

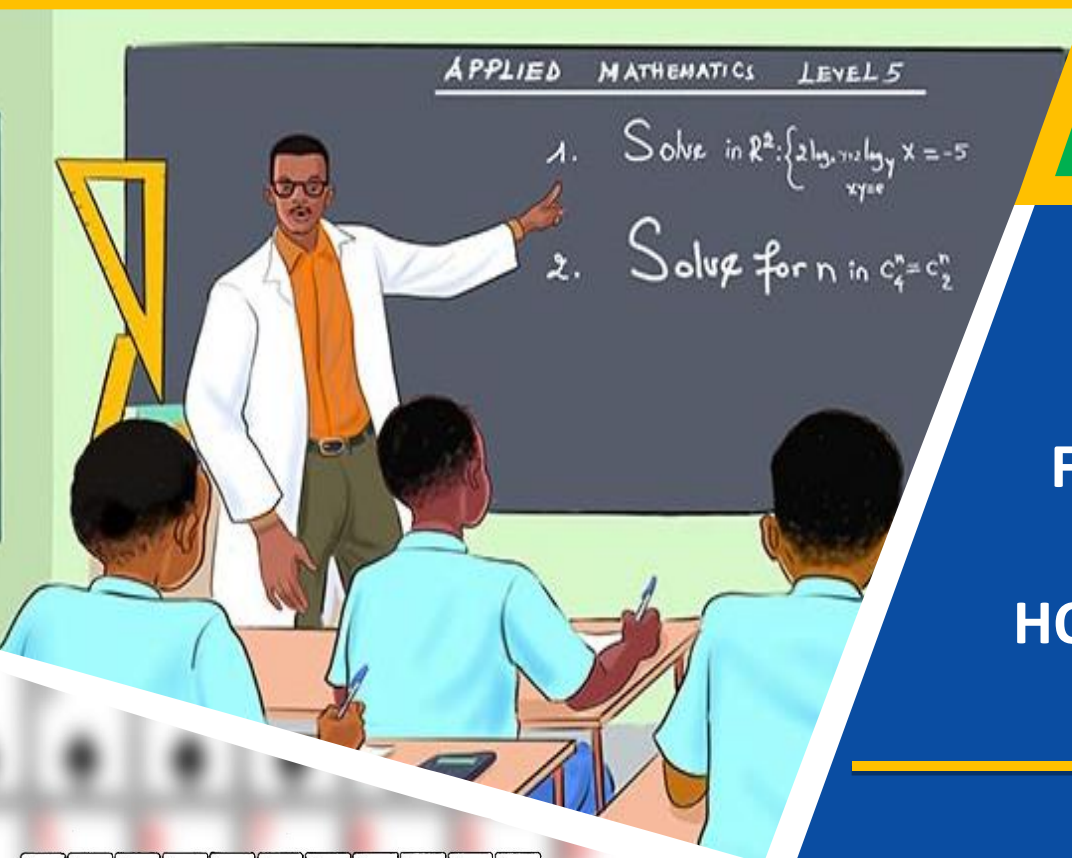


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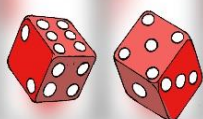
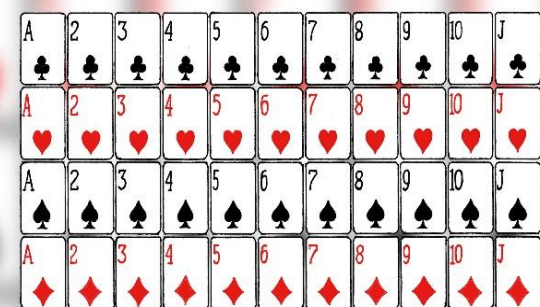
# RQF LEVEL 5



FRONT OFFICE  
AND  
HOUSEKEEPING  
OPERATIONS

**GENEP502**

Applied  
Mathematics



**TRAINEE'S MANUAL**

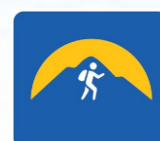
April 2025



Republic of Rwanda  
Ministry of Education



# APPLY LOGARITHMS, EXPONENTIAL EQUATIONS AND PROBABILITY



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**KIGALI-RWANDA**

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## LIST OF ABBREVIATIONS AND ACRONYMS

**CBET:** Competence Base Education and Training

**P(A):** Probability of event A occurring

**P(A|B):** Conditional probability (probability of A given B)

**RQF:** Rwanda Qualification Framework

**RTB:** Rwanda TVET Board

**TVET:** Technical and Vocational Education and Training

## INTRODUCTION

This trainee's manual encompasses all necessary skills, knowledge and attitudes required to **apply Logarithms, Exponential Equations and Probability**. Students undertaking this module shall be exposed to practical activities that will develop and nurture their competences. The writing process of this training manual embraced competency-based education and training (CBET) philosophy by providing practical opportunities reflecting real life situations.

The trainee's manual is subdivided into units, each unit has got various topics. You will start with a self-assessment exercise to help you rate yourself on the level of skills, knowledge and attitudes about the unit.

A discovery activity is followed to help you discover what you already know about the unit.

After these activities, you will learn more about the topics by doing different activities by reading the required knowledge, techniques, steps, procedures and other requirements under the key facts section, you may also get assistance from the trainer. The activities in this training manual are prepared such that they give opportunities to students to work individually and in groups.

After going through all activities, you shall undertake progressive assessments known as formative and finally conclude with your self-reflection to identify your strengths, weaknesses and areas for improvement.

Do not forget to read the point to remember the section which provides the overall key points and takeaways of the unit.

## Module Units:

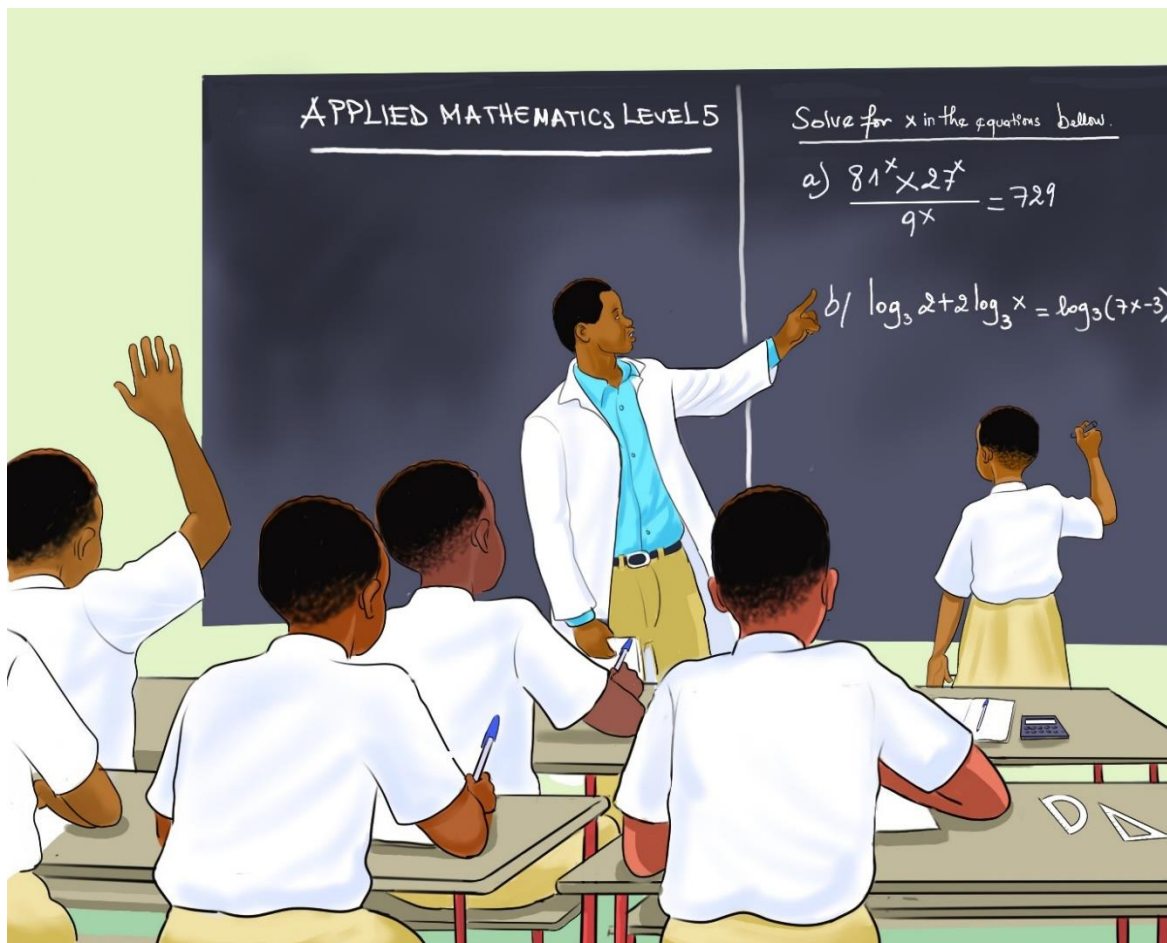
Unit 1: Solve exponential and logarithmic equations

