) = \frac{1}{6}$, $P(B) = \frac{3}{6} = \frac{1}{2}$. Therefore,

  $$P(A \cap B) = P(A) \bullet P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

- **Solution for question2**

  Let A be the event: "the number is a 4", then $A = \{4\}$

  B be the event: "the number is greater than 2", then $B = \{3,4,5,6\}$.Here we asked to find the conditional probability $P(A \setminus B) = \frac{P(A \cap B)}{P(B)}$

  $$A \cap B = \{4\} = P(A \cap B) = \frac{1}{6},$$

  $$P(B) = \frac{4}{6} = \frac{2}{3}$$

  Therefore,

$$P(A \setminus B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$$

- **Solution for question3**

  Let A= event that first candy is pink and

  B= event that second candy is pink

  Since the candies are taken out with replacement, this implies that the given events A and B are independent.

  $P(A) = \frac{3}{10}$, $P(B) = \frac{3}{10}$

  P(Pink and Pink)=P(A and B)=$P(A) \times P(B) = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$

- **Solution for question 4**

  A Let be the event that the number is a 5 and

  B be the event that the number is greater than 3. We are looking We are looking for P(A|B), the conditional probability of A Given B. Using the conditional probability formula $P(A \setminus B) = \frac{P(A \cap B)}{P(B)}$

  A∩B: The event that the number is 5 and greater than 3. This is just A, since 5 satisfies being greater than 3. So $P(A \cap B) = \frac{1}{6}$

  P(B): The probability that the number is greater than 3. The numbers greater than 3 are $\{4,5,6\}$. So $P(B) = \frac{3}{6} = \frac{1}{2}$ .Therefore $P(A \setminus B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{6} \times \frac{2}{1} = \frac{1}{3}$,

  The probability that the number obtained is a 5, given that the number is greater than 3, is: $\frac{1}{3}$

- **Solution for question5**

  Let A be the event that the first card is a King. And

  B be the event that the second card is a King.

  Since the first card is not replaced, the events are **dependent**, as the removal of one card changes the composition of the deck.

  ✓ **Probability of the First Event P(A):** There are 4 Kings in a deck of 52 cards : $P(A)=\frac{4}{52}=\frac{1}{13}$

  ✓ **Probability of the Second Event Given the First P(B|A):** If the first card is a King, only 3 Kings remain, and the deck now has 51 cards: $P(B\backslash A)=\frac{3}{51}=\frac{1}{17}$

  The probability that both cards are Kings (P(A∩B) is: $P(A)\times P(B\backslash A)=\frac{1}{13}\times\frac{1}{17}=\frac{1}{221}$

Activity 2: Guided Practice

Using the formulas in key fact 3.3a, answer the following questions:

Task 32:

1. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.
2. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.
3. A bag contains 3 pink candies and 7 green candies. Two candies are taken out from the bags with replacement. Find the probability that both candies are pink.
4. A die is tossed. Find the probability that the number obtained is a 5 given that the number is greater than 3.
5. Suppose you have a standard deck of 52 cards. You draw two cards one after the other without replacing the first card. What is the probability that the first card is a King and the second card is also a King?

Activity 3: Application

A hospital uses a diagnostic test for a rare disease that affects 1 in 1,000 people. The test has the following accuracy:

- True Positive Rate (Sensitivity): If a person has the disease, the test correctly detects it 99%of the time P (Positive | Disease) =0.99
- False Positive Rate**:** If a person does not have the disease, the test incorrectly gives a positive result 5% of the time. P (Positive | No Disease) =0.05 A patient tests positive for the disease.

Task 33:

1. What is the probability that the patient actually has the disease?
2. Two discs are selected one at a time without replacement from a box containing 5 red and 3 blue discs. Find the probability that
   a. the discs are of the same colour
   b. if the discs are of the same colour, both are red.

Formative Assessment

1. A tour package includes 5 major attractions:
   a. A National Park
   b. A Wildlife Safari
   c. A Lake Cruise
   d. A Cultural Village
   e. A Local Market

The tour company wants to create different itineraries by arranging these attractions in different orders.

i. How many different ways can the attractions be arranged if all 5 must be included?

ii. If only 3 attractions are selected for a half-day tour, how many arrangements are possible?

2. A survey among 100 tourists reveals their preferences for three types of activities:
   - **A:** Wildlife Safari (60 tourists)
   - **B:** Lake Cruise (50 tourists)
   - **C:** Cultural Village (40 tourists)

   Overlap data:
   - 20 tourists like **both A and B**
   - 15 tourists like **both A and C**
   - 10 tourists like **both B and C**
   - 5 tourists like **all three activities**
   - **C:** Cultural Village (40 tourists)

   a. How many tourists like at least one of the activities?
   b. What is the probability that a randomly selected tourist likes exactly two activities?

3. The tourism company offers a bonus trip to 3 tourists selected from the 100 surveyed tourists. What is the probability that all 3 selected tourists like at least one activity?

4. At a middle school, 18% of all students performed competition on Taxation and auditing, and 32% of all students performed competition on Taxation. What is the probability that a student who performed competition on Taxation also performed auditing?

5. A football team has the following probabilities of match outcomes based on their recent performance:
   - Win: 50%
   - Draw: 30%
   - Lose: 20%

   a. What is the probability that the team does not lose a match?
   b. What is the probability that the team loses two matches in a row?

6. A manager has three substitute players:
   - Player A: 70% chance of improving the team's performance.
   - Player B: 50% chance of improving the team's performance.

- Player C: 30% chance of improving the team's performance.
- The manager randomly selects one player for substitution.

a. What is the probability that the substitution improves the team's performance?
b. If the manager chooses Player A, what is the probability that the team's performance does not improve?

