Republic of Rwanda
Ministry of Education

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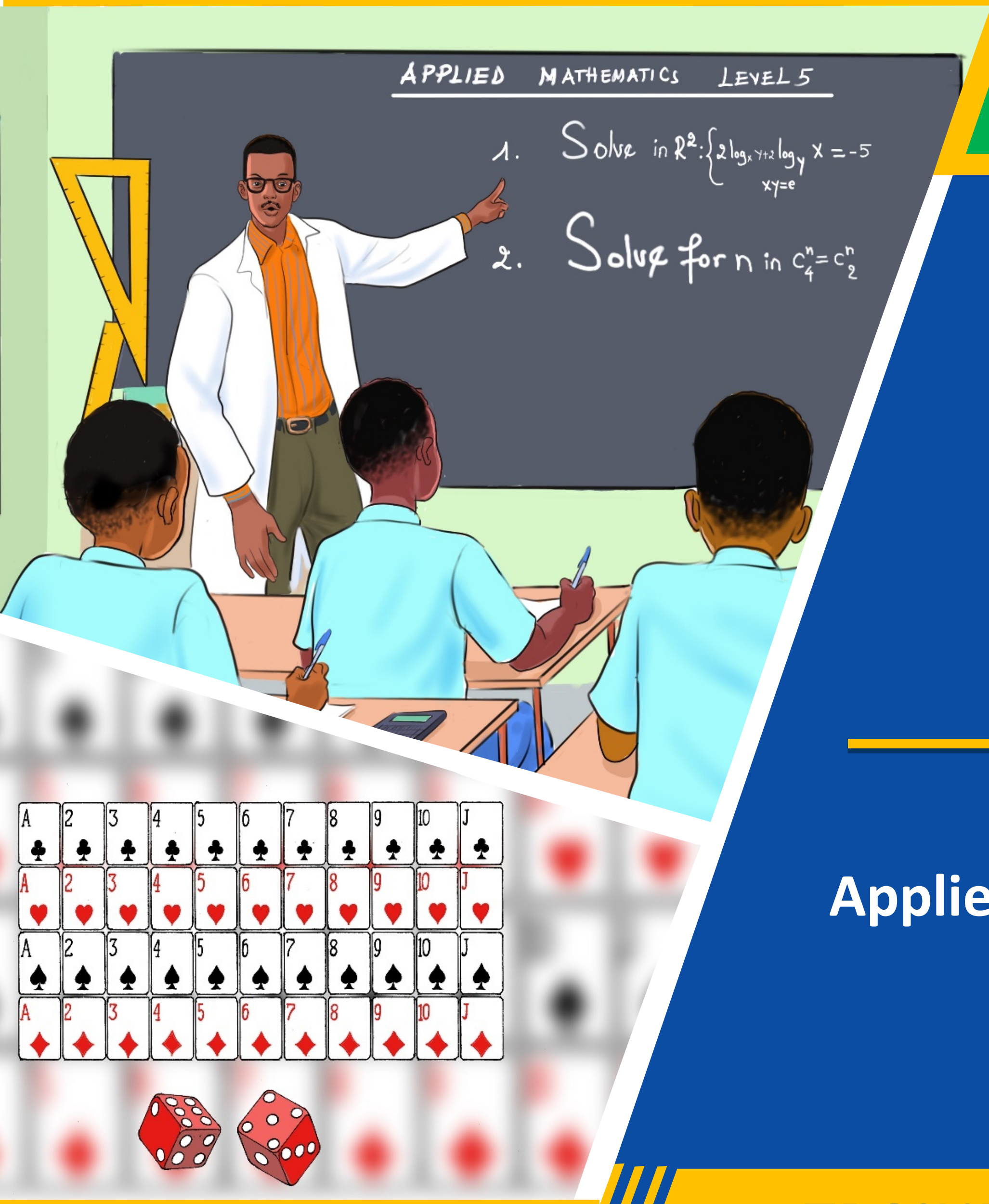
RQF LEVEL 5



TOURISM

GENMA501

Applied Mathematics

TRAINER'S MANUAL

April 2025

Republic of Rwanda
Ministry of Education

RTB | RWANDA TVET BOARD

APPLY LOGARITHMS, EXPONENTIAL EQUATIONS AND PROBABILITY

2025

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LIST OF ABBREVIATIONS AND ACRONYMS

CBET: Competence Base Education and Training

ln: Natural logarithm (logarithm to base e)

log: Logarithm (commonly base 10 if unspecified)

P(A): Probability of event A occurring

P(A|B): Conditional probability (probability of A given B)

RQF: Rwanda Qualification Framework

RTB: Rwanda TVET Board

TVET: Technical and Vocational Education and Training

INTRODUCTION

This trainer manual encompasses all methodologies necessary to guide you to properly deliver the module titled: "**Applied Mathematics"**. Trainees undertaking this module shall be exposed with practical activities that will develop and nurture their competences, the writing process of this training manual embraced competency-based education and training (CBET) philosophy by providing enough practical opportunities reflecting real life situations.

The trainer manual is subdivided into Learning outcomes, each outcome has got various topics, you will start guiding a self-assessment exercise to help trainees rate themselves on their level of skills, knowledge, and attitudes about the unit.

The trainer manual will give you the information about the objectives, learning hours, didactic materials, proposed methodologies and crosscutting issues.

A discovery activity is followed to help trainees discover what they already know about the unit.

This manual will give you tips, methodologies, and techniques about how to facilitate trainees to undertake different activities as proposed in their trainee manuals. The activities in this training manual are prepared such that they give opportunities to trainees to work individually and in groups.

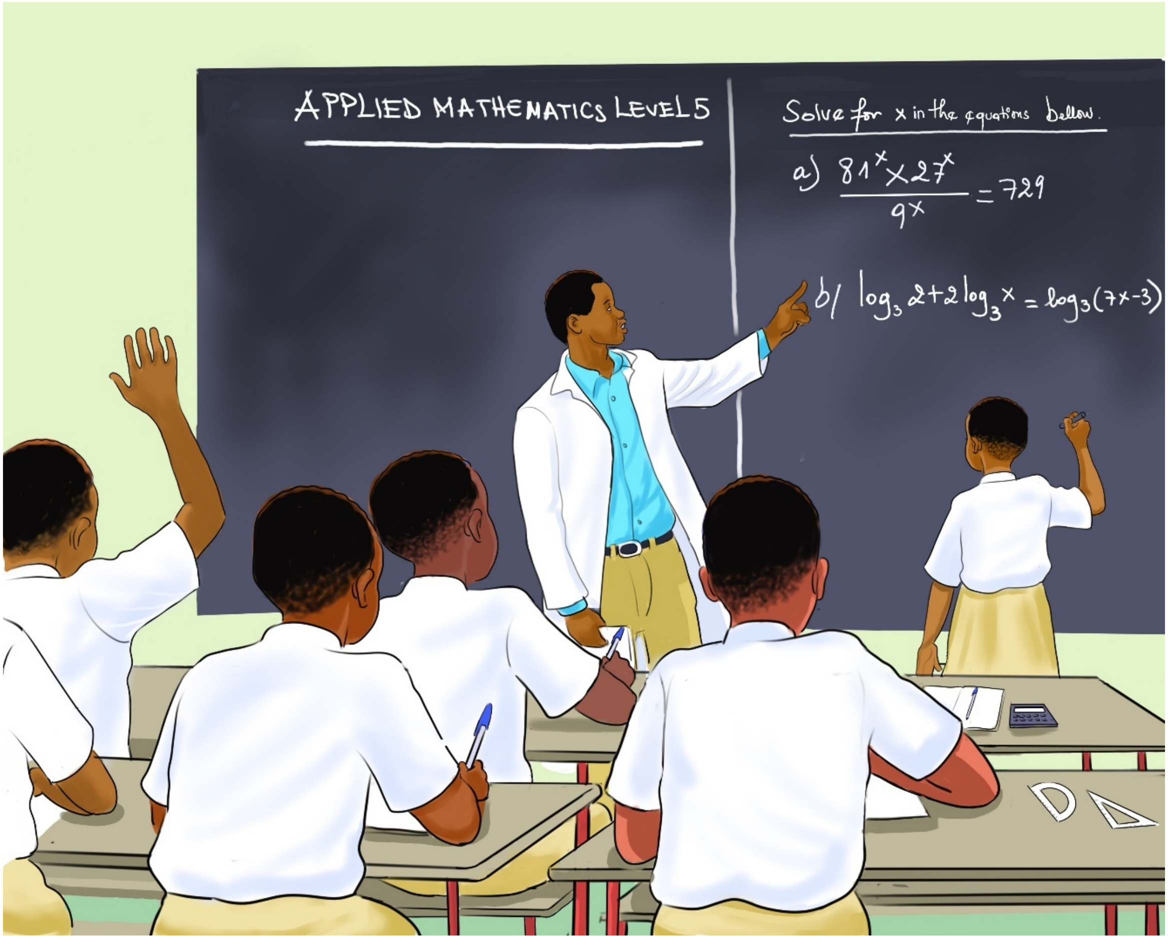
After going through all activities, you shall help trainees to undertake progressive assessments known as formative and finally facilitate them to do their self-reflection to identify strength, weaknesses, and areas for improvements.

Remind them to read point to remember section which provide the overall key points and take ways of the unit.

APPLIED MATHEMATICS.

Learning Outcomes	Learning Hours	Learning Outcomes/topics
1. Solve exponential and logarithmic equations	10	1.1. Description of properties for exponential expressions
		1.2. Description of properties for logarithmic expression
		1.3. Solving exponential equation
		1.4. Solving logarithmic equations
2. Apply exponential and logarithmic expression	20	2.1. Solving problems related to compound interest.
		2.2. Solving problems related to population growth.
		2.3. Solving problems related to population decay.
3. Apply fundamentals of Probability	30	3.1. Application of counting techniques
		3.2. Computation of probabilities
		3.3. Calculation of the conditional probability

LEARNING OUTCOME 1: SOLVE EXPONENTIAL AND LOGARITHMIC EQUATIONS



Learning outcome 1: Self-Assessment

1. Ask trainees to answer the following questions in their Trainee's Manuals.

   a. What is exponential function?

   b. What is a logarithmic function?

   c. What is the difference between exponential and logarithmic expressions?

   d. Differentiate exponential to logarithmic expressions?

   e. Without using calculators, solve each of the following equations for X:

      i. $2^{x+1} = 8$

      ii. $3^{2x} = 81$

      iii. $5^x = \frac{1}{25}$

      iv. $e^{2x} = 20$

      v. $\ln(x - 3) = 2$

      vi. $\log(x + 1) + \log(x - 1) = 1$

      vii. $\log(x^2 - 1) - \log(x + 1) = 0$

2. After the discussion, inform trainees that this unit is intended to provide them with the knowledge, skills and attitudes to apply Logarithms, Exponential Equations and Probability. They will cover the description of properties for exponential expressions, the description of properties for logarithmic expression, solving exponential equation and solving logarithmic equations
3. Ask trainees to fill out the self-assessment at the beginning of the unit in their Trainee's Manuals. Explain that:

   a. The purpose of the self-assessment is to become familiar with the topics in the unit and for them to see what they know or do not know at the beginning.

   b. There are no right or wrong ways to answer this assessment. It is for their own reference and self-reflection on the knowledge, skills and attitudes acquisition during the learning process.

c. They should think about themselves: do they think they have the knowledge, skills or attitudes to do this? How well?

d. They read the statements across the top and put a check in column that best represents their level of knowledge, skills or attitudes.

e. At the end of the unit, they will do a self-reflection, which includes re-taking the self-assessment and identifying their strengths, areas of improvement and actions to be taken.

Key Competencies:

Knowledge	Skills	Attitudes
1. Define properly exponential expressions	1. Isolates Exponential terms correctly	1. Pay attention to details while isolating exponential and logarithmic expression.
2. Define properly logarithmic expression	2. Isolates logarithmic terms correctly	2. Pay attention to details while logarithmic expression.
3. Identify properties of exponential expressions	3. Apply properties of exponential and logarithmic expression	3. Think logically when applying properties of exponential expressions.
4. Identify properties of logarithmic expressions	4. Apply properties of logarithmic expression	4. Think logically when applying properties of logarithmic expressions.
5. Describe different technics to solve exponential equations	5. Solve exponential equations based on their properties.	5. Be patient while solving exponential equations.
6. Describe different technics to solve logarithmic equations.	6. Solve exponential expressions and logarithmic equations based on their properties.	6. Be patient while solving exponential expressions and logarithmic equations.
7. Demonstrates exponential and logarithmic expressions.	7. Analyse correctly exponential expressions and logarithmic forms.	

Steps:

Discovery activity

Task 1

1. Using question and answer methodology, guide trainees to share their prior experience from their understanding regarding solve exponential and logarithmic equations. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are given.

    a. What is the best function of calculating the number of people who received that image?

    b. If each person was sharing the image to 10 people, how can we call this 10 that is keeping repeating?

    c. If each time the image was shared is termed level, Calculate the number of people that received the image at level 4.

    d. If each this picture was shared to 10 people in one minute, how long will it take for the picture to be shared to 900,000 people?

2. Encourage all students to give their views.

3. After sharing session, inform students that this activity was not intended for them to give the right answers but to give them a picture of what they will cover in the unit.

4. Introduce Topic 1.1: Description of properties for exponential expressions

Topic 1.1: Properties for exponential expressions

Objectives:

By the end of the topic, trainees will be able to:

a. Define correctly exponential expressions
b. Recognize and state the properties of exponents.
c. Apply Zero and Negative Exponents.
d. Simplify Exponential Expressions.

Time Required: 2hours.

Learning Methodology:

Trainer guided, small group work, group discussion, Individual work, Trial example.

Materials, Tools and Equipment Needed:

✓ Algebra or mathematics textbooks covering exponential expressions.

✓ Handouts summarizing the properties of exponents with examples

✓ Charts or posters illustrating exponential rules.

✓ Practice sheets with problems on simplifying, expanding exponential expressions.