Unit 2: Apply exponential and logarithmic expression

Unit 3: Apply fundamentals of probabilities

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## UNIT 1: SOLVE EXPONENTIAL AND LOGARITHMIC EQUATIONS



### Unit summary

This unit provides you with the knowledge, skills and attitudes required to solve exponential equations and logarithmic Equations. It covers the description of properties for exponential expressions, the description of properties for logarithmic expression, solving exponential equation and solving logarithmic equations

## Self-Assessment: Unit 1

1. Answer the following questions:
  - a. What is exponential function?
  - b. What is a logarithmic function?
  - c. What is the difference between exponential and logarithmic expressions?
  - d. Differentiate exponential to logarithmic expressions?
  - e. Without using calculators, solve each of the following equations for X:
    - i.  $2^{x+1} = 8$
    - ii.  $3^{2x} = 81$
    - iii.  $5^x = \frac{1}{25}$
    - iv.  $e^{2x} = 20$
    - v.  $\ln(x - 3) = 2$
    - vi.  $\log(x + 1) + \log(x - 1) = 1$
    - vii.  $\log(x^2 - 1) - \log(x + 1) = 0$
2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.
  - a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
  - b. Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
  - c. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

3. At the end of this unit, you will assess yourself again.

| Experience                                                 | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                           |                                          |                             |                                    |                                       |                                          |
| Define properly exponential expressions                    |                                          |                             |                                    |                                       |                                          |
| Isolate exponential terms correctly                        |                                          |                             |                                    |                                       |                                          |
| Define properly logarithmic expression                     |                                          |                             |                                    |                                       |                                          |
| Isolate logarithmic terms correctly                        |                                          |                             |                                    |                                       |                                          |
| Identify properties of exponential expressions             |                                          |                             |                                    |                                       |                                          |
| Identify properties of logarithmic expressions             |                                          |                             |                                    |                                       |                                          |
| Apply properties of logarithmic expression                 |                                          |                             |                                    |                                       |                                          |
| Describe different technics to solve exponential equations |                                          |                             |                                    |                                       |                                          |

| Experience                                                                        | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-----------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                  |                                          |                             |                                    |                                       |                                          |
| Solve exponential equations based on their properties.                            |                                          |                             |                                    |                                       |                                          |
| Describe different technics to solve logarithmic equations.                       |                                          |                             |                                    |                                       |                                          |
| Solve exponential expressions and logarithmic equations based on their properties |                                          |                             |                                    |                                       |                                          |



### Key Competencies:

| Knowledge                                         | Skills                                                        | Attitudes                                                                           |
|---------------------------------------------------|---------------------------------------------------------------|-------------------------------------------------------------------------------------|
| 1. Define properly exponential expressions        | 1. Isolates Exponential terms correctly                       | 1. Pay attention to details while isolating exponential and logarithmic expression. |
| 2. Define properly logarithmic expression         | 2. Isolates logarithmic terms correctly                       | 2. Pay attention to details while logarithmic expression.                           |
| 3. Identify properties of exponential expressions | 3. Apply properties of exponential and logarithmic expression | 3. Think logically when applying properties of exponential expressions.             |
| 4. Identify properties of logarithmic expressions | 4. Apply properties of logarithmic expression                 | 4. Think logically when applying properties of logarithmic expressions.             |

| Knowledge                                                      | Skills                                                                                | Attitudes                                                                      |
|----------------------------------------------------------------|---------------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| 5. Describe different technics to solve exponential equations  | 5. Solve exponential equations based on their properties.                             | 5. Be patient while solving exponential equations.                             |
| 6. Describe different technics to solve logarithmic equations. | 6. Solve exponential expressions and logarithmic equations based on their properties. | 6. Be patient while solving exponential expressions and logarithmic equations. |
| 7. Demonstrates exponential and logarithmic expressions.       | 7. Analyse correctly exponential expressions and logarithmic forms.                   |                                                                                |



### Discovery activity:

#### Scenario 1:

Nowadays every trending information is almost known by anyone in short time as information is being spread rapidly through social media. Take an example of one person who shares a photo to 10 people and each of these 10 people share it to another 10, these 10 also share it to another 10 and so on. Within short time, this photo will reach to many people.



#### Task 1:

1. What is the best function of calculating the number of people who received that image?
2. If each person was sharing the image to 10 people, how can we call this 10 that is keeping repeating?
3. If each time the image was shared is termed level, Calculate the number of people that received the image at level 4.
4. If each this picture was shared to 10 people in one minute, how long will it take for the picture to be shared to 900,000 people?

## Topic 1.1: Properties for exponential expressions



### Activity 1: Problem Solving



#### Task 2:

Answer the questions below:

1. What do you understand by term exponential rate?
2. Write the following number  $2 \times 2 \times 2 \times 2$  in exponential expression/equation.
3. From your answer in question 2, what is the appropriate term of 2 in the exponential equation?
4. State the exponent rules for solving exponential equations

#### Key Facts 1.1a: Description of properties for exponential expression

- **Definition of exponential expressions**

An **exponential expression** is a mathematical expression in which a base is raised to an exponent (also called a power). The base number represents the number being multiplied repeatedly, while the exponent represents the number of times the base number is multiplied by itself.

The general form of an exponential expression is written as  $y^n$ , where "y" is the base number and "n" is the exponent.

$$y^n = y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \dots (n \text{ times})$$

- **Key Concepts in Exponential Expressions:**

1. A positive exponent means you multiply the base that number of times.

Example:  $2^4 = 16$

In the above example, the base is 2 and the exponent is 4, Therefore the result is 16 because 2 is multiplied by itself four times  $2 \times 2 \times 2 \times 2 = 16$

2. A negative exponent means you take the reciprocal of the base raised to the positive exponent. Which is written as  $y^{-x} = \frac{1}{y^x}$

Example:  $2^{-4} = \frac{1}{16}$

In the above example, the base is 2 and the exponent is - 4, therefore the result is

$$2^{-4} = \frac{1}{2^4}, \text{ you remember that } 2^4 = 16 \text{ therefore,}$$

$$2^{-4} = \frac{1}{16}$$

3. For any non-zero base, when an exponent of 0 it means the value of the expression is always 1

Example:  $X^0 = 1$ ,  $100^0 = 1$ ,  $-1^0 = 1$ , etc....