7. A bag contains 6 blue balls, 5 green balls and 4 red balls. Three balls are selected at random without replacement. Find the probability that
   a. they are all blue
   b. 2 are blue and 1 is green
   c. there is one of each color.
8. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:
   a. Three boys being chosen.
   b. Exactly two boys and a girl being chosen.
   c. Exactly two girls and a boy being chosen.
   d. Three girls being chosen.
9. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colors noted. The first disk is not replaced before the second is selected. Find the probability that the discs will be:
   a. both red
   b. of different colors
   c. the same colors.
10. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be:
    a. all red
    b. all blue
    c. one of each color.
11. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

12. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.
13. At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?
14. Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone User	45	405	450
Total	70	675	755

Calculate the following probabilities using the table:

a. P (person is a car phone user).
b. P (person had no violation in the last year).
c. P(person had no violation in the last year AND was a car phone user).
d. P(person is a car phone user OR person had no violation in the last year).
e. P(person is a car phone user GIVEN person had a violation in the last year).
f. P(person had no violation last year GIVEN person was not a car phone user).

15. Suppose that machines M_1, M_2, M_3 and produce respectively 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M3 ?
16. Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

17. In a certain college, 5% of the men and 1% of the women are taller than 180 cm. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 cm, what is the probability that this student is a woman?
18. A certain federal agency employs three consulting firms (A, B and C) with probabilities 0.4, 0.35, 0.25, respectively. From past experience, it is known that the probabilities of cost overrun for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.
    a. What is the probability that the consulting firm involved is company C?
    b. What is the probability that it is company A?
19. A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P(A), P(B), $P(A \cap B)$ and $P(A \cup B)$.
20. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?
21. A nationwide survey found that 72% of people in the United States like pizza. If 3 people are selected at random, what is the probability that all the three like pizza?
22. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?
23. A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight?
24. A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If A is the event: "a pen is red" and B is the event: "a pen is black". Find P(A), P(AUB).
25. Two discs are selected one at a time without replacement from a box containing 5 red and 3 blue discs. Find the probability that
    a. the discs are of the same color
    b. if the discs are of the same color, both are red.

Points to Remember

- The set S of all possible outcomes of a given experiment is called the sample space.
- Any subset of the sample space is called an event. The event {a} consisting of a single element of S is called a simple event.
- The probability of an event E, denoted by P(E) or Pr(E), is a measure of the possibility of the event occurring as the result of an experiment.
- The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.
- Identify the sample space (SSS), which includes all possible outcomes.
- Types of Probability:
  - ✓ Theoretical Probability: Based on equally likely outcomes, calculated as:

    $$Number\ of\ Favorable\ Outcomes \div Total\ Number\ of\ Outcomes$$

  - ✓ Experimental Probability: Based on observations or experiments, calculated as:

    $$Number\ of\ Times\ Event\ Occurs \div Total\ Number$$

  - ✓ Subjective Probability: Based on intuition, experience, or belief.
  - ✓ Permutations $P_r^n = \frac{n!}{(n-r)!}$
  - ✓ Combinations $C_r^n = \frac{n!}{r!(n-r)!}$
- Range of Probability: The probability of any event E is always between 0 and 1: $0 \le P(E) \le 1$
- Sum of Probabilities: The total probability of all possible outcomes in the sample space is always 1: $\sum$ P(Outcomes in S)=1
- Addition Rule
  - ✓ For mutually exclusive events (E_1 and E_2), where they cannot happen simultaneously: $P(E_1, or, E_2) = P(E_1) + P(E_2)$
  - ✓ For non-mutually exclusive events:

    $P(E_1, or, E_2) = P(E_1) + P(E_2) - P(E_1, and, E_2)$,

- Independent Events: The occurrence of one event does not influence the other.
- Mutually Exclusive Events: Events cannot happen at the same time.
- Exhaustive Events: The events together cover all possible outcomes.
- Multiplication Rule:
  - ✓ For independent events (A and B), where one does not affect the other: P(A and B)=P(A)×P(B)
  - ✓ For dependent events: P(A and B)=P(A)×P(B|A)
- Complement Rule: The probability of an event not happening (E′) is: P(E′)=1−P(E)
- Conditional Probability: The probability of event AAA occurring, given that BBB has occurred, is: $P(A/B)=\frac{P(A, and, B)}{P(B)}, P(B) \succ 0$

Self-Reflection

1. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

Experience **Knowledge, skills, and attitudes**	**I do not have any experience doing this.**	**I know a little about this.**	**I have some experience doing this.**	**I have a lot of experience with this.**	**I am confident in my ability to do this.**
Define correctly probability and its associated terms					
Determine the number of permutations and combinations of "n" items, "r" taken at a time.					
Pay attention to details while calculating probability					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Describe the application of probability to different fields including tourism, social sciences etc…					
Use counting techniques to solve related problems.					
Approach problems systematically and logically.					
Distinguish properly between combination and permutation					
Use properties of combinations					
Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios.					
Define the combinatorial Analysis.					
Analyse problems to determine relevant probabilities.					
Be patient while solving complex probability problems.					
Describe different formulas used to solve problems related to					

Experience Knowledge, skills, and attitudes	I do not have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
calculation of probability					
Apply concepts to solve real-world problems and scenarios.					
Describe uses of Unions, Intersections and Complements in probability contexts.					
Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment.					
Interpret data to make probabilistic inferences					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1. 2. 3.	1. 2. 3.	1. 2. 3.

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APPENDIX

$\Re$ **:** Real number

∑: Summation symbol

n! :Factorial of n

e: Euler’s number (~2.718)

exp(x): Exponential function (e^x)

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