✓ Whiteboard or chalkboard with markers or chalks

Preparation:

- ❑ Gather some handouts, worksheets on exponent properties.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

- ✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.
- ✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.
- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

- Basic Mathematical Skills like Arithmetic Operations and Order of Operations (BODMAS)
- Introduction to Exponents
- Understanding Variables and Expressions
- Familiarity with Multiplication and Division Rules
- Understanding Number Properties

Activity 1: Problem-Solving

Task 2

1. Use small groups, guide trainees to read and answer the questions provided under task 2 in their trainee's manuals.
   a. What do you understand by term exponential rate?
   b. Write the following number 2x2x2x2 in exponential expression/equation

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Encourage all trainees to give their views by share their answers to the class and write their responses for reference.
4. After the sharing session, refers trainees to Key facts 1.1 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task 3

1. Use an individual work methodology and guide trainees to solve the questions provided under task 3 in their trainee's manuals.

   a. $3^4 x 3^2$=
   b. $\frac{5^7}{5^3}$ =
   c. $(2^3)^4$=
   d. 5^{-2}=
   e. $16^{\frac{1}{2}}$ =
   f. $\frac{3^5 x\ 3^2}{3^6}$ =
   g. $\frac{(2x^3\ y^2)^2 * x^4}{(xy)^3}$ =

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.
4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.
5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.
6. After the sharing session, refer trainees to Key Facts 1.1.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 4

1. **Ask trainees to perform the tasks provided in their trainee's manuals as described below:**

   With the current social media, information sharing grows at an exponential rate. Let's assume that a shared trending message doubles at each 5 minutes. If the message is currently seen by 10,000 people (at zero minute) what will the number of people who will see it in 20 minutes? Using exponent rules, solve the asked question.

Topic 1.2: Properties for logarithmic expressions

Objectives:

By the end of the topic, trainees will be able to:

a. Define clearly logarithmic expressions
b. Recognize that a logarithm is the inverse of exponentiation.
c. Understand how logarithms relate to exponential functions.
d. Apply the properties of logarithms
e. Understand how to convert logarithms between different bases.
f. Understand that the logarithm of 1 is always 0, regardless of the base (except when the base is 1).

Time Required: 2hours.

Learning Methodology:

Group discussion, trainer guided, small group work, Individual work, and Trial examples.

Materials, Tools and Equipment Needed:

- ✓ A textbook covering logarithmic expressions and their properties.
- ✓ Supplementary workbooks with exercises for practicing logarithmic expressions.
- ✓ A reference sheet with the key logarithmic properties (product, quotient, power, change of base formula, etc.).
- ✓ Customizable worksheets with exercises designed to practice each logarithmic property (product, quotient, and power rules) and simplify complex expressions.

- ✓ Whiteboard or chalkboard with markers or chalks

Preparation:

- ❑ Gather some handouts, worksheets on logarithms properties.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

- ✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.
- ✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.
- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will learn better this lesson if they have a good background on

- The rules of exponents since logarithms are inverse operations of exponentiation.
- Understanding the foundational logarithmic properties
- Change of base formula.

Activity 1: Problem-Solving

Task 5

1. Use small groups guides trainees to read and answer the questions provided under task 5 in their trainee's manuals.
   a. What is the logarithmic identity, and how is it used?
   b. What is the difference between logarithmic expressions and exponential expressions?
   c. What is the difference between the two kinds of logarithm?
   d. State and explain the properties used for solving logarithmic equations
   e. How does the change of base formula for logarithms work?
   f. What is the logarithm of 1 for any base?
   g. How is the logarithm of a base raised to an exponent simplified?
2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Encourage all trainees to give their views by share their answers to the class and write their responses for reference.
4. After the sharing session, refers trainees to Key facts 1.2 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task 6

1. Use an individual work methology and guides trainees to use the logarithmic rules to simplify and evaluate the logarithmic expressions under task 6 in their trainee's manuals.

   a. $\log_3(20\times 5)$

   b. $\log\left(\frac{8}{4}\right)^3$

   c. $\log(50\times 25)^5$

   d. $\ln\left(\frac{10}{20}\right)$

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.

3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.

4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.

5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.

6. After the sharing session, refer trainees to Key Facts 1.2.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 7

1. **Request trainees to perform tasks provided in their trainee's manuals as described below:**

   You are tour guide, and you are with a geologist studying volcanic earthquakes in Virunga National park caused by the movement of magma beneath the surface of the Earth. He explained to you Richter scale (measurement of the magnitude of an earthquake) is used to compare the energy released by different earthquakes. The formula for the Richter scale is: $M = \log_{10}\left(\frac{A}{A_0}\right)$ Where: M = magnitude of the earthquake A= amplitude of the seismic waves and A0= reference amplitude (a very small constant value).

   The geologist found out that the two recent earthquakes A and B had the reference amplitude of 10,000 and 1,000,000 respectively. Help him to calculate the magnitude of Earthquake A and Earthquake B. Which earthquake has released much energy?

Topic 1.3: Exponential equations

	Objectives: By the end of the topic, trainees will be able to: a. Define exponential equations b. Recognize the structure of exponential equations c. Determine domain and range of exponential functions d. Solve equations of the form $a^x = b$by applying logarithms or rewriting with the same base.
	Time Required: 2hours
	Learning Methodology: Trainer guided, small group work, group discussion, Individual work, Trial examples.

Materials, Tools and Equipment Needed:

✓ Student's book and other reference books to facilitate research.

✓ Scientific calculators capable of computing logarithms, exponents, and complex calculations.

✓ Whiteboard or Blackboard for explaining concepts, writing examples, and solving problems interactively.

✓ Markers/Chalk: For use with the whiteboard or blackboard.

Preparation:

- ❑ Gather some handouts, worksheets on exponent functions.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.

✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.

✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.

✓ **Critical thinking:** Give activities which enhance critical thinking

✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will easily learn this topics, if they have a good background on:

- Simplifying algebraic expressions.
- Familiarity with exponential notation
- Understanding logarithms as the inverse of exponents.
- Ability to interpret and manipulate mathematical expressions
- Understanding exponential functions ($f(x) = a^x$

Activity 1: Problem-Solving

Task 8

1. Use small groups, guide trainees to read and answer the questions provided under task 2 in their trainee's manuals.

   a. What is an exponential function? How is it written?

   b. Consider functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$, complete the following table

x	-2	-1	0	1	2
$f(x)$					
$g(x)$					

   c. Sketch the graph of function $f(x)$ and $g(x)$ from question b.

   d. What is domain and range of validity for exponential functions?

   e. Discuss whether $\forall x \in \Re, f(x) \in \Re$ and deduce the domain of $f(x)$

   f. Discuss whether $f(x)$ can be negative or not and deduce the range of $f(x)$

   g. What are the methods for solving exponential functions? What there their steps?

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Encourage all trainees to give their views by sharing their views answers to the class and write their responses for reference.
4. After the sharing session, refers trainees to Key facts 1.3 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task 9

1. Use an individual work methodology and guides trainees solve the questions provider under task 9 using the appropriate methods.

Using like base	Using logarithm
$3^{x+2} = 27^x$	$2e^{x+1} - 4 = 12$
$4^{x-1} = 16$	$e^x = 5$
$e^{-x^2} = (e^x)^2 * \frac{1}{e^3}$	$10 + e^{0.1} = 14$
$2^{2y} + 3(2^y) = 4$	$5 + 3^{t-4} = 7$
	$3(2^{4x}) - 7(2^{2x}) + 4 = 0$

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.
4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.

5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.
6. After the sharing session, refer trainees to Key Facts 1.3.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Using an appropriate methodology such as individual work, pairs or small groups trainees read the following given exponential functions:

Task 10

In a population growth model, the number of individuals in a species is given by the equation $4^{2x} - 6.2^{4x} + 6.2^{2x} - 1 = 0$,where x represents the time in years. Solve this equation to determine the critical points in time when the population reaches certain thresholds.

Topic 1.4: Logarithmic equations

Objectives:

By the end of the topic, trainees will be able to:

a. Define logarithmic equations

b. Recognize the structure of logarithmic equations

c. Determine domain and range of logarithmic functions

d. Distinguish between equations that can be simplified using properties of logarithms and those requiring more advanced techniques.

e. Solve simple logarithmic equations using basic algebraic methods

f. Solve complex equations by isolating the logarithmic term and rewriting it in exponential form.

g. Evaluate solutions to logarithmic equations to ensure they do not result in undefined expressions (e.g., logarithms of negative numbers or zero).

	Time Required: 4hours
	Learning Methodology: Trainer guided, small group work, group discussion, Individual work, Trial examples.
	Materials, Tools and Equipment Needed: ✓ Student’s book and other reference books to facilitate research. ✓ Scientific calculators capable of computing logarithms, exponents, and complex calculations. ✓ Whiteboard or Blackboard for explaining concepts, writing examples, and solving problems interactively. ✓ Markers/Chalk: For use with the whiteboard or blackboard.
	Preparation: ❑ Gather some handouts, worksheets on logarithmic functions. ❑ Assign practice problems for trainees to solve individually or in groups. ❑ Conduct a quick quiz or interactive activity to check understanding.
	Cross Cutting Issues: ✓ **Gender balance**: Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership. ✓ **Inclusive education:** Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak. ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict. ✓ **Critical thinking:** Give activities which enhance critical thinking ✓ Communication: Encourage every group member to participate in discussions.

Prerequisites:

Trainees will easily learn this topics, if they have a good background on:

- Simplifying algebraic expressions.
- Familiarity with solving equations using basic algebraic methods.
- properties and operations common logarithms learnt in previous lessons.
- domain and range of logarithmic functions and decimal logarithmic function
- Awareness that logarithmic equations may have restrictions (e.g., no negative arguments).
- Awareness that logarithms are the inverse of exponential functions.
- Understand that logarithmic functions are defined only for positive arguments (x>0).

Activity 1: Problem-Solving

Task 11

By using small groups methodology and guide trainees to read and answer the questions provided under task 11 in their trainee's manuals.

1. What is a logarithmic equation?
2. How is a logarithmic equations solved?
3. For which value(s) of x, each function is defined, and use the properties for logarithm to determine the value of x in the given expressions:
   a. $\ln x = 10$
   b. $\ln x = 3$
   c. $\log x = 2$
   d. $\log(100x) = 2 + \log 4$
   e. $\log_2 x = -3$

Activity 2: Guided Practice

Task 12

1. By means of an individual work methodology and guide trainees to solve the questions provided under task 12 in their trainee's manuals.

   a. $\log(x+2)=2$

   b. $\log x+\log(x^2+2x-1)-\log 2=0$

   c. $\log(35-x^3)=3\log 5-x)$

   d. $\log(3x-2)+\log(3x-1)=\log(4x-3)^2$

   e. $\log(1-x)=-1$

Activity 3: Application

Demands trainees to perform the tasks provided in their trainee's manuals as described below:

Task 13

1. In a biological study, the growth of bacteria population is modeled by the following logarithmic functions, where t represents the time in days. Determine for which value(s) of t each function is defined to understand the time periods during which the population growth can be accurately modeled.

   a. $f(t)=\ln(t+4)$

   b. $f(t)=\ln t$

   c. $f(t)=\ln(t^2-5t+16)$

2. Solve the following system: In an environmental study, the relationship between two species' populations is modeled by a system of logarithmic equations, where x and y

represent the populations of the two species. Solve the system to find the equilibrium populations:

$$\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$$

Formative Assessment

1. Write each of the following in logarithmic form

   a. $4^3 = 64$

   b. $2^{-3} = \frac{1}{8}$

   c. $\left(\frac{1}{2}\right)^x = y$

   d. $5^{-p} = q$

2. Use the properties of logarithms to rewrite each expression as a single logarithm:

   a. $2\log_b x + \frac{1}{2}\log_b(x+4)$

   b. $4\log_b(x+2) - 3\log_b(x-5)$

   c. $\log_b\left(\frac{x\sqrt{y}}{z^5}\right)$

3. Find the numerical value

   a. $\log_2 32$

   b. $\log_4 8$

   c. $\log_6 7$

   d. $\log_5 \sqrt{125}$

   e. $\log_5 0.008$

   f. $\log_9 10$

4. Find the exact value of x,showing your working steps:

a. $\log_2 8 = x$

b. $\log_x 125 = 3$

c. $\log_x 64 = 0.5$

d. $\log_4 64 = x$

e. $\log_9 x = 3$

5**.** Solve the following exponential equations

a. $2^{3x} = 3^{2x-1}$

b. $2e^{2x} - e^x - 6 = 0$

c. $3^{x+2} = 27^x$

d. $e^x - 12 = \dfrac{-5}{e^{-x}}$

6. Find the domain of definition of the functions:

a. $f(x) = \ln(x+1)$

b. $f(x) = \dfrac{1+\ln x}{-1+\ln x}$

7**.**Solve the following in real number:

$$\log x + \log(x^2 + 2x - 1) - \log 2 = 0$$

8. Evaluate the value of x and y for the following system:

a. $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$

b. $\begin{cases} \ln(xy) = 7 \\ \ln y\left(\dfrac{x}{y}\right) = 1 \end{cases}$

Answers:

1. Write each of the following in logarithmic form

   a. $4^3 = 64$ **:Answer :** $\log_4 64 = 3$

   b. $2^{-3} = \frac{1}{8}$**:** **Answer :** $\log_2 \frac{1}{8} = -3$

   c. $\left(\frac{1}{2}\right)^x = y$ **Answer:** $\log_{\frac{1}{2}} y = x$

   d. $5^{-p} = q$**:** **Answer:** $\log_5 q = -p$

2. Use the properties of logarithms to rewrite each expression as a single logarithm

   a. $2\log_b x + \frac{1}{2}\log_b (x+4)$

   Answer:

   $$\Rightarrow \log_b x^2 + \log_b x^2 (x+4)^{\frac{1}{2}} \Leftrightarrow \log_b x^2 \left(\sqrt{x+4}\right)$$

   b. $4\log_b (x+2) - 3\log_b (x-5)$

   Answer:

   $$\log_b (x+2)^4 - \log_b (x-5)^3 \Leftrightarrow \log_b \frac{(x+2)^2}{(x-5)^3}$$

   c. $\log_b \left(\frac{x\sqrt{y}}{z^5}\right)$

   Answer:

   $$\log_b \frac{\frac{y}{x}}{z^5}$$

3. Find the numerical value

   a. $\log_2 32$

Answer:

$\Rightarrow \log_2 2^5 \Leftrightarrow 5\log_2 2 = 5$

b. $\log_4 8$

Answer:

Use the change of base formula and properties of logarithms

$\Rightarrow \log_4 8 = \frac{\log 8}{\log 4}$,Express 8 and 4 as power of 2 $8 = 2^3, 4 = 2^2$

Then using the power rule of logarithm ($\log b^a = a\log b$ so

$\log_4 8 = \frac{\log(2^3)}{\log(2^2)} \Leftrightarrow \frac{3\log 2}{2\log 2} = \frac{3}{2}$

c. $\log_6 7$

d. $\log_5 \sqrt{125}$

Answer:

$\log_5 \sqrt{5^3} = \log_5 5^{\frac{3}{2}}$, Then use the power rule of logarithm to get

$\frac{3}{2}\log_5 5 \Leftrightarrow \frac{3}{2}$

e. $\log_5 0.008$

f. $\log_9 10$

For (c), (e) and (f) use the same strategy as (b)

4. Find the exact value of x,showing your working steps:

   a. $\log_2 8 = x$, **Answer:** $8 = 2^x \Leftrightarrow 2^x = 2^3 \Leftrightarrow x = 3$

   b. $\log_x 125 = 3$, **Answer:** $125 = x^3 \Rightarrow x = \sqrt[3]{125} \Leftrightarrow x = 5$

   c. $\log_x 64 = 0.5$, **Answer:** $64 = x^{0.5} \Rightarrow x = 64^2 \Leftrightarrow x = 4096$

   d. $\log_4 64 = x$, **Answer:** $64 = 4^x \Rightarrow 4^3 = 4^x \Leftrightarrow x = 3$

   e. $\log_9 x = 3$, **Answer:** $x = 9^3 \Leftrightarrow x = 729$

5. Solve the following exponential equations

a. $2^{3x} = 3^{2x-1}$

Answer:

Domain of definition $x \in \Re$

Use natural logarithm (ln) or base 10 logarithm (log)

$\ln 2^{3x} = \ln 3^{2x-1} \Leftrightarrow 3x\ln 2 = (2x-1)\ln 3$,use approximations

$\ln 2 \approx 0.6931, \ln 3 \approx 1.0986$

$\Rightarrow 3x(0.6931) = 1.0986(2x-1)$after solving this equation $x \approx 9.318$

b. $3^{x+2} = 27^{x}$

Answer:

Domain of validity $x \in \Re$**,**

$3^{x+2} = 27^{x} \Rightarrow 3^{x+2} = (3^{3})^{x}$

$\Rightarrow 3^{x+2} = 3^{3x} \Leftrightarrow x+2 = 3x \Rightarrow 3x = 2 \Leftrightarrow x = 1$

Therefore, solution is $S = \{1\}$

c. $2e^{2x} - e^{x} - 6 = 0$

Answer:

Let $e^{x} = t, t \succ 0$**,**then $2t^{2} - t - 6 = 0$**,**Solve this new equation by using determinant

$\Rightarrow \Delta = b^{2} - 4ac \Leftrightarrow (-1)^{2} - 4 \bullet 2(-6) = 49 \Leftrightarrow \sqrt{49} = 7$

$$t_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-(-1)+7}{2 \bullet 2} = \frac{8}{4} = 2$$

$$t_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-(-1)-7}{4} = \frac{-3}{2}$$

If $t = 2, e^{x} = 2 \Leftrightarrow x = \ln 2 \Leftrightarrow S = \{\ln 2\}$

If $t = \frac{-3}{2}$,no solution

d. $e^{x}-12=\frac{-5}{e^{-x}}$

Answer:

$$\Rightarrow e^{x}\times e^{-x}-12e^{-x}=-5$$
$$e^{0}-12e^{-x}=-5\Leftrightarrow 1-12e^{-x}=-5$$
$$-12e^{=x}=-6\Leftrightarrow e^{-x}=\frac{-6}{-12},e^{-x}=\frac{1}{2}$$
$$\Rightarrow -x\ln e=\ln\frac{1}{2}\Leftrightarrow -x=-\ln 2\Leftrightarrow x=\ln 2, S=\{\ln 2\}$$

6. Find the domain of definition of the functions:

a. $f(x)=\ln(x+1)$

Answer:

The function $f(x)=\ln(x+1)$ is defined if and only if $x+1\succ 0\Leftrightarrow x\succ -1$

and gives that $x\in\left]-1,+\infty\right[$

b. $f(x)=\frac{1+\ln x}{-1+\ln x}$

Answer:

The function $f(x)=\frac{1+\ln x}{-1+\ln x}$ is defined if and only if $x\succ 0$ and $-1+\ln x\neq 0\Leftrightarrow \ln x\neq 1, x\neq e$ gives that $\Re\setminus\{e\}$

7. Solve the following in real number:

$$\log x+\log(x^{2}+2x-1)-\log 2=0$$

Answer:

Domain of validity **:** $x\succ 0$ and $x^{2}+2x-1\succ 0$

$$\log x+\log(x^{2}+2x-1)-\log 2=0$$

$$\Rightarrow \log \frac{x(x^2+2x-1)}{2} = 0 \Leftrightarrow \log \frac{x(x^2+2x-1)}{2} = \log 1$$

$$\Rightarrow \frac{x(x^2+2x-1)}{2} = 1 \Leftrightarrow x^3 + 2x^2 - x = 0$$

Let solve this obtained equation by factorization

	1	**2**	**-1**	**-2**
1		**1**	**3**	**2**
	1	**3**	**2**	**0**

$$(x-1)(x^2+3x+2 \Leftrightarrow \Delta = b^2 - 4ac) \Rightarrow \Delta = (3)^2 - 4 \bullet 2 = 9 - 8 = 1 \Leftrightarrow \sqrt{1} = 1$$

$x_1 = 1, x_2 = \frac{-3+1}{2} = -1$, $x_3 = \frac{-3-1}{2} = -2$ **,** both x2 and x3 are rejected because they are negative then,

$S = \{1\}$

8. Evaluate the value of x and y for the following system:

a. $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$

Answer:

let multiply -5 in equation (1) and (3) in equation (2) to obtain:

$\begin{cases} -10\ln x - 15\ln y = 10 \\ 9\ln x + 15\ln y = -12 \end{cases} \Rightarrow -\ln x = -2, \ln x = 2 \Leftrightarrow x = e^2$,replace the value of x in equation(1) **to obtain:**

$$2\ln e^2 + 3\ln y = -2 \Rightarrow 4\ln e + 3\ln y = -2$$
$$3\ln y + 4 = -2 \Rightarrow 3\ln y = -6$$
$$\ln y = -2 \Leftrightarrow y = e^{-2}$$

The solution is $S = \{e^2, e^{-2}\}$

b. $\begin{cases} \ln(xy) = 7 \\ \ln y\left(\frac{x}{y}\right) = 1 \end{cases}$

Answer:

$$\begin{cases} \ln(xy) = 7 \\ \ln\left(\dfrac{x}{y}\right) = 1 \end{cases} \Leftrightarrow \begin{cases} \ln x + \ln y \\ \ln x - \ln y \end{cases}$$

solve the system of equations started eliminating variable y

$$\begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases} \Leftrightarrow 2\ln x = 8$$

$$\Rightarrow \ln x = 4 \Rightarrow x = e^4$$

replacing the value of x in equation (1), $\ln e^4 + \ln y = 7$ then by solving this equation the

$$\begin{aligned} &\ln e^4 + \ln y = 7 \\ &\Rightarrow 4\ln e + \ln y = 7 \\ &\Rightarrow 4 + \ln y = 7 \\ &\Rightarrow \ln y = 7 - 4 \Leftrightarrow \ln y = 3 \end{aligned}$$

solution for y is given by: $y = e^3$

Therefore, the solution of the system is. $S = \{(e^4, e^3)\}$

Self-Reflection

1. Ask learners to re-take the self-assessment at the beginning of the unit. They should then fill in the table in their Trainee's Manual to Identify their areas of strength, areas for improvement and actions to take to improve.
2. Discuss trainees' results with them. Identify any areas that are giving many trainees difficulties and plan to give additional support as needed (ex. use class time before you begin the next learning outcome to go through commonly identified difficult concepts).

Points to Remember

These are the key learning points from all activities in this learning outcome.

- Definition of exponential expression
- Definition of logarithmic functions
- Properties of exponential functions.
- Properties of logarithmic functions
- Change of base law.
- Change from logarithm into exponential and vice
- Don't use calculators while solving exponential and logarithmic expressions
- Before solving any exponential and logarithmic expressions, think of the appropriate rule/property to use for simplification
- Exponential equations.
- Logarithmic equations.
- Domain and range of exponential functions
- Domain and range of logarithmic functions

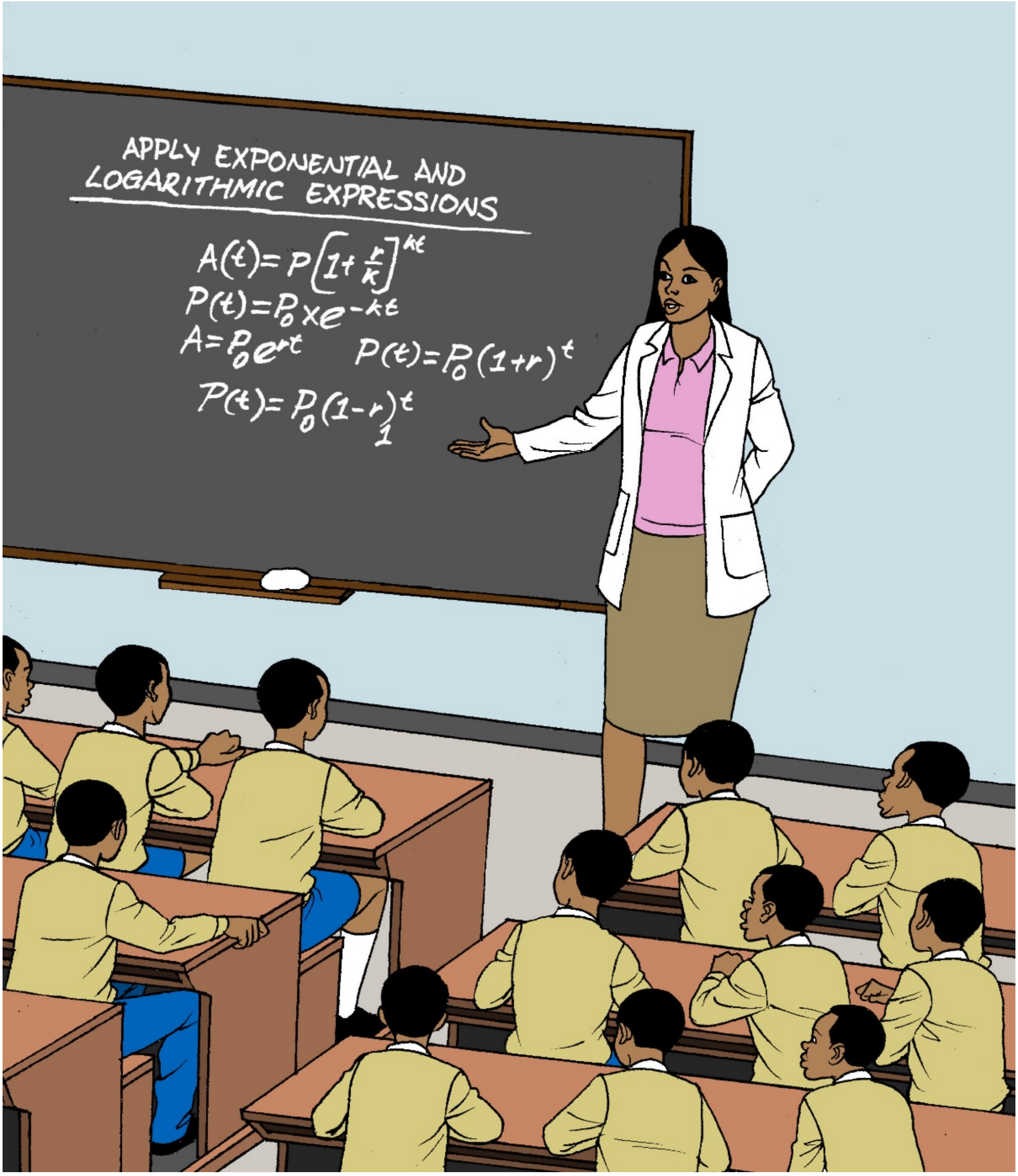
Further Information for the Trainer

Look information on:

Key steps for solving exponential equations.

Key steps for solving logarithmic equations

LEARNING OUTCOME2: APPLY EXPONENTIAL AND LOGARITHMIC EXPRESSIONS.



Learning outcome 2: Self-Assessment

1. Ask trainees to answer the following questions in their Trainee's Manuals.
   a. What is compound interest?
   b. Given the formula for calculating the compound interest below, what does each letter stands for?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   c. What is population growth and its importance?
   d. What is the formula for calculation of population growth?
   e. What is population decay?
   f. What is the formula for calculation of population decay?
2. After the discussion, inform trainees that this unit is intended to provide them with the knowledge, skills and attitudes to apply Logarithms, Exponential Equations and Probability. They will cover the description of properties for exponential expressions, the description of properties for logarithmic expression, solving exponential equation and solving logarithmic equations
3. Ask trainees to fill out the self-assessment at the beginning of the unit in their Trainee's Manuals. Explain that:
   a. The purpose of the self-assessment is to become familiar with the topics in the unit and for them to see what they know or do not know at the beginning.
   b. There are no right or wrong ways to answer this assessment. It is for their own reference and self-reflection on the knowledge, skills and attitudes acquisition during the learning process.
   c. They should think about themselves: do they think they have the knowledge, skills or attitudes to do this? How well?
   d. They read the statements across the top and put a check in column that best represents their level of knowledge, skills or attitudes.
   e. At the end of the unit, they will do a self-reflection, which includes re-taking the self-assessment and identifying their strengths, areas of improvement and actions to be taken.

Key Competencies:

Knowledge	Skills	Attitudes
1. Define correctly simple and compound interest.	1. Calculate simple and compound interest accurately using formulas.	1. Pay attention to details while calculating simple and compound interest
2. Recognize the meaning of each variable needed in calculation of compound interest (e.g. principal value, rate, time, and compounding frequency or periods)	2. Rearrange logarithmic and exponential equations to solve for unknown variables (e.g., time t, rate r)	2. Think logically when applying formula of compound interest.
3. Explain exponential growth using its formula.	3. Apply exponential growth formulas to real-world problems (e.g., predicting future population size).	3. Be patient while solving problems related to population growth.
4. Recognize the meaning of each variable needed in calculation of population growth (growth rate, initial population etc.).	4. Predict future population sizes using exponential models.	4. Develop a problem solving mindset when working with population growth formula.
5. Explain exponential decay using its formula.	5. Solve real-world problems involving population reduction using exponential decay formula.	5. Think logically when applying the population decay formula to ensure accurate calculations and meaningful interpretations.