- **Exponent rules**

Exponent rules are those laws that are used for simplifying expressions with exponents. Many arithmetic operations like addition, subtraction, multiplication, and division can be conveniently performed in quick steps using the laws of exponents.

| Rule                            | Description                                                                                                 | Formula                                                                                   |
|---------------------------------|-------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------|
| <b>Product of Powers Rule</b>   | When multiplying two exponential expressions with the same base, add the exponents.                         | $a^m \times a^n = a^{m+n}$                                                                |
| <b>Quotient of Powers Rule</b>  | When dividing two exponential expressions with the same base, subtract the exponents.                       | $\frac{a^m}{a^n} = a^{m-n}$                                                               |
| <b>Power of a Power Rule</b>    | When raising an exponential expression to another power, multiply the exponents                             | $(a^m)^n = a^{mn}$                                                                        |
| <b>Power of a Product Rule</b>  | When raising a product to a power, apply the exponent to both factors                                       | $(ab)^m = a^m b^m$                                                                        |
| <b>Power of a Quotient</b>      | When raising a quotient to an exponent, you apply the exponent to both the numerator and the denominator    | $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$                                            |
| <b>Negative Exponent Rule</b>   | A base raised to a negative exponent is equal to the reciprocal of the base raised to the positive exponent | $a^{-m} = \frac{1}{a^m}$                                                                  |
| <b>Zero Exponent Rule</b>       | Any non-zero base raised to the power of zero is equal to 1                                                 | $a^0 = 1$                                                                                 |
| <b>Fractional Exponent Rule</b> | A base raised to a fractional exponent can be rewritten as a root                                           | $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ is<br>written as<br>$b^{\frac{m}{n}} = (\sqrt[n]{b})^m$ |

## Key Facts 1.1b: Simplifying and solving exponential expressions using exponent rules

- The first equation, is resolved by the product of powers rule

$$\begin{aligned}\text{From this, } 3^4 x 3^2 &= 3^{(4+2)} \\ &= 3^6 \\ &= 729\end{aligned}$$

- The second equation, is resolved by the quotient of powers rule,

$$\begin{aligned}\text{Using this rule, } \frac{5^7}{5^3} &= 5^{(7-3)} \\ &= 5^4 \\ &= 625\end{aligned}$$

- The third equation, is resolved by the power of powers rule,

$$\begin{aligned}\text{From this rule, } (2^3)^4 &= 2^{(3 \times 4)} \\ &= 2^{12} \\ &= 4096\end{aligned}$$

- The forth equation, is resolved by the negative exponential rule,

$$\begin{aligned}\text{From this rule, } 5^{-2} &= \frac{1}{5^2} \\ &= \frac{1}{25}\end{aligned}$$

- The fifth equation, is resolved by the fractional exponent rule,

$$\begin{aligned}\text{From this, } 16^{\frac{1}{2}} &= (\sqrt{16})^1 \\ &= \sqrt{16}\end{aligned}$$

- The sixth equation, is resolved by more than one rule,

$$\text{First, you use the product of powers property in the numerator: } 3^5 x 3^2 = 3^{(5+2)} = 3^7$$

$$\text{With the above step, now we have } \frac{3^7}{3^6}$$

$$\text{Second, you use quotient of powers rule on the whole fraction, } \frac{3^7}{3^6} = 3^{(7-6)}$$

$$\text{Now we get } 3^1 = 3$$

- For simplifying this  $\frac{(2x^3 y^2)^2 \cdot x^4}{(xy)^3}$

**Step 1:** Apply the power rule for products to expression  $(2x^3 y^2)^2$

$$(2x^3y^2)^2 = 2^2 \times (x^3)^2 \times (y^2)^2 = 4x^6y^4$$

**Step 2:** Apply the power rule for quotients to  $(xy)^3 = x^3y^3$

$$\frac{4x^6y^4 \times x^4}{x^3y^3}$$

**Step 3:** Substitute back into the original expression

**Step 4:** Apply the product rule for exponents for the  $x$ -terms and the  $y$ -terms:

$$\frac{4x^{6+4}y^4}{x^3y^3} = \frac{4x^{10}y^4}{x^3y^3}$$

**Step 5:** Apply the quotient rule for exponents to simplify further:  $4x^{10-3}y^{4-3} = 4x^7y$

Final simplified expression now is  $4x^7y$



## Activity 2: Guided Practice



### Task 3:

As seen above, exponent rules are used to simplify and manipulate expressions with exponents. These rules help in performing operations like multiplication, division, and powers of exponents more efficiently. By understanding and applying exponent rules, we can easily solve complex algebraic equations and problems. Using the exponent rules, simplify and evaluate the following exponential expressions

1.  $3^4x^3^2 =$

2.  $\frac{5^7}{5^3} =$

3.  $(2^3)^4 =$

4.  $5^{-2} =$

5.  $16^{\frac{1}{2}} =$

6.  $\frac{3^5x^3^2}{3^6} =$

7.  $\frac{(2x^3y^2)^2 \times x^4}{(xy)^3} =$



## Activity 3: Application



#### **Task 4:**

With the current social media, information sharing grows at an exponential rate. Let's assume that a trending post of a photo of a new local cuisine menu in X HOTEL doubles at each 2 minutes. If the message is seen by 1000 people 10:00 AM, Using exponent rules, what will the number of people who will see it at 10:20 AM?

## Topic 1.2: Properties for logarithmic expressions



### Activity 1: Problem Solving



#### Task 5:

Answer the questions below:

- a. What is the logarithmic identity, and how is it used?
- b. What is the difference between logarithmic expressions and exponential expressions?
- c. What is the difference between the two kinds of logarithm?
- d. State and explain the properties used for solving logarithmic equations
- e. How does the change of base formula for logarithms work?
- f. What is the logarithm of 1 for any base?
- g. How is the logarithm of a base raised to an exponent simplified?

### Key Fact 1.2a: Description of properties for logarithmic expressions

- **Definition of logarithmic expression**

A **logarithmic expression** is a mathematical expression that represents the logarithm of a number or variable. A logarithm is defined as the power to which a number must be raised to get some other values. Briefly, a logarithm is the inverse operation of exponentiation. Fundamental properties of logarithms that simplify or manipulate logarithmic expressions are called **logarithmic identities**. These identities are based on the relationship between exponents and logarithms.

The general form of a logarithmic expression is written as  $\log_b(x)$  where **b** (the base) is a positive real number not equal to 1, **x** (the argument) is a positive real number,  $\log_b(x)$  represents the exponent to which the base b must be raised to get **x**.

- **Kinds of logarithms**

There are two kinds of logarithms: Common/decimal (or Briggian) logarithms and Natural (or Napierian) logarithms.

A **common logarithm**, is the logarithm with base **10**. It is denoted as  $\log_{10}(x)$  or simply  $\log(x)$  in many contexts (especially on calculators).  $\log_{10}(x) = y$  if and only if  $x = 10^y$

A **Natural logarithm**, is a logarithm with the base **e**, where  $e \approx 2.718$ . It is denoted as  $\ln(x)$ .

If for any positive value **b** and different to zero equal to  $e^x$  ( $= e^x$  and  $b > 0$ ), means that  $\ln b = x$

- **Key concepts in Logarithmic expressions:**

1. For any base b we have  $b^0 = 1$ , and  $b^1 = b$ . Hence  $\log_b 1 = 0$  and  $\log_b b = 1$ .

**Example.**  $\log_{10} 10 = 1, \log_3 3 = 1, \log_{10} 1 = 0$

2. The logarithm of negative number and zero are not defined means doesn't exist.

- **Examples of exponential expressions vs logarithmic expressions**

| Exponential expression | Logarithmic expression |
|------------------------|------------------------|
| $3^4 = 81$             | $\log_3 81 = 4$        |
| $2^5 = 32$             | $\log_2 32 = 5$        |

|             |               |
|-------------|---------------|
| $e^0 = 1$   | $\ln 1 = 0$   |
| $e^2 = 7.4$ | $\ln 7.4 = 2$ |

### 3. Change of bases

The Change of Base Formula for logarithms allows you to rewrite a logarithm in terms of logs of another base. It is expressed as:

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$$

This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm which always has base of 10.