Steps:

Discovery activity

Task 14

1. Using question and answer methodology, ask trainees to share their prior experience under task 14 in their Trainee's Manuals. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are give.
   a. What is the appropriate term for the initial amount of money the park has in the bank?
   b. What is the formula used to calculate the final amount of money?
   c. How much money will the park be having after 10 years?
   d. What will be the number of carnivores in 10 years?
   e. What will be the number of herbivores in 10 years?

2. Encourage all students to give their views.
3. After the presentations/sharing session, inform trainees that this activity was not intended for them to give the right answers but to give them a picture of what they will cover in the unit.
4. Introduce Topic 2.1: Calculation of Compound interest

Topic 2.1: Compound interest

Objectives:

By the end of the topic, trainees will be able to:

e. Define correctly compound interest
f. Differentiate compound to simple interest.
g. Differentiate initial to future value.
h. Apply the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$

i. Use logarithmic and exponential functions to solve compound interest problems accurately.

Time Required: 7hours.

Learning Methodology:

Trainer guided, small group work, group discussion, Individual work, Trial example.

Materials, Tools and Equipment Needed:

✓ Exercise book, pen and textbooks or internet if available.

✓ Formulas sheets, or guides detailing compound interest formulas and logarithmic rules.

✓ A scientific calculator capable of computing exponents and logarithms.

✓ Whiteboard or chalkboard with markers or chalks

Preparation:

- ❑ Gather some handouts, worksheets application of logarithm and exponential in calculation of compound interest.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.

✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.

- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

- Understanding of the Compound Interest Formula.
- Understanding powers and how to work with exponential expressions is crucial because compound interest grows exponentially.
- Rearrange equations algebraically to isolate variables like t or r.

Activity 1: Problem-Solving

Task 15

1. Utilizes small group methodology and guides trainees to read and answer the questions provided under task 15 in their trainee's manuals.
   a. What does the compound interest in the above scenario mean?
   b. Explain how Mr. MUSINGUZI can discover that power which raised his money.
   c. What is the total interest earned over the 3 years?
   d. How much interest does Mr. MUSINGUZI earns each year?
   e. What was the growth factor (1 + r) over the 3 years?
   f. What would the final amount be if Mr. MUSINGUZI left his money for 5 years instead of 3?
2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Motivate all trainees to give their views by sharing their answers to the class and write their responses for reference.

4. After the sharing session, refers trainees to Key facts 2.1a Calculation of Compound interest and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task 16

1. Uses an individual work methodology and guide trainees to answer the questions provided under task 16 in their trainee's manuals.
    a. How much Interest do you get if you put 10,000 FRW in a saving account that pays simple interest at 9% per annum for two (2) years?
        i. How many interest if you invest in the account in 2 years?
        ii. How many interest if you invest only the money in the account for half a year?
    b. Suppose 500,000Frw principal earns 150,000Frw interest after 6 years.
        i. If compound interest is used, what was the interest rate?
        ii. What if the simple interest is used?
    c. A tourist saves $1,000 in a park's eco-tourism savings plan that offers an annual interest rate of 5% compounded annually. What will the amount grow to in 3 years?
    d. You deposit 5,000 in a bank account earning 6% interest per year, compounded monthly. How much will you have after 5 years?
2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.
4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.

5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.
6. After the sharing session, refer trainees to Key Facts 2.1.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 17

1. Urges trainees to perform the tasks provided in their trainee's manuals as described below:

   A tourism company needs 1,500,000,000 frw to expand its business by increasing number of cars in its fleet as well as building a guest house. They have 300,000,000 and plan to invest it at an annual interest rate of 7.5% compounded annually. How long will it take to reach their goal?

Topic 2.2: Population growth

Objectives:

By the end of the topic, trainees will be able to:

a. Explain what is population growth.

b. Differentiate initial to future population

c. Apply the formula $P(t) = P_0 e^{rt}$ or $P(t) = P_0 (1+r)^n$

d. Use logarithmic and exponential functions to solve population growth problems accurately.

	Time Required: 7hours.
	Learning Methodology: Group discussion, trainer guided, small group work, Individual work, and Trial examples.
	Materials, Tools and Equipment Needed: ✓ Exercise book, pen and textbooks or internet if available. ✓ Worksheets with guided exercises and practice problems. ✓ **Scientific Calculators** for calculating logarithms, exponentials, and growth rates. ✓ Whiteboard or chalkboard with markers or chalks
	Preparation: ❑ Gather some handouts, worksheets on population growth ❑ Prepare trial examples on applying logarithm and exponential in calculation of population growth. ❑ Assign practice problems for trainees to solve individually or in groups. ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

- ✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.
- ✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.
- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will learn better this lesson if they have a good background on

- Understanding of the population growth formula.
- Understanding powers and how to work with exponential expressions is crucial because population grows exponentially.
- Rearrange equations algebraically to isolate variables like t or r.
- Understanding the foundational logarithmic properties

Activity 1: Problem-Solving

Task18

1. Use small groups, guide trainees to read and answer the questions provided under task 18 in their trainee's manuals.

a. What is the initial population of zebras in the reserve?

b. What does the growth rate of 5% per year mean?

c. How many years will we track the population to observe its growth?

d. What will the population be after 1 year? (Use the formula $P = P_O x e^{rt}$, where e≈2.718).

e. What happens to the population as time increases?

f. If the growth rate decreases to 2% per year, will the population grow faster or slower?

g. Why is it important to know the rate of growth and time of growth?

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.

3. Encourage all trainees to give their views by share their answers to the class and write their responses for reference.

4. After the sharing session, refers trainees to Key facts 2.2 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task19

1. Utilizes an individual work methodology and guide trainees to solve the questions provided under task 19 in their trainee's manuals.

a. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.

i. Assuming an exponential growth model, estimate the population in 1980.

ii. What is the doubling time for the town's population?

b. The population, P, of an island t years after January 1st 2016 is given by this formula

$P = 4200 \times (1.04)^t$

i. What was the population of the island on January 1st 2016?

ii. What is the constant rate?

iii. Work out the population of the island on January 1st 2021.

c. If the tourist population grew from 10,000 to 15,000 in 5 years,
   i. Calculate the rate of this growth.
   ii. If the current population is 8,000, growing at 4% annually, how long to reach 12,000?
   iii. If the tourist population is 5,000, growing at 3% annually, calculate the number of tourist population after 5 years?

d. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.
   i. Assuming an exponential growth model, estimate the population in 1980.
   ii. What is the doubling time for the town's population?

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.

3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.

4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.

5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.

6. After the sharing session, refer trainees to Key Facts 2.2.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 20

Ask trainees to perform the tasks provided in their trainee's manuals as described below:

1. The population of a city is growing at a rate of 5% per year. If the current population is 10,000, what will the population be in 10 years?

2. A tourist destination has a current annual visitor count of 5,000, and the number of tourists is increasing at a rate of 7% per year. How long will it take for the number of tourists to double?
3. The number of tourists visiting a park increased from 2,000 to 3,000 in 5 years. What is the annual growth rate?
4. According to United Nation data, the world population at the beginning of 1975 was approximately 4 billion and growing at rate of about 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2030.
5. The population of a country grows according to the law $P = Ae^{0.06t}$ where P million is the population at time t years and A is a constant. Given that at time t=0, the population is 27.3 million, calculate the population when

   a) t=10 b) t=15 c) t=25

6. The population of a country grows according to the law $P = 12e^{kt}$ where p million is the population at time t years and k is the constant .Given that when t=7, P=15. Find the time for which the population will be a) 20 million b) 30 million c) 35 million
7. A city in Texas had a population of 75,000 in 1970 and a population of 200,000 in 1995. The growth between the years 1970 and 1995 followed an exponential pattern of the form $f(t) = A \times e^{\alpha t}$

   a. Find the values of A and α

   b. Using the given model, estimate the population for the year 2010

Topic 2.3: Population decay

Objectives:

By the end of the topic, trainees will be able to:

a. Define what is population decay

b. Understand the mathematical model of exponential decay and its key components (e.g., initial population, decay rate, time).

c. Use the exponential decay formula $P(t) = P_0 \bullet e^{-kt}$ or $P(t) = P_0(1-r)^t$ to calculate the remaining population at a given time.

d. Calculate the half-life of a population or substance and use it in decay calculations.

e. Determine the decay constant k from given data

Time Required: 6hours

Learning Methodology:

Trainer guided, small group work, group discussion, Individual work, Trial examples.

Materials, Tools and Equipment Needed:

✓ Student's book and other reference books to facilitate research.

✓ Pre-prepared exercises on exponential decay and logarithmic calculations.

✓ Whiteboard or Blackboard for explaining concepts, writing examples, and solving problems interactively.

✓ Markers/Chalk: For use with the whiteboard or blackboard.

Preparation:

- ❑ Gather some handouts, worksheets on population decay.

- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.

✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.

✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.

✓ **Critical thinking:** Give activities which enhance critical thinking

✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will easily learn this topics, if they have a good background on:

- Simplifying algebraic expressions.
- Ability to solve linear and nonlinear equations.
- Proficiency in isolating variables in equations.
- Understanding the properties of exponents.
- Familiarity with negative and fractional exponents.
- Understanding that population decay follows an exponential trend where the rate of decline is proportional to the current population size.
- Familiarity with the concept of half-life, the time required for a quantity to reduce to half its initial value.

- Ability to use a scientific calculator for exponentiation (e^x),natural logarithms ($\ln x$).

Activity 1: Problem-Solving

Task 21

1. By using small group methodology, guide trainees to read and answer the questions provided under task21 in their trainee's manuals.

   During an experiment, a scientist notices that the number of bacteria halves every second. If there were 2.3×10^{30} bacteria at the start of the experiment, how many bacteria were left after 5 seconds? Give your answer in standard form correct to two significant figures.

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Using an appropriate methodology such as question and answer in a large group, pair work or small group work, trainees share their answers to the class.
4. Encourage all trainees to give their views by sharing their answer to the class and write their responses for reference.
5. After the sharing session, refers trainees to Key facts 2.3 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task22

1. Use an individual work methodology and guide trainees to solve the questions provided under task 22 in their trainee's manuals.
   a. A population of bacteria starts with 10,000 individuals and decreases at a decay rate of 8% per year. What will the population be after 5 years?
   b. If a population of 50,000 decreases by 6% per year, how long will it take for the population to decay to 20,000?
   c. If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 1,000 cells?
2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.
4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.
5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.
6. After sharing the session, refer trainees to Key Facts 2.3.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 23

Request trainees to perform the tasks provided in their trainee's manuals as described below:

1. A city's population is decreasing exponentially by 4% per year. If the current population is 100,000, how long will it take for the population to halve?

2. In a certain experiment, the number of bacteria reduces by a quarter each second. If the number of bacteria initially was X, write a formula that can be used to calculate the number of bacteria, V, remaining after t seconds
3. The population of a particular town on July 1, 2011 was 20,000. If the population decreases at an average annual rate of 1.4%, how long will it take for the population to reach 15,300?
4. A certain town has an initial population of 40,000 people. If the population is decreasing by 2% per year, how long will it take for the population to fall below 30,000?
5. A population of a certain species of fish in a lake is initially 5,000. If the population decreases at a rate of 3% per year, how many years will it take for the population to reduce to 2,000?
6. A certain species of plant in a controlled environment has a population of 500. If the population decreases at a rate of 4% per month, how many plants will remain after 6 months?

Formative Assessment

1. Multiple Choice questions

a. The formula for compound interest is given by $A = P\left(1+\frac{r}{n}\right)^{nt}$. What does the variable **t** represent?

i. Time in years
ii. Interest rate
iii. Principal amount
iv. Number of times interest is compounded

b. Population grows according to the formula $P(t) = P_0 e^{kt}$. If $k \succ 0$, what does it indicate?

i. Population decay
ii. Population stability
iii. Population growth
iv. None of the above

c. If the half-life of a substance is 5 years, what is the value of the decay constant k (rounded to 3 decimal places)?

   i. 0.693

   ii. −0.693

   iii. 0.138

   iv. −0.138

d. Which of the following equations is used to calculate continuous compound interest?

   i. $A = P(1+r)^t$

   ii. $A = P\left(1+\frac{r}{n}\right)^{nt}$

   iii. $A = Pe^{rt}$

   iv. $A = P + \Pr t$

2. Differentiate Initial population to Future population
3. Differentiate simple to Compound Interest.
4. A \$1000 deposit is made at a bank that pays 12% compounded weekly. How much will you have on your account at the end of 10 years?
5. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.
   a. Assuming an exponential growth model, estimate the population in 1980.
   b. What is the doubling time for the town's population?
6. A principal amount of \$10,000 is invested at an annual interest rate of 5%, compounded quarterly. Calculate the amount after 3 years. (Use $A = P\left(1+\frac{r}{n}\right)^{nt}$)
7. A radioactive substance has a half-life of 10 years. How much of a 100g sample remains after 30 years? Use $P(t) = P_0 e^{-kt}$ and calculate $k = \frac{\ln(2)}{half-life}$
8. A population of bacteria decreases from 100,000 to 25,000 in 8 hours. Calculate the decay constant k and the time it takes for the population to reduce to 12,500.
9. A loan of \$50,000 is taken out with an annual interest rate of 6%, compounded continuously. Calculate the total amount to be repaid after 10 years. (Use $A = Pe^{rt}$)

10. The population of a country is 1 million, and it is observed to double every 35 years. Derive the exponential growth equation for this population and predict the population after 70 years.
11. Solve for t in the compound interest equation: $2000 = 1000(1+0.05)^t$
12. A population decreases by 3% annually. If the current population is 12,000, how long will it take for the population to reduce to 8,000? Use the formula ($P(t) = P_0 e^{-rt}$

Answers:

1.

a. is (i) Time in years

b. is (iii) Population growth

c. is (iii) 0.138

d.is (iii) $A = Pe^{rt}$

Questions **2** and **3** cfr trainee's manual

4. To calculate the final amount in the account, we use the compound interest formula for periodic compounding: $A = P\left(1+\dfrac{r}{n}\right)^{nt}$

Where:

- A:is the final amount,
- P=1000 is the principal deposit,
- r=0.12 is the annual interest rate (12% = 0.12),
- n=52: is the number of compounding periods per year (weekly compounding = 52 weeks per year).
- t=10: is the time in years.

Substitute the values into the formula to get

$$A = 1000\left(1+\frac{0.12}{52}\right)^{52\times 10} \Rightarrow A = 1000(1+0.0023077)^{520} \Rightarrow 1000(1.0023077)^{520} \Leftrightarrow A = 1822.12$$

At the end of 10 years, the account will have approximately **$1822.12**.

5.

a. Estimating the population in 1980

The exponential growth model is: $P(t) = P_0 e^{kt}$ Where:

- P(t) is the population at time t
- P0 is the initial population (P0=10,000),
- k is the growth rate,
- t is the time elapsed since 1960.
    - Let Calculate k using the data for 1970

In 1970 (t=10), the population is P(10)=12,000. Substituting into the equation to get

$12{,}000 = 10{,}000e^{10k}$,divide both sides by 10,000: $1.2 = e^{10k}$.

Take the natural logarithm (ln) of both sides:

$$\ln(1.2) = \ln e^{10k} \Rightarrow \ln 1.2 = 10k \Leftrightarrow k = \frac{\ln 1.2}{10} \Leftrightarrow k \approx 0.0182$$

Calculate the population in 1980 (t=20), substitute k=0.0182 and t=20 into the growth model**:**

$$P(20) = 10{,}000e^{0.0182\times 20} \Rightarrow P(20) = 10{,}000e^{0.364} \Leftrightarrow P(20) = 14{,}390 \approx 14400$$

. Hence the estimated population in 1980 is approximately 14400.

b. Doubling Time

The formula for doubling time T in exponential growth is :

$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.0182} \Leftrightarrow T \approx 38.08\,years$. The doubling time for the town's population is approximately **38.1 years**.

7. Question 6 can be performed using same strategy as question4. (sol is **A= $11,611.12**)

7. To calculate the remaining amount of a radioactive substance we use the formula for exponential decay.

$$P(t) = P_0 e^{-kt}$$

Where:

- P = remaining amount

- P0=initial amount = 100g
- t= time elapsed = 30years
- TTT = half-life = 10years

$k = \frac{\ln(2)}{half - life} \Leftrightarrow k = \frac{\ln 2}{10} \Leftrightarrow k = 0.069$, then

$$P(t) = P_0 e^{-kt} \Rightarrow P(30) = 100e^{-0.069\times 30} \Leftrightarrow 100e^{-2.07} \Leftrightarrow P(30) = 12.6g$$

After 30 years, **12.6 g** of the radioactive substance remains.

8. The decay of population can be modeled by the exponential decay formula :

$$P(t) = P_0 e^{-kt}$$

Where:

- P(t)=population at time t
- P0=initial population=100,000
- K=decay constant
- T=time elapsed.

Find the decay constant k

From the problem, after 8 hours (t=8), the population is $P(t) = 25{,}000$ Substituting these values into the formula:

$$25{,}000 = 100{,}000e^{-k\times 8} \Rightarrow \frac{25{,}000}{100{,}000} = e^{-8k} \Leftrightarrow 0.25 = e^{-8k} \Rightarrow \ln 0.25 = -8k \Leftrightarrow k = \frac{\ln 0.25}{-8} \Leftrightarrow k = 0.1733$$

9. To calculate the total amount to be repaid with continuously compounded interest, we use the formula: $A = Pe^{rt}$

Where:

- A: total amount to be repaid,
- P=50,000 is the principal loan amount,

- r=0.06 is the annual interest rate (6% = 0.06),
- t=10 is the time in years,
- e is Euler's number (≈2.718).

$$A = 50{,}000e^{0.06\times 10} \Rightarrow A = 50{,}000e^{0.6} \Leftrightarrow A \approx \$91{,}106$$

The total amount to be repaid after 10 years is approximately **$91,106**.

10.The exponential growth of a population is modeled as : $P(t) = P_0 e^{kt}$

Where:

- P(t): Initial population
- K is growth rate constant
- T is time elapsed

Let find the growth rate constant k

The population doubles every 35 years, so when t=35 P(t)=2P0

$$2P_0 = P_0 e^{35\times k} \Leftrightarrow \frac{2P_0}{P_0} = e^{35k} \Rightarrow e^{35k} = 2$$

Take natural logarithm of both sides :

$$\ln 2 = \ln e^{35k} \Leftrightarrow 35k = \ln 2 \Leftrightarrow k = \frac{\ln 2}{35} \Leftrightarrow k \approx 0.0198$$

Exponential growth equation is $P(t) = 1{,}000{,}000e^{0.0198t}$

- **Predict the population after 70 years.**

$$P(t) = 1{,}000{,}000e^{0.0198\times 70} \Leftrightarrow 1{,}000{,}000e^{1.386} \Leftrightarrow P(t) \approx 4{,}000{,}000$$

After 70 years, the population will be approximately **4,000,000**.

11. Solve for t in the compound interest equation: $2000 = 1000(1+0.05)^t$

$$2000 = 1000(1+0.05)^t$$

Step 1: Simplify the equation

$$\frac{2000}{1000} = (1.05)^t$$

$$2 = (1.05)^t$$

Step 2: Take the logarithm of both sides

$$\log(2) = \log((1.05)^t)$$

$$\log(2) = t \cdot \log(1.05)$$

Step 3: Solve for t

$$t = \frac{\log(2)}{\log(1.05)}$$

Now compute the values:

$$t = \frac{0.3010}{0.0212} \approx 14.2$$

Final Answer:

$$t \approx \boxed{14.2 \text{ years}}$$

12. A population decreases by 3% annually. If the current population is 12,000, how long will it take for the population to reduce to 8,000? Use the formula ($P(t) = P_0 e^{-rt}$

$$P(t) = P_0 e^{-rt}$$

Where:

- $P(t) = 8000$ (final population),
- $P_0 = 12000$ (initial population),
- $r = 0.03$ (decay rate),
- t is the time in years.

Step 1: Plug in the known values

$$8000 = 12000 \cdot e^{-0.03t}$$

Step 2: Divide both sides by 12000

$$\frac{8000}{12000} = e^{-0.03t}$$

$$\frac{2}{3} = e^{-0.03t}$$

Step 3: Take the natural logarithm (ln) of both sides

$$\ln\left(\frac{2}{3}\right) = -0.03t$$

Step 4: Solve for t

$$t = \frac{\ln\left(\frac{2}{3}\right)}{-0.03}$$

Now calculate:

$$\ln\left(\frac{2}{3}\right) \approx \ln(0.6667) \approx -0.4055$$

$$t = \frac{-0.4055}{-0.03} \approx 13.52$$

$t \approx \boxed{13.5 \text{ years}}$

Self-Reflection

1. Ask learners to re-take the self-assessment at the beginning of the unit. They should then fill in the table in their Trainee's Manual to Identify their areas of strength, areas for improvement and actions to take to improve.
2. Discuss trainees' results with them. Identify any areas that are giving many trainees difficulties and plan to give additional support as needed (ex. use class time before you begin the next learning outcome to go through commonly identified difficult concepts).

Points to Remember

These are the key learning points from all activities in this learning outcome.

- Exponential and logarithmic functions are used in population growth, half-life, and decay in radioactivity. Logarithmic functions are also used to find interest rate problems. A quantity is said to have an exponential growth (decay) model if at each instant of time its rate of increase (decrease) is proportional to the amount of the quantity present.
- Formula for compound interest $A = P\left(1+\frac{r}{n}\right)^{nt}$ or $A = Pe^{rt}$ for continuous compounding. Where:
  - ✓ A: Final amount
  - ✓ P: Principal amount
  - ✓ r: Annual interest rate (decimal form)
  - ✓ n: Number of compounding periods per year
  - ✓ t: Time in years
- Exponential growth is given by $P(t) = P_0e^{rt}$ .Where :
  - ✓ P(t): Population at time t
  - ✓ P0: Initial population.
  - ✓ k: Growth rate constant (positive for growth).
  - ✓ t: Time elapsed.
  - ✓ Use $k = \ln(1+r)$ if the growth rate is given in percentage.
  - ✓ The population double when $t = \frac{\ln(2)}{k}$ (known as doubling time)
- Exponential decay is given by $P(t) = P_0e^{-rt}$ .Where:
  - ✓ P(t): Population or amount remaining at time t.
  - ✓ P0: Initial Population
  - ✓ k: Decay rate
  - ✓ t: Time elapsed.

- ✓ The decay constant k is calculated using the half-life: $k = \frac{\ln(2)}{half - life}$

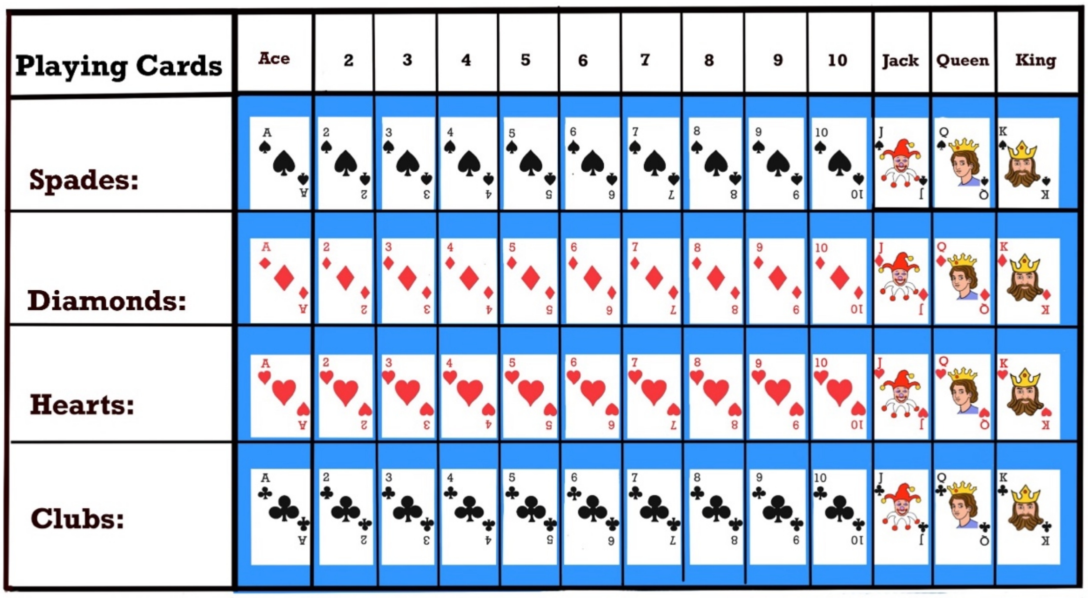
- For exponential growth model, the time required for it to double in size is called the doubling time. Similarly, for exponential decay model, the time required for it to reduce in value by half is called the halving time. For radioactive elements, halving time is called half-life.
- Remember the distinction between growth $(k \succ 0)$ and decay $(k \prec 0)$
- Logarithms can help determine how long it takes for a substance to decay to a specific amount.

Further Information for the Trainer

Look information on :

- Key applications of Exponential expressions
- Key applications of Logarithmic Expressions.
- Applications Involving both Exponential and Logarithmic Equations

LEARNING OUTCOME3: APPLY FUNDAMENTALS OF PROBABILITY



Learning outcome 3: Self-Assessment

1. Ask trainees to answer the following questions in their Trainee's Manuals.
   a. Define probability, events, sample spaces, and outcomes.
   b. What are some real life examples of probability?
   c. How to calculate probability?
   d. Ivan rolls a fair dice, with sides labelled A,B,C,D,E and F. What is the probability that the dice lands on a vowel?
   e. Max tested a coin to see whether it was fair. The table shows the results of his coin toss experiment

Head	Tail
26	**41**

   What is the relative frequency of the coin landing on heads?
2. After the discussion, inform trainees that this unit is intended to provide them with the knowledge, skills and attitudes to **apply Fundamentals of Probability**. They will cover the application of counting techniques, computation of probabilities and calculation of the conditional probability.
3. Ask trainees to fill out the self-assessment at the beginning of the unit in their Trainee's Manuals. Explain that:
   a. The purpose of the self-assessment is to become familiar with the topics in the unit and for them to see what they know or do not know at the beginning.
   b. There are no right or wrong ways to answer this assessment. It is for their own reference and self-reflection on the knowledge, skills and attitudes acquisition during the learning process.
   c. They should think about themselves: do they think they have the knowledge, skills or attitudes to do this? How well?
   d. They read the statements across the top and put a check in column that best represents their level of knowledge, skills or attitudes.
   e. At the end of the unit, they will do a self-reflection, which includes re-taking the self-assessment and identifying their strengths, areas of improvement and actions to be taken.

Key Competencies:

Knowledge	Skills	Attitudes
1. Define correctly probability and its associated terms	1. Determine the number of permutations and combinations of "n" items, "r" taken at a time.	1. Pay attention to details while calculating probability
2. Describe the application of probability to different fields including tourism, social sciences etc…	2. Use counting techniques to solve related problems.	2. Approach problems systematically and logically.
3. Distinguish properly between combination and permutation	3. Use properties of combinations	3. Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios.
4. Define the combinatorial Analysis.	4. Analyse problems to determine relevant probabilities.	4. Be patient while solving complex probability problems.
5. Describe different formulas used to solve problems related to calculation of probability	5. Apply concepts to solve real-world problems and scenarios.	
6. Describe uses of Unions, Intersections and Complements in probability contexts.	6. Interpret data to make probabilistic inferences	
7. Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment.		

Steps:

Discovery activity

Task 24

1. Applying question and answer methodology, guide trainees to share their prior experience from their understanding regarding solve exponential and logarithmic equations. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are given.
   a. What is the meaning of term probability?
   b. What is the probability of spotting at least one species (either chimpanzees or colobus monkeys)?
   c. What is the probability of not spotting either species during the tour?
   d. If 3 tours are conducted on different days, what is the probability of spotting chimpanzees on all 3 tours?

Topic 3.1: Counting techniques in probability

	Objectives: By the end of the topic, trainees will be able to: j. Define correctly what is probability k. Identify basic formula of probability. l. Identify and explain basic branches of probability. m. Apply the counting techniques in probability
	Time Required: 10hours.
	Learning Methodology: Trainer guided, small group work, group discussion, Individual work, Trial example.

Materials, Tools and Equipment Needed:

- ✓ Textbooks/Handouts contain probability with examples of counting techniques (permutations, combinations, etc.) or internet if available.
- ✓ . Pre-made or blank cards to visually explain terms like "permutations," "combinations," or scenarios for problem-solving.
- ✓ Probability trees, factorial calculation charts, or Venn diagrams.
- ✓ Whiteboard or chalkboard with markers or chalks for drawing diagrams, writing formulas, and explaining steps.