**Example:**  $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

**Note:** Let b be a positive number. The logarithm of any positive number x to be the base written as  $\log_b(x)$  represents the exponent to which b must be raised to x.

That is  $y = \log_b(x)$  and  $b^y = x$

**Example:**

a.  $\log_2 8 = 3$  since  $2^3 = 8$

b.  $\log_2 64 = 6$  since  $2^6 = 64$

### 4. Rules of Decimal and Natural logarithms

These are those laws that are useful for simplifying and solving logarithmic expressions operations like addition, subtraction, Multiplication and division can be appropriately achieved in fast steps using the laws of logarithms. The logarithm of a base raised to an exponent can be simplified using the **power rule of logarithms**

| Rule                 | Description                                                                                                                        | Formula                                                              |
|----------------------|------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------|
| <b>Product Rule</b>  | The <b>Product Rule of Logarithms</b> states that the logarithm of a product is equal to the sum of the logarithms of its factors. | $\log(ab) = \log(a) + \log(b)$<br>$\ln(x \cdot y) = \ln(x) + \ln(y)$ |
| <b>Quotient rule</b> | The Quotient Rule of Logarithms states that the                                                                                    | $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$                   |

|                                  |                                                                                                                                                                                                                         |                                                                                                                                                                                                             |
|----------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                                  | logarithm of a quotient is equal to the difference of the logarithms of the numerator and denominator.                                                                                                                  | $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$                                                                                                                                                             |
| <b>Power rule</b>                | The <b>Power Rule of Logarithms</b> states that the logarithm of a number raised to an exponent is equal to the exponent multiplied by the logarithm of the base number.                                                | $\log(x)^n = n \log x$<br>$\ln(y)^n = n \ln y$                                                                                                                                                              |
| <b>Power of the product Rule</b> | The logarithm of a product raised to a power can be rewritten as the power (n) multiplied by the logarithm of the product (x•y). It combines the Power Rule and the Product Rule of logarithms into a single statement. | $\log(x \bullet y)^n = n \log(x \bullet y) \Leftrightarrow n \log(x) + n \log(y)$<br>$\ln(x \bullet y)^n = n \ln(x \bullet y) \Leftrightarrow n[\ln(x) + \ln(y)]$                                           |
| <b>Power of a Quotient</b>       | This rule says that the logarithm of a quotient raised to a power is equal to the power (n) multiplied by the logarithm of the quotient $\left(\frac{x}{y}\right)$                                                      | $\log\left(\frac{x}{y}\right)^n = n \log\left(\frac{x}{y}\right) \Leftrightarrow n[\log(x) - \log(y)]$<br>$\ln\left(\frac{x}{y}\right)^n = n \ln\left(\frac{x}{y}\right) \Leftrightarrow n[\ln(x) - \ln y]$ |
| <b>Reciprocal rule</b>           | The logarithm of the reciprocal of $x$ is the negative of the logarithm of $x$                                                                                                                                          | $\log\left(\frac{1}{x}\right) = -\log(x)$                                                                                                                                                                   |

|  |  |                                         |
|--|--|-----------------------------------------|
|  |  | $\ln\left(\frac{1}{x}\right) = -\ln(x)$ |
|--|--|-----------------------------------------|

### Key Facts 1.2b: using logarithm rules to simplify and solve logarithmic Expressions

- The first equation, is resolved by the product rule

$$\text{From this, } \log_3(20 \times 5) = \log_3(20) + \log_3(5)$$

$$\log_2(3 \times 5) = \log_2(3) + \log_2(5)$$

- The second equation, is resolved by the quotient of powers rule, Using this rule,

$$\log\left(\frac{8}{4}\right)^3 = 3\log\left(\frac{8}{4}\right) \Leftrightarrow 3[\log 8 - \log 4]$$

$$\ln\left(\frac{x}{2}\right)^4 = 4[\ln(x) - \ln(2)]$$

- The third equation, is resolved by the power of a product rule,

From this rule,

$$\log(50 \times 25)^5 = 5\log(50) + 5\log(25)$$

$$\log(3 \times 4)^2 = 2\log(3 \times 4) = 2[\log(3) + \log(4)]$$

$$\ln(5 \times 9)^5 = 5\ln(5 \times 9) \Leftrightarrow 5[\ln(5) + \ln(9)]$$

- The fourth equation, is resolved by negative logarithm rule, From this rule,

$$\ln\left(\frac{10}{20}\right) = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\log\left(\frac{1}{9}\right) = -\log 9$$

$$\ln\left(\frac{1}{5}\right) = -\ln 5$$



## Activity 2: Guided Practice



### Task 6:

Using the logarithmic rules, simplify and evaluate the following logarithmic expressions

1.  $\log_3(20 \times 5)$       2.  $\log\left(\frac{8}{4}\right)^3$       3.  $\log(50 \times 25)^5$       4.  $\ln\left(\frac{10}{20}\right)$



## Activity 3: Application

The food Technologist want to get the piece of food. If each person got one piece of food, the equation becomes  $150000 = 7(2)^t$



### Task 7:

- What is the type of the above equation?
- Demonstrate how to calculate t
- Determine the value of t

## Topic 1.3: Exponential equation



### Activity 1: Problem Solving



#### Task 8:

- a) What is an exponential function? How is it written?
- b) Consider functions  $f(x) = 2^x$  and  $g(x) = \left(\frac{1}{2}\right)^x$ , complete the following table

| $x$    | -2 | -1 | 0 | 1 | 2 |
|--------|----|----|---|---|---|
| $f(x)$ |    |    |   |   |   |
| $g(x)$ |    |    |   |   |   |

- c) Sketch the graph of function  $f(x)$  and  $g(x)$  from question b.
- d) What is domain and range of validity for exponential functions?
- e) Discuss whether  $\forall x \in \mathbb{R}, f(x) \in \mathbb{R}$  and deduce the domain of  $f(x)$
- f) Discuss whether  $f(x)$  can be negative or not and deduce the range of  $f(x)$
- g) What are the methods for solving exponential functions? What are their steps?

### Key Facts 1.3a: Solving exponential equation

#### • Definition

An exponential function is written as function in the form  $f(x) = a^x$ , with  $a \neq 0$  and  $a \neq 1$ . where  $a$  is the base and  $x$  is an exponent. An exponential function is also defined as  $\exp_a : \mathbb{R} \rightarrow \mathbb{R}_0^+ : x \rightarrow y = \exp_a x$ .