Preparation:

- ❑ Gather some handouts, worksheets application of counting techniques in probability.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

- ✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.
- ✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.
- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

- Basic Mathematical Skills on addition, subtraction, multiplication, and division.
- Understanding what factorials represent and how to compute them.
- Understanding the basic formula of probability
- Knowing how to list all possible outcomes of an experiment.
- Understanding what events are (e.g., simple events, compound events).
- Knowledge of operations like union and intersection.
- Recognizing complementary events.
- Recognizing that for independent events, the total number of outcomes is the product of individual outcomes.

Activity 1: Problem-Solving

1. What is probability?
2. What is the formula for probability calculation?
3. What are the branches of probability?
4. What are the Techniques for counting the number of elements in a sample space?
5. Provide formula for each technique.

Task 25

1. Utilizes groups, guide trainees to read and answer the questions provided under task 25 in their trainee's manuals.
   a. What is probability?
   b. What is the formula for probability calculation?
   c. What are the branches of probability?
   d. What are the Techniques for counting the number of elements in a sample space?
   e. Provide formula for each technique.

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.
3. Encourage all trainees to give their views by share their answers to the class and write their responses for reference.
4. After the sharing session, refers trainees to Key facts 1.1 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task 26

1. Use an individual work methodology and guide trainees to solve the questions provided under task 26 in their trainee's manuals.

   Using the techniques for counting the number of elements in a sample space, answer the following questions:

   a. Let $U=\{1,2,3,4,5,6,7\}$, $A=\{1,3,5,7\}$, $B=\{4,5,6,7\}$ and $C=\{2,3,5,7\}$. List the elements in each of the following sets:

$a. A\cup B$	$d. A\cup B'$
$b. A\cap C$	$e. C'-A'$
$c. B-C$	$f. A'\cap C$

2. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Using three diagram, find the probability that the ball drawn will be:
   a. red followed by green,
   b. red and green in any order,
   c. of the same color.
3. Habimana has 4 shirts, 3 pair of trousers and 2 pairs of shoes. He chooses a shirt, a pair of trousers and a pair of shoes to wear every day. Using multiplication rule, find the maximum number of days he does not need to repeat his clothing.

4. Using multiplication rule, find how many ways the first three places (with no ties) can be filled in a race with 5 contestants.
5. Using permutation, how many ways can 4 students from a group of 15 be lined up for a photograph?
6. Using permutation, How many three digit numbers can be formed with the digits: 1,2,3,4.5 with repletion allowed?
7. In how many ways can six letters of the word “mammal” be arranged in a row using permutation?
8. Using combination technique, how many different committees of 3 people can be chosen from a group of 12 people?
9. A committee of 5 is to be chosen from 12 men and 8 women. In how many ways can this be done if there are to be 3 men and 2 women on the committee using conditional combination?

Activity 3: Application

Task 27

Ask trainees to perform the tasks provided in their trainee’s manuals as described below:

Suppose your company has 8 qualified tour guides, and you need to select a team of 3 guides for a safari tour.

1. How many different teams of 3 guides can you form?

2. If the roles of team leader, assistant, and driver need to be assigned, how many arrangements are possible?

Topic 3.2: Probabilities for an event

Objectives:

By the end of the topic, trainees will be able to:

a. Explain what is sample space, possible outcome and event**.**

b. Explain types of event

c. Apply the formula of infusive events, mutually exclusive events,

Time Required: 10hours.

Learning Methodology:

Group discussion, trainer guided, small group work, Individual work, and Trial examples.

Materials, Tools and Equipment Needed:

- ✓ Books on probability theory and statistics
- ✓ Summarized materials covering key concepts such as probability rules, permutations, combinations, and probability distributions.
- ✓ Practice sheets with problems on probability computation.
- ✓ Scientific Calculator for basic probability calculations, factorials, combinations, and permutations.
- ✓ White or black board and Chalks or Markers for illustrating concepts and working through problems in real-time.
- ✓ Dice for experiments involving simple events like rolling outcomes.
- ✓ Coins to illustrate probabilities of binary events (e.g., heads or tails).
- ✓ Decks of Cards for hands-on examples involving probabilities with cards.
- ✓ Marbles or Beads for experiments demonstrating conditional probability and sampling.

Preparation:

- ❑ Gather some handouts, worksheets on description of formula for computation of probability for an event.
- ❑ Prepare trial examples on applying logarithm and exponential in calculation of population growth.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

- ✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.
- ✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.
- ✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.
- ✓ **Critical thinking:** Give activities which enhance critical thinking
- ✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will learn better this lesson if they have a good background on

- Proficiency in addition, subtraction, multiplication, and division.
- Understanding how to work with fractions, decimals, percentages and convert between these formats.

- Familiarity with factorials (n!) as they are often used in permutations and combinations.
- Knowledge of sets, subsets, unions, intersections, and complements.
- Understanding the fundamental principle of counting, permutations, and combinations.
- Ability to use logical thinking to analyze problems.
- Some exposure to simple experiments like rolling dice, flipping coins, or drawing cards, which serve as hands-on examples of probability concepts.

Activity 1: Problem-Solving

Task28

1. Use small groups, guide trainees to read and answer the questions provided under task 28 in their trainee's manuals.
   a. What do you understand by term event in probability?
   b. What are the types of events in probability?
   c. How many tourists like at least one of the activities?
   d. What is the probability that a randomly selected tourist likes exactly two activities?

Activity 2: Guided Practice

Task29

1. Use an individual work methodology and guide trainees to solve the questions provided under task 29 in their trainee's manuals. An ordinary die of 6 sides is rolled once. With the aid of definition of complement event, answer the following questions. Determine the probability of:
   a. Obtaining 5 b. Not obtaining 5
   c. Obtaining 3 or 4 d. Not obtaining 3 or 4
2. A letter is chosen from the letters of the word ''MATHEMATICS''. What is the probability that the letter chosen is an ''A''?

3. A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P (A), P(B), $P(A \cap B)$ and $P(A \cup B)$.
4. A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find P(A), P(B) and $P(A \cup B)$.
5. A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Activity 3: Application

Task 30

Demands trainees to perform the tasks provided in their trainee's manuals as described below:

A survey among 100 tourists reveals their preferences for three types of activities:

- **A:** Wildlife Safari (60 tourists)
- **B:** Lake Cruise (50 tourists)
- **C:** Cultural Village (40 tourists)

Overlap data:

- 20 tourists like **both A and B**
- 15 tourists like **both A and C**
- 10 tourists like **both B and C**
- 5 tourists like **all three activities**

a. How many tourists like at least one of the activities?

b. What is the probability that a randomly selected tourist likes exactly two activities?

Topic 3.3: The conditional probability

Objectives:

By the end of the topic, trainees will be able to:

a. Define conditional probability

b. apply the conditional probability formula : $P(A/B)=\frac{P(A\cap B)}{P(B)}, P(B)\succ 0$

c. Differentiate between conditional probability and independent events.

d. use tree diagram to find probability of events.

e. . finds probability of independent events.

f. find probability of one event given that the other event has occurred.

g. use and apply Bayes theorem.

Time Required: 10hours

Learning Methodology:

Trainer guided, small group work, group discussion, Individual work, Trial examples.

Materials, Tools and Equipment Needed:

- ✓ Notes or handouts with explanations and examples of conditional probability.
- ✓ Practice worksheets with step-by-step problems involving conditional probability.
- ✓ Scientific calculator for performing calculations efficiently.
- ✓ White or blackboard and chalks or markers: For teaching and solving problems interactively.
- ✓ Dice, coins, or cards for hands-on experiments to demonstrate conditional probability concepts.

Preparation:

- ❑ Gather some handouts, worksheets on calculation of conditional probability.
- ❑ Assign practice problems for trainees to solve individually or in groups.
- ❑ Conduct a quick quiz or interactive activity to check understanding.

Cross Cutting Issues:

✓ **Gender balance:** Mix girls and boys in order to promote cross-gender interaction. Encourage both genders to take on roles of leadership.

✓ **Inclusive education**: Put trainees into different mixed-ability groups. If there are, trainees with disabilities mix them with others. If there are some with hearing disabilities or communication difficulties, you should always get their attention on before you begin to speak and encourage them to look at your face when you speak.

✓ **Peace and value education:** Discourage negative behavior such as booing or laughing at others if they give incorrect answers. Encourage learners to work in peace and harmony in order to avoid conflict.

✓ **Critical thinking:** Give activities which enhance critical thinking

✓ **Communication:** Encourage every group member to participate in discussions.

Prerequisites:

Trainees will easily learn this topics, if they have a good background on:

- Understanding probability as a measure of likelihood.
- Knowledge of terms such as "event," "sample space," "mutually exclusive events," and "independent events."
- Understanding sets, subsets, intersections ($A \cap B$) and, unions ($A \cup B$)
- Distinction between independent events and dependent events

Activity 1: Problem-Solving

Task 31

1. Use small groups, guide trainees to read and answer the questions provided under task 31 in their trainee's manuals.

   A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event “the first pen is red” and B be the event “the second pen is blue”. Is the occurrence of event B affected by the occurrence of event A? Explain.

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.

3. Encourage all trainees to give their views by share their answers to the class and write their responses for reference.
4. After the sharing session, refers trainees to Key facts 3.3 and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 2: Guided Practice

Task32

1. Utilies an individual work methodology and guide trainees to solve the questions provided under task 32 in their trainee's manuals.

   a. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.
   b. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.
   c. A bag contains 3 pink candies and 7 green candies. Two candies are taken out from the bags with replacement. Find the probability that both candies are pink.

d. A die is tossed. Find the probability that the number obtained is a 5 given that the number is greater than 3.
e. Suppose you have a standard deck of 52 cards. You draw two cards one after the other without replacing the first card. What is the probability that the first card is a King and the second card is also a King?

2. Make sure instructions are understood, all the trainees are actively participating and necessary materials/tools are provided and being used.

3. During the task, trainees should be given a degree of independence to apply the knowledge and skills acquired in activity 1. Your role is to guide them by using probing questions such as Why? What? How? to enable them to come to informed responses.

4. During the task, use this opportunity to discuss or address any cross-cutting issues that may arise such as gender, inclusivity, financial education among others. Also attitudes and behaviour changes should be handled during this activity.

5. Using an appropriate methodology such as question and answer in a large group, pair or small group work, trainees share their answers to the class. Write their responses for reference. Encourage all trainees to give their views.

6. After the sharing session, refer trainees to Key Facts 3.3.b and discuss them together while harmonizing their responses provided in the sharing session and answer any questions they have.

Activity 3: Application

Task 33

Requests trainees to perform the tasks provided in their trainee's manuals as described below:

A hospital uses a diagnostic test for a rare disease that affects 1 in 1,000 people. The test has the following accuracy:

1. True Positive Rate (Sensitivity): If a person has the disease, the test correctly detects it 99%of the time P (Positive | Disease) =0.99

2. False Positive Rate**:** If a person does not have the disease, the test incorrectly gives a positive result 5% of the time. P (Positive | No Disease) =0.05 A patient tests positive for the disease.
   a. What is the probability that the patient actually has the disease?
   b. Two discs are selected one at a time without replacement from a box containing red and 3 blue discs. Find the probability that
      i. the discs are of the same colour
      ii. if the discs are of the same colour, both are red.

Formative Assessment

1. A tour package includes 5 major attractions:

   a. A National Park
   b. A Wildlife Safari
   c. A Lake Cruise
   d. A Cultural Village
   e. A Local Market

The tour company wants to create different itineraries by arranging these attractions in different orders.

i. How many different ways can the attractions be arranged if all 5 must be included?
ii. If only 3 attractions are selected for a half-day tour, how many arrangements are possible?

Answer:

a. If all 5 attractions are included:

   When all 5 attractions are included, the number of arrangements is the total number of permutations of the 5 attractions:

   $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

b. If only 3 attractions are selected:

If 3 attractions are selected out of 5, and we consider different orders, the number of arrangements is given by permutations:

$$P(n,r)=\frac{n!}{(n-r)!}$$

where n=5 (total attractions) and r=3 (selected attractions).

$$P(5,3)=\frac{5!}{(5-3)!}=\frac{5\times4\times3\times2\times1}{2\times1}=5\times4\times3=60$$

Final Answers:

a. 120 ways

b. b) 60 ways

2. A survey among 100 tourists reveals their preferences for three types of activities:

- **A:** Wildlife Safari (60 tourists)
- **B:** Lake Cruise (50 tourists)
- **C:** Cultural Village (40 tourists)

Overlap data:

- 20 tourists like **both A and B**
- 15 tourists like **both A and C**
- 10 tourists like **both B and C**
- 5 tourists like **all three activities**
- **C:** Cultural Village (40 tourists)

a. How many tourists like at least one of the activities?

b. What is the probability that a randomly selected tourist likes exactly two activities?

Answer:

This problem involves using the principles of inclusion –exclusion for counting. Let's solve it step by step.

Notation:

- Tourists who like wildlife safari $n(A)=60$
- Tourists who like Lake Cruise $n(B)=50$
- Tourists who like Cultural Village $n(C)=40$
- Tourist who like both wildlife Safari and Lake Cruise $n(A\bigcap B)=20$
- Tourists who like both wildlife Safari and Cultural Village $n(A\bigcap C)=15$
- Tourists who like both Lake Cruise and Cultural Village $n(B\bigcap C)=10$
- Tourists who like all three activities $n(A\cap B\cap C)=5$

a. **Tourists who like at least one of the activities.**

Using the principle of inclusive-exclusion:

$$n(A\bigcup B\bigcup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)$$
$$\Rightarrow 60+50+40-20-15-10+50=110$$

i.e 110 Tourists like at least one of the activities.

b. **Probability that a randomly selected tourist likes exactly two activities.**

We need to count the tourists who like exactly two activities. This can be computed as follows:

- Tourists who like A and B only :

$$n(A\cap B)/C=n(A\cap B)-n(A\cap B\cap C)\Rightarrow 20-5=15$$

- Tourists who like A and C only:

$$n(A\cap C)/B=n(A\cap C)-n(A\cap B\cap C)\Rightarrow 15-5=10$$

- Tourists who like B and C only :

$$n(B\cap C)/A=n(B\cap C)-n(A\cap B\cap C)\Rightarrow 10-5=5$$

The total number of tourists who like exactly two activities is

n(exactly two)= $n(A\cap B)/C+n(A\cap C)/B\cap n(B\cap C)/A\Rightarrow 15+10+5=30$

The probability that a randomly selected tourist likes exactly two activities is:

P(exactly two)=n(exactly two)/Total tourist= $\dfrac{30}{100}=0.3$

3. The tourism company offers a bonus trip to 3 tourists selected from the 100 surveyed tourists. What is the probability that all 3 selected tourists like at least one activity?