For simplicity we write  $\exp_a x = a^x$ . If the base  $a$  of exponential function  $f(x) = a^x$  is equal to  $e^x$  or  $f(x) = \exp(x)$ . The exponential function  $y = e^x$  has graph which is like the graph of  $f(x) = a^x$  where  $a > 1$

#### Example 1

Given exponential functions  $f(x) = 2^x$  and  $g(x) = \left(\frac{1}{2}\right)^x$ , the table below can be filled as:

| $x$ | -2 | -1 | 0 | 1 | 2 |
|-----|----|----|---|---|---|
|-----|----|----|---|---|---|

|        |               |               |   |               |               |
|--------|---------------|---------------|---|---------------|---------------|
| $f(x)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2             | 4             |
| $g(x)$ | 4             | 2             | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ |

These functions' graphs can be plotted as shown below:

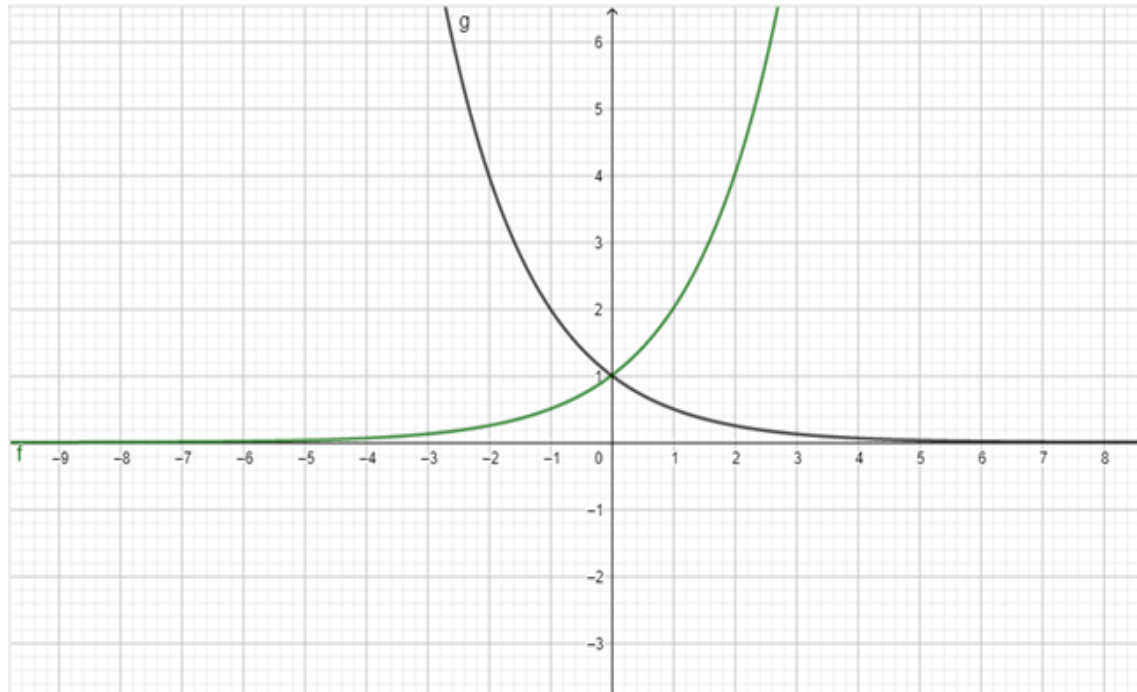


Figure 1: graph of  $f(x) = 2^x$  and  $g(x) = \left(\frac{1}{2}\right)^x$

From this, graph  $f(x)$  is increasing while  $g(x)$  decrease and they meet at point (0,1)

. It is very important to note that exponential function  $f(x) = a^x$  is:

- Ever zero
- Always positive
- Only taking value 1 when  $x=0$
- **Domain and range of exponential functions**

With the definition  $y = f(x) = a^x$  and the restrictions that  $a > 0$  and that  $a \neq 1$ , **the domain** of an exponential function is the set of all real numbers or  $]-\infty, +\infty[$ . The range is the set of all positive real numbers or  $]0, +\infty[$  because there is no exponent that can turn  $y = f(x) = a^x$  into a zero nor into a negative result.

**Note:** Generally, if  $u(x)$  is a defined function of  $x$ , the domain and range of  $f(x)$  depend on the values of  $x$  that make the expression valid and the resulting outputs.

**Example**

Determine the domain and the range of the function  $f(x) = 3^{\sqrt{2x}}$

**Solution**

Condition for the existence of  $\sqrt{2x}$  in  $\mathbb{R} : x \geq 0$ . Thus,  $Domf = [0, +\infty[$  and the range is  $[1, +\infty[$ .

**Example2**

Find the domain and the range of  $f(x) = 3^{\left(\frac{x+1}{x-2}\right)}$

**Solution**

Condition:  $x - 2 \neq 0 \Rightarrow x \neq 2$ . Thus  $Domf = \mathbb{R} - \{2\}$  and

the range is given by is  $]0, 3[ \cup ]3, +\infty[$ .

**Example3**

Find the domain and the range of  $f(x) = 4^{\sqrt{x^2-4}}$

**Solution**

Condition:  $x^2 - 4 \geq 0 \Rightarrow x \in ]-\infty - 2] \cup [2, +\infty[$ . Thus,  $Domf = ]-\infty - 2] \cup [2, +\infty[$  and the range is  $[1, +\infty[$

the function  $f(x) = a^{u(x)}$  will depend on  $u(x)$ .

- Solving exponential equations**

Equations that involves powers (exponents) as terms of their expressions are referred to as exponential equations. Such equations can sometimes be solved by 1) appropriately applying the properties of exponents or 2) introducing logarithms within expression.

**Solve exponential equations using one- to- one property (Using like bases)**

If an exponential equation can be written so that both bases are the same, the equation can be solved by comparing the exponents. To solve such exponential, follow the three steps:

- Rewrite both sides of the equation as an exponential expression with the same base.
- Since the bases are equal, then the exponents must be equal. Set the exponents equal to each other

c. Solve the equations and check answers

**Solve an exponential equation by taking logarithms for both sides**

To solve an exponential equation using logarithms, four steps are followed:

- Isolate the exponential expression
- Taking logarithms for both sides
- Rewriting exponential side as a linear expression
- Solving the obtained equation and check the answer.

**Key Facts 1.3b: Solving exponential equation using like bases and logarithm**

• **Using like bases**

Exponential equation 1:  $3^{x+2} = 27^x$  is solved as follows:

For this equation, the domain of validity  $x \in \mathbb{R}$ , therefore

$$3^{x+2} = 27^x \Rightarrow 3^{x+2} = (3^3)^x$$

$$3^{x+2} = 3^{3x}$$

$$x + 2 = 3x \Leftrightarrow 2 = 2x, \text{ then, } x = 1$$

Therefore, **solution is**  $s = \{1\}$

Exponential equation 2:  $4^{x-1} = 16$  is solved as follows:

Domain of validity  $x \in \mathbb{R}$ ,

$$4^{x-1} = 16^x \Rightarrow 4^{x-1} = (4)^2$$

$$4^{x-1} = 4^2 \Leftrightarrow x-1 = 2 \Rightarrow x = 3,$$

Therefore, **solution**  $s = \{3\}$

Exponential equation 3:  $e^{-x^2} = (e^x)^2 * \frac{1}{e^3}$  is solved as follows:

**Domain of validity:**  $x \in \mathbb{R}$

$$e^{-x^2} = e^{2x} * e^{-3}$$

$$e^{-x^2} = e^{2x-3} \Rightarrow -x^2 = 2x-3$$

$$\text{then, } x^2 + 2x - 3 = 0$$

The solution of this equation  $x^2 + 2x - 3 = 0$  gives  $x = -3$  or  $x = 1$  The solution set is  $s = \{-3, 1\}$