Answer:

- Total tourists surveyed: 100
- Tourists who like at least one activity: Let this be x, and the remaining tourists who don't like any activity will be 100−x.
- Tourists selected: 3, randomly.
- Probability that one tourist likes at least one activity: P(Likes at least one activity)= $\frac{x}{100}$
- Therefore, the probability a tourist does **not** like any activity is: P(Does not like any activity)= $1-\frac{x}{100} \Leftrightarrow \frac{100-x}{100}$
- The probability of selecting 3 tourists who like at least one activity is: P(All 3 like at least one activity)= $\frac{x}{100} \bullet \frac{x-1}{99} \bullet \frac{x-2}{98}$. This accounts for the fact that after selecting one tourist, the pool of eligible tourists decreases. To compute the exact probability, substitute the value of x (the number of tourists who like at least one activity).

4. At a middle school, 18% of all trainees performed competition on Taxation and auditing, and 32% of all trainees performed competition on Taxation. What is the probability that a student who performed competition on Taxation also performed auditing?

Answer:

To solve this, we use the concept of **conditional probability**. The question asks for the probability that a student who performed a competition on Taxation also performed a competition on Auditing.

1. Let A = Event that a student performed a competition on **Taxation and Auditing**.

2.Let B = Event that a student performed a competition on **Taxation**.

We need to find $P(A/B)$, the probability that a student performed a competition on Auditing, given that they performed a competition on Taxation. The formula for conditional probability is:

$$P(A/B)=\frac{P(A\cap B)}{P(B)},$$

Known Data:

- $P(A \cap B)$ =18%=0.18 (trainees who performed both competitions).
- P(B)=32%=0.32 (trainees who performed a competition on Taxation).

Then $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.32} \Leftrightarrow P(A/B) = 0.5625$. The probability that a student who performed a competition on Taxation also performed a competition on Auditing is **56.25%**.

5. A football team has the following probabilities of match outcomes based on their recent performance:
    - Win: 50%
    - Draw: 30%
    - Lose: 20%

    a. What is the probability that the team does not lose a match?

    b. What is the probability that the team loses two matches in a row?

Answer:

Given Probabilities:

- P(Win)=50%=0.5
- P(Draw)=30%=0.3
- P(Lose)=20%=0.2

a. Probability that the team does not lose a match:

The event "does not lose" is the union of the events "Win" and "Draw." Therefore:

P (Does not lose) =P(Win)+P(Draw)

P (Does not lose) =0.5+0.3=0.8. The probability that the team does not lose a match is **0.8 (80%)**.

b. probability that the team loses two matches in a row:

To lose two matches in a row, the team must lose the first match and then lose the second match. Assuming independence between matches:

P(Lose two matches in a row) =P(Lose)·P(Lose)

P(Lose two matches in a row) =0.2·0.2=0.04. The probability that the team loses two matches in a row is **0.04 (4%)**.

6. A manager has three substitute players:
   a. Player A: 70% chance of improving the team's performance.
   b. Player B: 50% chance of improving the team's performance.
   c. Player C: 30% chance of improving the team's performance.
   d. The manager randomly selects one player for substitution.
   a) What is the probability that the substitution improves the team's performance?
   b) If the manager chooses Player A, what is the probability that the team's performance does not improve?

   Answer:

   a. The manager randomly selects one of the three players, so each has an equal probability of being chosen ($\frac{1}{3}$)

      P(Improvement)=P(A)*P(Improvement)/A) +P(B)*P(Improvement/B) +P(C)*P(Improvement/C).

      P(Improvement)= $\frac{1}{3}\times 0.7+\frac{1}{3}\times 0.5+\frac{1}{3}\times 0.3 \Rightarrow \frac{1}{3}\left(0.7+0.5+0.3\right)=0.5$.Thus the probability that the substitution improves the temea's performance is 0.5=(50%).

   b. P(No Improve/A)=1-P(Improve/A)= $1-0.7=0.3$.Thus the probability that player A does not improve the team's performance is 0.3 (30%

7. A bag contains 6 blue balls, 5 green balls and 4 red balls. Three balls are selected at random without replacement. Find the probability that

   a. they are all blue
   b. 2 are blue and 1 is green
   c. there is one of each color.

Answer:

The number of all possible outcomes is $\binom{15}{3} = \frac{15!}{3!\times 12!} = \frac{15\times 14\times 13}{3!} = 455$

a. The number of ways of obtaining 3 blue balls is

$$\binom{6}{3}\times\binom{4}{0} = \frac{6!}{3!3!}\times 1\times 1 = \frac{6\times 5\times 4}{3!} = \frac{120}{6} = 20$$

Thus (all are blue)= $\frac{20}{455} = \frac{4}{91}$

b. The probability of obtaining 2 blue balls and one green ball is:

$$\binom{6}{2}\times\binom{5}{1}\times\binom{4}{0} = \frac{6!}{2!4!}\times 5\times 1 = \frac{6\times 5}{2!}\times 5 = \frac{150}{2} = 75$$

Thus, P(2 blue and 1 green) = $\frac{75}{455} = \frac{15}{91}$

c. The probability of obtaining 1 blue ball, 1 green ball and 1 red ball is:

$$\binom{6}{1}\times\binom{5}{1}\times\binom{4}{1} = \frac{6!}{1!5!}\times\frac{5!}{1!4!}\times\frac{4!}{1!3!} = 6\times 5\times 4 = 120$$

Thus, P (1 blue, 1 green and 1 red)= $\frac{120}{455} = \frac{24}{91}$

8. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:

a. Three boys being chosen.
b. Exactly two boys and a girl being chosen.
c. Exactly two girls and a boy being chosen.

d. Three girls being chosen.

Answer:

To solve this problem, we use the concept of combinations. The number of ways to choose r item from n items is given by the formula: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

The class consists of 6 girls and 10 boys for a total of 6+10=16 trainees. The total number of ways to choose 3 trainees from 16 is:

$$\binom{16}{3} = \frac{16!}{3!(16-3)!} = \frac{16!}{3!13!} = \frac{16\times15\times14}{3\times2\times1} = 560$$

a. Probability of choosing three boys

To form a committee of 3 boys, we need to select all 3 members from the group of 10 boys. The number of ways to do this is:

$$\binom{10}{3} = \frac{10\times9\times8}{3\times2\times1} = 120$$

P(3 boys)= $\frac{\binom{10}{3}}{\binom{16}{3}} = \frac{120}{560} = \frac{3}{14} = 0.214$

b. Probability of choosing exactly Two boys and One girl

To form a committee which exactly 2 boys and 1, we choose 2 boys from 10 and 1 girl from 6.The number of ways to do this is: $\binom{10}{2}\times\binom{6}{1} = \frac{10\times9}{2}\times6 \Leftrightarrow 45\times6 = 270$

P(2 boys and 1 Girl)= $\left(\dfrac{\binom{10}{2}\times\binom{6}{1}}{\binom{16}{3}}\right)=\dfrac{270}{560}\Leftrightarrow\dfrac{27}{56}=0.482$

c. **Probability of choosing exactly Two Girls and One Boy**

To form a committee with exactly 2 girls and 1 boy, we choose 2girls from 6 and 2 boy from 10.The number of ways to do this is: $\binom{6}{2}\times\binom{10}{1}=\dfrac{6\times5}{2}\times10=15\times10\Leftrightarrow150$

P(2 Girls and 1 Boy)= $\dfrac{\binom{6}{2}\times\binom{10}{1}}{\binom{16}{3}}=\dfrac{150}{560}\Leftrightarrow\dfrac{15}{56}=0.268$

d. **Probability of choosing Three girls**

To form a committee of 3 girls ,we need to select all 3 members from the group of 6 girld.The number of ways to do this is : $\binom{6}{3}=\dfrac{6!}{3!3!}=\dfrac{6\times5\times4}{3\times2\times1}=20$

P(3 Girls)= $\dfrac{\binom{6}{3}}{\binom{16}{3}}=\dfrac{20}{560}\Leftrightarrow\dfrac{1}{28}=0.0357$

9. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colors noted. The first disk is not replaced before the second is selected. Find the probability that the discs will be:

a. both red
b. of different colors
c. the same colors.

Answer:

Given:

- Total discs are 7(2 red and 5 green)
- After the first selection, there are 6 discs remaining.

a. Both red.

The probability of selecting a red discs on the first draw is: **P(1st red)=** $\frac{2}{7}$.After the first red disc is drawn, only 1 red remains out of 6 total discs. Thus, the probability of selecting a red disc on the second draw is: P(2nd red /1st red) = $\frac{1}{6}$.The probability of both discs being red is:

P(both red)=P(1st red)*P(2nd red /1st red) $\frac{2}{7}\times\frac{1}{6}=\frac{1}{21}$

b. Of different colors

For this case, the two possible scenarios are:

1.The first disc is red, and the second is green

2. The first disc is green and the second is red

Scenario1: First red, second green

- P(1st red)= $\frac{2}{7}$, P(2nd green/1st red= $\frac{5}{6}$

Scenario2: First green, second green

- P(1st green)= $\frac{5}{7}$, P(2nd green/1st red)= $\frac{2}{6}$
- P(1st red,2nd green)= $\frac{5}{7}\times\frac{2}{6}=\frac{5}{21}$

Adding both scenarios gives the total probability of different colors:

P(Different colors)= $\frac{5}{21}+\frac{5}{21}=\frac{10}{21}$

c. The same colors

This occurs when both discs are red or both are green

- Probability of both red is :P(both red)= $\frac{1}{21}$

- For both green, P(1st green) = $\frac{5}{7}$, P(2nd green/1st green)= $\frac{4}{6}$

  P(both green)= $\frac{5}{7}\times\frac{4}{6}=\frac{20}{42}=\frac{10}{21}$

The total probability of the same colors is :P(both red) +P(both green) $\Rightarrow \frac{1}{21}+\frac{10}{21}=\frac{11}{21}$

Final answers:

a. P (both red): $\frac{1}{21}$

b. P (different colors): $\frac{10}{21}$

c. P (a=same colors): $\frac{11}{21}$

10. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be:

    a. all red
    b. all blue
    c. one of each color.

Answer:

To calculate the probabilities, we need to use combinations to determine the number of ways.

Total number of ways to choose 3 discs from the 18 discs (3 red,8 blue and 7 white is given by combination formula: $C(n,k)=\frac{n!}{k!(n-k)!}$.Where n is the number of items and k is the number of items chosen for this case:

$C(18,3)=\frac{18!}{3!(18-3)!}=\frac{18\times17\times16}{3\times2\times1}=816$. Now let's calculate the probability for each case:

a. The probability that all three discs chosen are red:

The number of ways to choose 3 red discs from e red discs is: $C(3,3) = \dfrac{3!}{3!(3-3)!}$

Thus, the probability is:P(all red)= $\dfrac{C(3,3)}{C(18,3)} = \dfrac{1}{816}$

b. Probability that all three discs chosen are blue is:

$C(8,3) = \dfrac{8!}{3!(8-3)!} = \dfrac{8\times7\times6}{3\times2\times1} = 56$. Thus, the probability is:

P(all blue)= $\dfrac{C(8,3)}{C(18,3)} = \dfrac{56}{816} = \dfrac{7}{102}$

c. Probability that one disc of each color is chosen:

The number of ways to choose 1 red disc from 3 red discs ,1 blue disc from 8 blue discs and 1 white discs from 7 white discs is:

$C(3,1)\times C(8,1)\times C(7,1) = 3\times8\times7 = 168$. Thus, the probability is:

P (one each other): $\dfrac{C(3,1)\times C(8,1)\times C(7,1)}{C(18,3)} = \dfrac{168}{816} = \dfrac{7}{34}$

Final Results

a. P (all red): $\dfrac{1}{816}$

b. P (all blue): $\dfrac{7}{102}$

c. P (One of each color): $\dfrac{7}{34}$

11. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Answer:

Let A be the event :'' a 4 is obtained on the first throw'', then $P(A)=\frac{1}{6}$.That is $A=\{4\}$

Let B be the event: 'an odd number is obtained on the second throw .That is B= $\{1,3,5\}$

Since the result on second throw is not affected by the result on the first throw, A and B are independent events. There are 3 odd numbers, then

$P(B)=\frac{3}{6}=\frac{1}{2}$. Therefore,

$$P(A\cap B)=P(A)\times P(B)=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12}$$

12. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Answer:

Let A be the event: 'the number is a 4'',the A= $\{4\}$

Let B the event: 'the number is greater than 2'',then $B=\{3,4,5,6\}$ and

$P(B)=\frac{4}{6}=\frac{2}{3}$.But $A\cap B=\{4\}$and $P(A\cap B)=\frac{1}{6}$.Therefore,

$$P(A/B)=\frac{P(A\cap B)}{P(B)}\Leftrightarrow P(A/B)=\frac{\frac{1}{6}}{\frac{2}{3}}=\frac{1}{6}\times\frac{3}{2}=\frac{1}{4}$$

13. At a middle school, 18% of all trainees play football and basketball, and 32% of all trainees play football. What is the probability that a student who plays football also plays basketball?

Answer:

Let A be a set of trainees who play football and B a set of trainees who play basketball; then the set of trainees who play both games is $A\cap B$. We have $P(A)=32\%=0.32$,

$P(A \cap B) = 18\% = 0.18$ We need the probability of B known that A has occurred. Therefore,

$$P(B/A) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.32} \Leftrightarrow 0.5625 = 56\%$$

14. Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone User	45	405	450
Total	70	675	755

Calculate the following probabilities using the table:

a. P (person is a car phone user).
b. P (person had no violation in the last year).
c. P(person had no violation in the last year AND was a car phone user).
d. P(person is a car phone user OR person had no violation in the last year).
e. P(person is a car phone user GIVEN person had a violation in the last year).
f. P(person had no violation last year GIVEN person was not a car phone user).