Exponential equation 4:  $2^{2y} + 3(2^y) = 4$  is solved as follows:

$$2^{2y} + 3(2^y) = 4 \Leftrightarrow (2^y)^2 + 3(2^y) = 4$$

**Let**  $2^y = x$ , then  $x^2 + 3x = 4$

$$\Rightarrow x^2 + 3x - 4 = 0$$

$$\Rightarrow x = -4 \text{ or } x = 1$$

Replacing the value of  $x$  in the equation  $2^y = x$

For  $x = -4$   $2^y = -4$  doesn't exist

For  $x = 1$ ,  $2^y = 1 \Leftrightarrow y = 0$

$$S = \{0\}$$

- **Using Logarithm**

Exponential equation 1:  $2e^{x+1} - 4 = 12$

Let first isolate the exponential expression,  $2e^{x+1} - 4 = 12$

$$2e^{x+1} = 16 \Leftrightarrow e^{x+1} = 8$$

Now, apply logarithm for both sides to get,  $\ln e^{x+1} = \ln 8$

$$(x+1)\ln e = \ln 8 \Rightarrow x+1 = \ln 8$$

$$x = \ln 8 - 1, x \approx 1.079$$

Therefore, solution  $S = \{1.079\}$

Exponential equation 2:  $e^x = 5$

*Domain of validity:*  $x \in \mathbb{R}$

$$\ln e^x = \ln 5$$

$$x \ln e = \ln 5 \quad \text{where } \ln e = 1$$

$$x = \ln 5$$

$$S = \{\ln 5\}$$

Exponential equation 3:  $10 + e^{0.1t} = 14$

Domain of validity:  $t \in \mathbb{R}$

$$e^{0.1t} = 4$$

$$\ln e^{0.1t} = \ln 4$$

$$0.1t = \ln 4$$

$$t = 10 \ln 4$$

$$S = \{10 \ln 4\}$$

Exponential equation 4:  $5 + 3^{t-4} = 7$

$$3^{t-4} = 2 \Rightarrow \ln 3^{t-4} = \ln 2$$

$$\Leftrightarrow (t-4) \ln 3 = \ln 2$$

$$\Rightarrow t = 4 + \frac{\ln 2}{\ln 3}$$

Exponential equation 5:  $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

$$3(2^{2x})^2 - 7(2^{2x}) + 4 = 0$$

$$3k^2 - 7k + 4 = 0 \Leftrightarrow \Delta = 1$$

Let  $2^{2x} = k$ , then  $\Rightarrow k = \frac{4}{3}, \text{ or } k = 1$

Replacing the value of k in the equation  $2^{2x} = k$ , for  $k = \frac{4}{3}$

$$2^{2x} = \frac{4}{3} \Leftrightarrow 2x \ln 2 \Leftrightarrow 2x \ln 2 = \ln 4 - \ln 3$$

$$\Rightarrow x = \frac{\ln 4 - \ln 3}{2 \ln 2}$$

$$\Rightarrow 2x \ln 2 = \ln 1$$

For  $k=1$ :  $x = 0$

$$S = \left\{ 0, \frac{\ln 4 - \ln 3}{2 \ln 2} \right\}$$



## Activity 2: Guided Practice



### Task 9:

Solve the given exponential functions below using appropriate method,

| Using like base                         | Using logarithm                    |
|-----------------------------------------|------------------------------------|
| 1. $3^{x+2} = 27^x$                     | 1. $2e^{x+1} - 4 = 12$             |
| 2. $4^{x-1} = 16$                       | 2. $e^x = 5$                       |
| 3. $e^{-x^2} = (e^x)^2 * \frac{1}{e^3}$ | 3. $10 + e^{0.1} = 14$             |
| 4. $2^{2y} + 3(2^y) = 4$                | 4. $5 + 3^{t-4} = 7$               |
|                                         | 5. $3(2^{4x}) - 7(2^{2x}) + 4 = 0$ |



## Activity 3: Application



### Task 10:

In a population growth model, the number of individuals in a species is given by the equation  $4^{2x} - 6.2^{4x} + 6.2^{2x} - 1 = 0$ , where  $x$  represents the time in years. Solve this equation to determine the critical points in time when the population reaches certain thresholds.

## Topic 1.4 Logarithmic equations



### Activity 1: Problem Solving



#### Task 11:

Answer the following questions:

1. What is a logarithmic equation?
2. How is a logarithmic equation solved?
3. For which value(s) of  $x$ , each function is defined, and use the properties for logarithm to determine the value of  $x$  in the given expressions:
  - a.  $\ln x = 10$
  - b.  $\ln x = 3$
  - c.  $\log x = 2$
  - d.  $\log(100x) = 2 + \log 4$
  - e.  $\log_2 x = -3$

### Key Facts 1.4: Solving logarithmic equations

- **Common logarithmic equations involving one unknown**

Logarithmic equation in  $\mathfrak{R}$  is the equation containing the unknown within the logarithmic expression. To solve logarithmic equations, the following steps are followed:

1. Set existence conditions for solution(s) of equation.
2. Express logarithms in the same base
3. Use logarithmic properties to obtain:
  - a.  $\log_a u(x) = \log_a v(x) \Leftrightarrow u(x) = v(x)$  where  $u(x)$  and  $v(x)$  are the functions in  $x$ .
4. Make sure that the value(s) of unknown verifies the conditions above. The properties of logarithms can be used to solve logarithmic equations.

**Example 1:** Solve the following equation  $\log_x 49 = 2$

**Solution:** condition of validity:  $x > 0, x \neq 1$

From the equation  $\log_x 49 = 2$  we change logarithmic equation to exponential equation to get  $x^2 = 49$

$x = \pm 7$  for  $x > 7$  therefore ,solution set is  $\{7\}$

**Example 2:** Solve each equation

- $\log_3(x+1) = \log_3 2$
- $\log_{x-2} 3 = 1$
- $\log_2(x+14) + \log_2(x+2) = 6$

**Solution:**

a.  $\log_3(x+1) = \log_3 2$

**Condition of existence :**  $x > 0 \Leftrightarrow x > -1$

**Then,**  $\log_3(x+1) = \log_3 2$  (*simplify*) to obtain

$$\Leftrightarrow x+1 = 2 \quad \text{Then, } x = 1, S = \{1\}$$

b.  $\log_{x-2} 3 = 1$

**Condition :**  $x-2 > 0 \Leftrightarrow x > 2$  and  $x-2 \neq 1 \Leftrightarrow x \neq 3$  this means

$$\log_{x-2} 3 = 1 \Leftrightarrow \log_{x-2}(x-2)$$

$$x \in ]2, 3[ \cup ]3, +\infty[, \text{Then, } \Leftrightarrow 3 = x - 2$$

$$\Leftrightarrow x = 5, S = \{5\}$$

c.  $\log_2(x+14) + \log_2(x+2) = 6$

**Condition:**  $x+14 > 0$  and  $x+2 > 0 \Leftrightarrow x > -14$  and  $x > -2$

$$\Leftrightarrow x \in ]-2, +\infty[$$

$$\log_2(x+14) + \log_2(x+2) = 6 \Leftrightarrow \log_2(x+14)(x+2) = 6 \log_2 2$$

$$\Leftrightarrow \log_2(x+14)(x+2) = 6 \log_2 2^6$$

$$\Leftrightarrow (x+14)(x+2) = 64$$

$$x^2 + 16x + 28 - 64 = 0 \Leftrightarrow x^2 + 16x - 36 = 0$$

$$\Leftrightarrow (x+18)(x-2) = 0$$

$$\Leftrightarrow x = 2 \text{ or } x = -18, S = \{2\}$$

- Systems of equations involving common logarithms**

In solving systems of equations using logarithms, like one-variable logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the logarithms in ways that make it easier to solve

for the variables. Algebraic procedures like substitution and elimination can be used in the creation of a one-variable equation that is simple to solve.

**Example:** solve the following system of equations

$$\begin{cases} \log x + \log y = 1 \\ \log(10x) - \log y = 2 \end{cases}$$

### Solution

Let us apply elimination method by eliminating the variable  $y$ ,

$$\begin{cases} \log x + \log y = 1 \\ \log(10x) - \log y = 2 \end{cases} \Leftrightarrow \begin{cases} \log x + \log y = 1 \\ \log(10x) - \log y = 2 \Leftrightarrow \log x + \log(10x) = 3 \end{cases}$$

Use logarithmic property of addition to solve for  $x$  the new equation

$$\log x + \log(10x) = 3 \Leftrightarrow \log(10x * x) = 3$$

$$\log 10x^2 = 3 \Leftrightarrow \log 10x^2 = 3 \log 10$$

$$\Rightarrow \log 10x^2 = \log 10^3$$

$$10x^2 = 10^3 \Rightarrow x = \pm 10$$

Since for  $-10$  is undefined, we reject it and keep  $x = 10$

### • Natural logarithmic equations

The Natural logarithmic equation in  $\mathbb{R}$  is the equation containing the unknown within the natural logarithmic expression. To solve Natural logarithmic equations, the following steps are followed:

- Set existence conditions for solution(s) of equation.
- Use logarithmic properties to obtain:  
 $\ln u(x) = \ln v(x) \Leftrightarrow u(x) = v(x)$ , where  $u(x)$  and  $v(x)$  are functions in  $x$
- Make sure that the value(s) of unknown verifies the conditions above.
- $y = \ln x = \log_e x \Leftrightarrow e^y = x$  (as inverse) The equation  $\ln x = 1$  has a unique solution, the rational number 2.71828182845904523536...and this number is represented by letter  $e$ . The properties of natural logarithms can be used to solve natural logarithmic equations.

**Example 1:**

Solve each equation

a.  $\ln x - \ln 5 = 0$

b.  $3 + 2\ln x = 7$

c.  $\ln 2x + \ln(x + 2) = \ln 6$

**Solution**

a.  $\ln x - \ln 5 = 0$

condition of validity:  $x > 0$ 

Then,  $\ln x = \ln 5 \Leftrightarrow x = 5$

Hence, the solution  $S = \{5\}$ 

b.  $3 + 2\ln x = 7$

condition of validity:  $x > 0$ , By solving:

$$2\ln x = 7 - 3 \Leftrightarrow 2\ln x = 4 \Leftrightarrow \ln x = 2 \Leftrightarrow x = e^2$$

$$S = \{e^2\}$$

c.  $\ln 2x + \ln(x + 2) = \ln 6$

Conditions of validity:  $2x > 0$  and  $x + 2 > 0$ 

$\Rightarrow x > 0$  and  $x > -2$

Domain of validity:  $] -2, +\infty[ \cap ]0, +\infty[ \Leftrightarrow ]0, +\infty[$ **Solving:**

$$\Leftrightarrow \ln 6 = \ln[2x(x + 2)]$$

$$\Leftrightarrow 2x(x + 2) = 6$$

$$\Leftrightarrow 2x^2 + 4x - 6 = 0$$

$$\Leftrightarrow x^2 + 2x - 3 = 0$$

$$\ln 2x + \ln(x + 2) = \ln 6 \Rightarrow (x - 1)(x + 3) = 0$$

$$x = 1 \text{ or } x = -3$$

**As**  $x \in ]0, +\infty[$ , **then**  $S = \{1\}$

- Systems of equations involving natural logarithms**

In solving systems of equations using natural logarithms, like one-variable natural logarithmic equations require the same set of techniques like logarithmic identities and

exponents, which help to rephrase the natural logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can be used in the creation of a one-variable equation that is simple to solve.

**Example1:** solve the following system of equations

$$\begin{cases} \ln(xy) = 7 \\ \ln\left(\frac{x}{y}\right) = 1 \end{cases}$$

**Solution**

$$\begin{cases} \ln(xy) = 7 \\ \ln\left(\frac{x}{y}\right) = 1 \end{cases} \Leftrightarrow \begin{cases} \ln x + \ln y \\ \ln x - \ln y \end{cases}$$

solve the system of equations started eliminating variable  $y$

$$\begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases} \Leftrightarrow 2 \ln x = 8$$

$$\Rightarrow \ln x = 4 \Rightarrow x = e^4$$

replacing the value of  $x$  in equation (1),  $\ln e^4 + \ln y = 7$  then by solving this equation

$$\begin{aligned} \ln e^4 + \ln y &= 7 \\ \Rightarrow 4 \ln e + \ln y &= 7 \end{aligned}$$

the solution for  $y$  is given by:  $\Rightarrow 4 + \ln y = 7$

$$\begin{aligned} \Rightarrow \ln y &= 7 - 4 \Leftrightarrow \ln y = 3 \\ y &= e^3 \end{aligned}$$

Therefore the solution of the system is.  $S = \{(e^4, e^3)\}$

**Example2:** Solve the following:

$$\begin{cases} 3 \ln x + 2 \ln y = 144 \\ 4 \ln x + \ln y = 152 \end{cases}$$

solve the system of equations started eliminating variable  $y$

$$\begin{cases} 3 \ln x + 2 \ln y = 144 \\ 4 \ln x + \ln y = 152 \end{cases}$$

By multiplying 1 to the first equation and -2 to the second equation of the system we will get

$$\begin{cases} 3\ln x + 2\ln y = 144 \\ -8\ln x - 2\ln y = -304 \end{cases} \Leftrightarrow -5\ln x = -160$$

$$\Rightarrow \ln x = \frac{160}{5} \Leftrightarrow \ln x = 32$$

$$\log_e x = \ln x, \text{ so } \Leftrightarrow \log_e x = 32$$

Remember that  $\Rightarrow x = e^{32}$

$$3\ln x + 2\ln y = 144$$

$$\Rightarrow 3\ln e^{32} + 2\ln y = 144$$

$$\Rightarrow 3 \times 32 \ln e + 2\ln y = 144$$

$$\Rightarrow 96 \ln e + 2\ln y = 144$$

$$\Rightarrow 96 + 2\ln y = 144$$

$$\Rightarrow 2\ln y = 144 - 96$$

$$\Rightarrow 2\ln y = 48 \Leftrightarrow \ln y = \frac{48}{2}$$

replacing the value of  $\ln y$  in equation(1)

$$\ln y = 24 \Leftrightarrow y = e^{24}$$

**Therefore, the solution of the system is:  $S = \{e^{32}; e^{24}\}$**



## Activity 2: Guided Practice



### Task 12:

For each of the following function solve for  $x$ .

- $\log(x + 2) = 2$
- $\log x + \log(x^2 + 2x - 1) - \log 2 = 0$
- $\log(35 - x^3) = 3\log 5 - x$
- $\log(3x - 2) + \log(3x - 1) = \log(4x - 3)^2$
- $\log(1 - x) = -1$

### Notes:

-For any clarifications you may contact your trainer.

-You may also refer to the **key facts 1.4** in this manual.



### Activity 3: Application



#### Task 13:

1. In a biological study, the growth of bacteria population is modeled by the following logarithmic functions, where  $t$  represents the time in days. Determine for which value(s) of  $t$  each function is defined to understand the time periods during which the population growth can be accurately modeled.

a.  $f(t) = \ln(t + 4)$

b.  $f(t) = \ln t$

c.  $f(t) = \ln(t^2 - 5t + 16)$

2. Solve the following system: In an environmental study, the relationship between two species' populations is modeled by a system of logarithmic equations, where  $x$  and  $y$  represent the populations of the two species. Solve the system to find the equilibrium populations:

$$\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$$



### Formative Assessment

1. Write each of the following in logarithmic form

a.  $4^3 = 64$

b.  $2^{-3} = \frac{1}{8}$

c.  $\left(\frac{1}{2}\right)^x = y$

d.  $5^{-p} = q$

2. Use the properties of logarithms to rewrite each expression as a single logarithm:

a.  $2\log_b x + \frac{1}{2}\log_b (x + 4)$

b.  $4\log_b (x + 2) - 3\log_b (x - 5)$

c.  $\log_b \left( \frac{x\sqrt{y}}{z^5} \right)$

3. Find the numerical value

- a.  $\log_2 32$
  - b.  $\log_4 8$
  - c.  $\log_6 7$
  - d.  $\log_5 \sqrt{125}$
  - e.  $\log_5 0.008$
  - f.  $\log_9 10$
4. Find the exact value of  $x$ , showing your working steps:
- a.  $\log_2 8 = x$
  - b.  $\log_x 125 = 3$
  - c.  $\log_x 64 = 0.5$
  - d.  $\log_4 64 = x$
  - e.  $\log_9 x = 3$
5. Solve the following exponential equations
- a.  $2^{3x} = 3^{2x-1}$
  - b.  $3^{x+2} = 27^x$
  - c.  $2e^{2x} - e^x - 6 = 0$
  - d.  $e^x - 12 = \frac{-5}{e^{-x}}$
6. Find the domain of definition of the functions:
- a.  $f(x) = \ln(x+1)$
  - b.  $f(x) = \frac{1 + \ln x}{-1 + \ln x}$
7. Solve the following in real number:
- $$\log x + \log(x^2 + 2x - 1) - \log 2 = 0$$
8. Evaluate the value of  $x$  and  $y$  for the following system:
- a. 
$$\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$$
  - b. 
$$\begin{cases} \ln(xy) = 7 \\ \ln y \left( \frac{x}{y} \right) = 1 \end{cases}$$