Answer:

a. P(person is a car phone user)=$number\ of\ car\ users\ divide\ by\ total\ number\ in\ study$

$$\Rightarrow \frac{305}{755} \approx 0.404$$

b. P (person had no violation in the last year=$number\ that\ had\ no\ violation\ divide\ by\ number\ in\ study$

$$\Rightarrow \frac{685}{755} \approx 0.907$$

c. P(person had no violation in the last year AND was a car phone user)

$$\Rightarrow \frac{280}{755}$$

d. P(person is a car phone user OR person had no violation in the last year)

$$\Rightarrow\left(\frac{305}{755}+\frac{685}{755}\right)-\frac{280}{755}=\frac{710}{755}\approx 0.9404$$

e. The sample space is reduced to the number of persons who had a violation. Then P (person is a car phone user GIVEN person had a violation in the last year)=

$$\frac{25}{70}\approx 0.357$$

f. The sample space is reduced to the number of persons who were not car phone users. Then P(person had no violation last year GIVEN person was not a car phone user)= $\frac{405}{450}\approx 0.9$

15. Suppose that machines M_1, M_2, M_3 and produce respectively 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M3 ?

Answer:

To solve this problem, we use **Bayes' theorem**, which helps us find the probability that the defective part was produced by machine M_3, given that it is defective.

- M_1, M_2, M_3: The part was produced by machine M_1, M_2, M_3, respectively.
- D: The part is defective.
- We want $P(M_3 / D)$, the probability that a defective part was produced by machine M_3

Write Bayes' theorem

$P(M_3 / D)=\dfrac{P(D / M_3)P(M_3)}{P(D)}$**.** Calculation the components

1. P(M1), P(M2), P(M3): Probability that a part is produced by each machine

- The machines produce 500, 1000, and 1500 parts, respectively, out of a total of 500+1000+1500=3000 parts.

$$P(M_1)=\frac{500}{3000}=\frac{1}{6},\ P(M_2)=\frac{1000}{3000}=\frac{1}{3},\ P(M_3)=\frac{1500}{3000}=\frac{1}{2}$$

2. $P(D/M_1), P(D/M_2), P(D/M_3)$: **Probability of defect given the machine**

- **Machine M1:** $P(D/M_1)$=5%=0.05
- Machine M2: $P(D/M_2)$=6%=0.06
- Machine M3: $P(D/M_3)$=7%=0.07

3. P(D): Total probability of a defective part. Using the law of total probability

$$P(D)=P(D/M_1)P(M_1)+P(D/M_2)P(M_2)+P(D/M_3)P(M_3)$$
$$\Rightarrow P(D)=\left(0.05\bullet\frac{1}{6}\right)+\left(0.06\bullet\frac{1}{3}\right)+\left(0.07\bullet\frac{1}{2}\right)\Leftrightarrow P(D)=\frac{0.05}{6}+\frac{0.06}{3}+\frac{0.07}{2}\Leftrightarrow P(D)\approx 0.0633$$

Substitute into Bayes' theorem:

$P(M_3/D)=\dfrac{P(D/M_3)P(M_3)}{P(D)}=\dfrac{0.07\bullet\frac{1}{2}}{0.0633}\approx 0.5528$. So The probability that the defective part was produced by machine M3 is approximately **55.28%**..

16. Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

Answer:

Let E be the event that the part came from machine A, C be the event that the part came from machine B and D be the event that the part is defective.

We require $P(E/D)$.Now,

$P(E)\times P(D/E)=0.6\times 0.03=0.018$and

$$P(D) = P(E \cap D) + P(C \cap D) = 0.018 + 04 \times 0.05 = 0.038.$$

Therefore, the required probability is $\dfrac{0.018}{0.038} = \dfrac{9}{19} \approx 0.4736$.

The probability that the defective part came from machine A is approximately **47.37%**

17. In a certain college, 5% of the men and 1% of the women are taller than 180 cm. Also, 60% of the trainees are women. If a student is selected at random and found to be taller than 180 cm, what is the probability that this student is a woman?

Answer:

Let's solve this problem using Baye's Theorem.

Given:

- P(W): Probability a student is a woman =60%=0.6
- P(M): Probability a student is a man :1-P(W)=1-0.6=0.4
- P(T/W): Probability a woman is taller than 180cm=1%=0.01
- P(T/M): Probability a man is taller than 180cm=5%=0.05
- P(T): Probability a randomly selected student is taller than 180 cm
- **P(W/T): Probability a taller –than 180cm student is a woman (what we want to find).**

  Total probability of being taller than 180cm: P(T)

  $$P(T) = P(T/W)P(W) + P(T/M)P(M)$$
  $$\Rightarrow (0.01)(0.6) + (0.05)(0.4) \Leftrightarrow 0.026$$

By applying Baye's Theorem:

$$P(W/T) = \frac{P(T/W)P(W)}{P(T)} \Leftrightarrow \frac{(0.01)(0.6)}{0.026} = \frac{0.06}{0.026}$$
$$\Rightarrow P(W/T) \approx 0.2308,$$

So, the probability that the student is a woman given that they are taller than 180 cm is approximately 23.08%.

18. A certain federal agency employs three consulting firms (A, B and C) with probabilities 0.4, 0.35, 0.25, respectively. From past experience, it is known that the probabilities of cost overrun for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.
    a. What is the probability that the consulting firm involved is company C?
    b. What is the probability that it is company A?

Answer:

To solve this, we can use Bayes'Theorem.Let:

- P(A), P(B), P(C): Probability of hiring firms A, B, C respectively
- P(O/A), P(O/B), P(O/C): Probability of cost overrun given that the firms are A, B, C respectively.
- P(C/O), P(A/O): Probability that the firms are C and A given a cost overrun.

Given data:

- P(A)=0.4, P(B)=0.35, P(C)=0.25
- P(O/A) =0.05, P(O/B) =0.03, P(O/C) =0.15

Total probability of a cost Overrun P(O)

$$P(O) = P(O/A)P(A) + P(O/B)P(B) + P(O/C)P(C)$$
$$\Rightarrow (0.05)(0.4) + (0.03)(0.35) + (0.15)(0.25) \Leftrightarrow 0.02 + 0.0105 + 0.0375$$
$$\Leftrightarrow P(O) = 0.068$$

a. The probability that the consulting firm involved is company C

$$P(C/O) = \frac{P(O/C)P(C)}{P(O)} = \frac{(0.15)(0.25)}{0.068}$$
$$\Leftrightarrow P(C/O) \approx 0.5515$$

b. probability that the firm is A given a cost Overrun (P(A/O)

$$P(A/O) = \frac{P(O/A)P(A)}{P(O)} = \frac{(0.05)(0.4)}{(0.068)} = \frac{0.02}{0.068}$$
$$\Leftrightarrow P(A/O) \approx 0.2941$$

19. A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P(A), P(B), P(A ∩ B) and P(A ∪ B).

Answer:

P(A)=P(an ace)= $\frac{4}{52}=\frac{1}{13}$

P(B)=P(a spade)= $\frac{13}{52}=\frac{1}{4}$

P(A ∩ B)=P(the ace of spades)= $\frac{1}{52}$

P(A ∪ B)=P(A)+P(B)-P($A \bigcap B$)= $\frac{4}{52}+\frac{13}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$

20. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

Answer:

We need to find the probability that none of the balls drawn is blue when two balls are drawn randomly from a bag containing:

- 2 red balls
- 3 green balls
- 2 blue balls

The total number of balls is: 2+3+2=7

Total number of ways to choose 2 balls: The total number of ways to choose 2 balls out of 7 is given by the combination formula $\binom{n}{r}=\frac{n!}{r!(n-r)!}$. Total ways= $\binom{7}{2}=\frac{7!}{2!(7-1)!}=\frac{7\times 6}{2\times 1}=21$

If none of the balls is blue, then both balls must be either red or green. The total number of red and green balls is: 2+3=5.

The number of ways to choose 2 balls from these 5 is: $\binom{5}{2} = \frac{5\times4}{2\times1} = 10$

The probability is the ratio of favorable outcomes to total outcomes: $\frac{10}{21}$.

The probability that none of the balls drawn is blue is: $\frac{10}{21}$

21. A nationwide survey found that 72% of people in the United States like pizza. If 3 people are selected at random, what is the probability that all the three like pizza?

Answer:

The probability that a randomly selected person likes pizza is:0.72

Since the selection of each person is independent, the probability of all three liking pizza is the product of the probabilities for each individual:

P (all three like pizza) =P (likes pizza) ×P (likes pizza) ×P (likes pizza)

P (all three like pizza) =0.72×0.72×0.72= 0.373248, So

The probability that all three selected people like pizza is:0.373 or approximately **37.3%.**

22. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?

Answer:

The jar contains:

- 3 red marbles
- 5 green marbles
- 2 blue marbles
- 6 yellow marbles

The total number of marbles is: 3+5+2+6=16

The probability of selecting a green marble on the first draw is:$P(green) = Number\ of\ green\ barble: Total\ number\ of\ marbles$

P(green)= $\frac{5}{16}$.Since the marble is replaced, the total number of marbles remains the same for the second draw. The probability of selecting a yellow marble on the second draw is:P(yellow)= $\frac{6}{16}$

The probability of both events (green first and yellow second) occurring is the product of their individual probabilities:

P(green then yellow)=P(green)×P(yellow)= $\frac{5}{16} \times \frac{6}{16} = \frac{30}{256} \Leftrightarrow \frac{15}{128}$.

The probability of choosing a green marble and then a yellow marble is: $\frac{15}{128}$ or approximately **0.117** (11.7%).

23. A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight?

Answer:

In a standard deck of 52 cards, there are 4 jacks. The probability of selecting a jack is:

P(jack)=$Number\ of\ jacks \div Total\ number\ of\ cards$= $\frac{4}{52} = \frac{1}{13}$

Similarly, there are 4 eights in the deck. The probability of selecting an eight is:

P(eight)=$Number\ of\ eights \div Total\ number\ of\ cards$= $\frac{4}{52} = \frac{1}{13}$

Since the events are independent (because the card is replaced), the probability of both events (choosing a jack and then an eight) is the product of their individual probabilities:

P(jack then eight)=P(jack)×P(eight)= $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$. The probability of choosing a jack and then an eight is: $\frac{1}{169}$ or approximately **0.00592** (0.592%).

24. A pen is drawn from a basket containing 10 pens of which 5 are red and 3 are black. If A is the event: "a pen is red" and B is the event: "a pen is black". Find P(A), P(AUB).

Answer:

There are 5 red pen, then $P(A)=\frac{5}{10}=\frac{1}{2}$

There are 3 black pens, then $P(B)=\frac{3}{10}$

Since the pen cannot be red and black at the same time, then A B ∩ =∅ and two events are mutually exclusive so

$$P(A\cup B)=P(A)+P(B)=\frac{1}{2}+\frac{3}{10}=\frac{4}{5}$$

25. Two discs are selected one at a time without replacement from a box containing 5 red and 3 blue discs. Find the probability that

(a) the discs are of the same color

(b) if the discs are of the same color, both are red.

Answer:

To solve this, let's consider the probabilities step by step.

Total number of ways to select 2 discs:

The box contains 555 red and 333 blue discs, making 888 discs in total. The total number of ways to select 2 discs from 888 is: $\binom{8}{2}=\frac{8\times 7}{2}=28$

a. **Probability that the discs are of the same color**

The number of ways to select 2 red discs from 5 is: $\binom{5}{2}=\frac{5\times 4}{2}=10$

The number of ways to select 2 blue discs from 3 is: $\binom{3}{2} = \frac{3\times 2}{2} = 3$

Total number of favorable outcomes: 10+3=13. So

P(same color)=favorable outcomes/total outcomes= $\frac{13}{28}$

b. Probability that both are red given the discs are of the same color

This is a conditional probability problem. We use the formula:

Points to Remember

These are the key learning points from all activities in this learning outcome.

- The set S of all possible outcomes of a given experiment is called the sample space.
- Any subset of the sample space is called an event. The event {a} consisting of a single element of S is called a simple event.
- The probability of an event E, denoted by P(E) or Pr(E), is a measure of the possibility of the event occurring as the result of an experiment.
- The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.
- Identify the sample space (SSS), which includes all possible outcomes.
- Types of Probability:
  - ✓ Theoretical Probability: Based on equally likely outcomes, calculated as:

    $$Number\ of\ Favorable\ Outcomes \div Total\ Number\ of\ Outcomes$$
  - ✓ Experimental Probability: Based on observations or experiments, calculated as:

    $$Number\ of\ Times\ Event\ Occurs \div Total\ Number$$
  - ✓ Subjective Probability: Based on intuition, experience, or belief.
  - ✓ Permutations $P_r^n = \frac{n!}{(n-r)!}$
  - ✓ Combinations $C_r^n = \frac{n!}{r!(n-r)!}$

- Range of Probability: The probability of any event E is always between 0 and 1: $0 \le P(E) \le 1$
- Sum of Probabilities: The total probability of all possible outcomes in the sample space is always 1: $\sum$ P(Outcomes in S)=1
- Addition Rule
  - ✓ For mutually exclusive events ($E_1$and E_2), where they cannot happen simultaneously: $P(E_1, or, E_2) = P(E_1) + P(E_2)$
  - ✓ For non-mutually exclusive events:

    $P(E_1, or, E_2) = P(E_1) + P(E_2) - P(E_1, and, E_2)$,
- Independent Events: The occurrence of one event does not influence the other.
- Mutually Exclusive Events: Events cannot happen at the same time.
- Exhaustive Events: The events together cover all possible outcomes.
- Multiplication Rule:
  - ✓ For independent events (A and B), where one does not affect the other: P(A and B)=P(A)×P(B)
  - ✓ For dependent events: P(A and B)=P(A)×P(B|A)
- Complement Rule: The probability of an event not happening (E') is: P(E')=1−P(E)

- Conditional Probability: The probability of event AAA occurring, given that BBB has occurred, is: $P(A/B) = \dfrac{P(A, and, B)}{P(B)}, P(B) \succ 0$

Self-Reflection

1. Ask learners to re-take the self-assessment at the beginning of the unit. They should then fill in the table in their Trainee's Manual to Identify their areas of strength, areas for improvement and actions to take to improve.
2. Discuss trainees' results with them. Identify any areas that are giving many trainees difficulties and plan to give additional support as needed (ex. use class time before

you begin the next learning outcome to go through commonly identified difficult concepts).

ⓘ Further Information for the Trainer

Look information on:

- Basic probability formula
- Types of probability
- Key probability Rules and Concepts
- Common probability Distributions
- Real-world applications of Probability

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APPENDIX

$\Re$: Real number

∑: Summation symbol

n! :Factorial of n

e: Euler’s number (~2.718)

exp(x): Exponential function (e^x)

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