## Points to Remember

- The exponential equation form is  $f(x) = a \cdot b^x$  where  $a$  is the initial amount,  $b$  is the base, and  $x$  is the exponent.
- The natural exponential function,  $e^x$ , is especially important due to its unique properties.
- The logarithm is the inverse of exponential function, if  $f(x) = b^x$ , then  $x = \log_b(y)$
- Common logarithms ( $\log_{10}$ ) and natural logarithms ( $\ln$ , base  $e$ ) are widely used.
- Use logarithms to solve exponential equations and vice versa.
- Practice isolating the variable you need to solve for.
- Before solving any exponential and logarithmic expressions, think of the appropriate rule/property to use for simplification
- Work on a variety of problems to get comfortable with these concepts.
- Don't use calculators while solving exponential and logarithmic expressions



## Self-Reflection

1. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

| Experience                              | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-----------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes        |                                          |                             |                                    |                                       |                                          |
| Define properly exponential expressions |                                          |                             |                                    |                                       |                                          |
| Isolate exponential terms correctly     |                                          |                             |                                    |                                       |                                          |

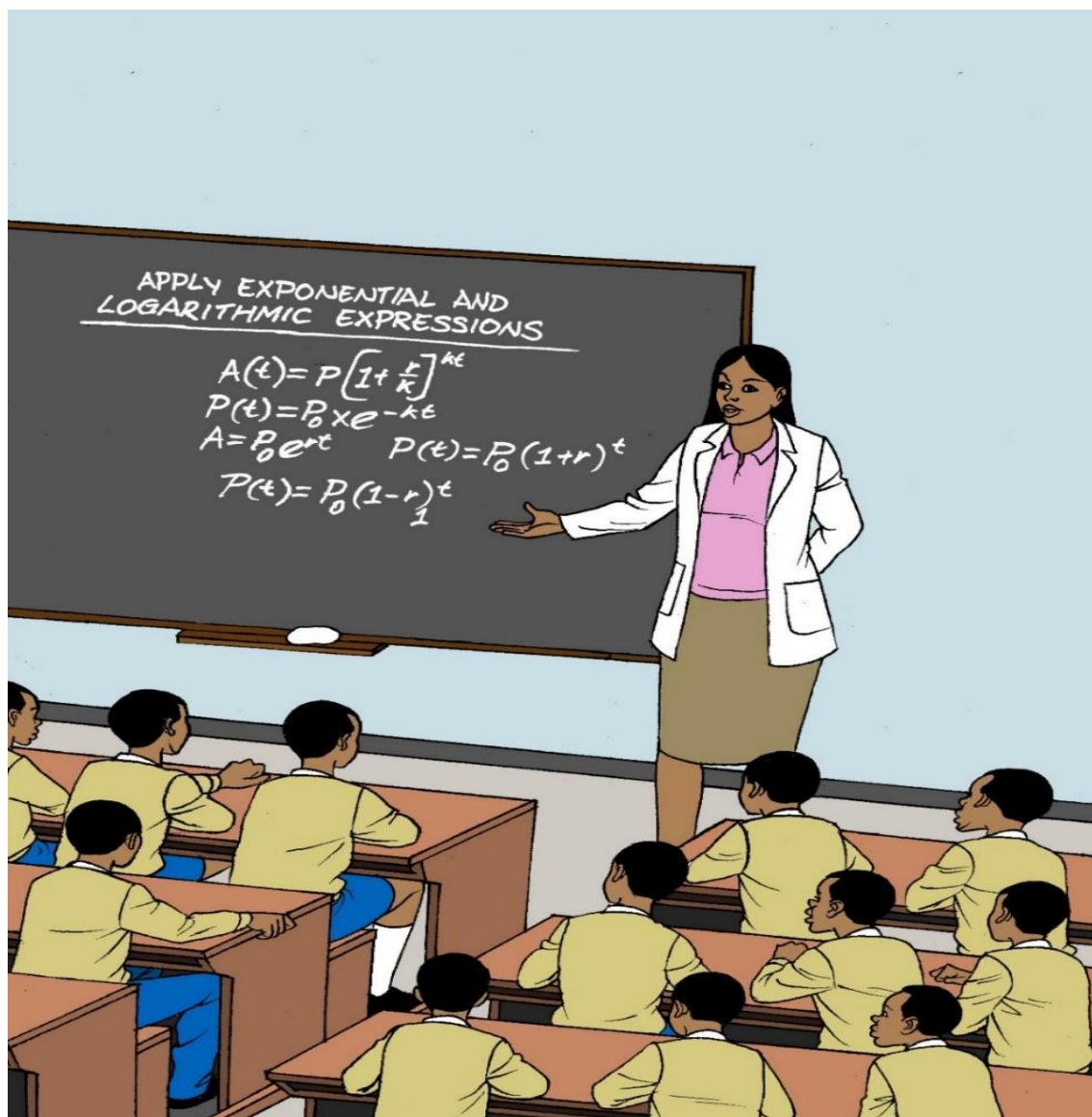
| Experience                                                  | I do not                        | I know a           | I have                      | I have a lot             | I am                                |
|-------------------------------------------------------------|---------------------------------|--------------------|-----------------------------|--------------------------|-------------------------------------|
| Knowledge, skills, and attitudes                            | have any experience doing this. | little about this. | some experience doing this. | of experience with this. | confident in my ability to do this. |
| Define properly logarithmic expression                      |                                 |                    |                             |                          |                                     |
| Isolate logarithmic terms correctly                         |                                 |                    |                             |                          |                                     |
| Identify properties of exponential expressions              |                                 |                    |                             |                          |                                     |
| Identify properties of logarithmic expressions              |                                 |                    |                             |                          |                                     |
| Apply properties of logarithmic expression                  |                                 |                    |                             |                          |                                     |
| Describe different technics to solve exponential equations  |                                 |                    |                             |                          |                                     |
| Solve exponential equations based on their properties.      |                                 |                    |                             |                          |                                     |
| Describe different technics to solve logarithmic equations. |                                 |                    |                             |                          |                                     |

| Experience                                                                        | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-----------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                  |                                          |                             |                                    |                                       |                                          |
| Solve exponential expressions and logarithmic equations based on their properties |                                          |                             |                                    |                                       |                                          |

2. Fill in the table above and share results with the trainer for further guidance.

| Areas of strength | Areas for improvement | Actions to be taken to improve |
|-------------------|-----------------------|--------------------------------|
| 1.                | 1.                    | 1.                             |
| 2.                | 2.                    | 2.                             |
| 3.                | 3.                    | 3.                             |

## UNIT 2: APPLY EXPONENTIAL AND LOGARITHMIC EXPRESSIONS.



### Unit summary

This unit provides you with the knowledge, skills and attitudes required to apply exponential equations and logarithmic Expressions. It covers calculation of compound interest, calculation of population growth and calculation of population decay.

## Self-Assessment: Unit2

1. Answer the following questions:

- What is compound interest?
- Given the formula for calculating the compound interest below, what does each letter stands for?

$$A = P \left( 1 + \frac{r}{n} \right)^{nt}$$

- What is population growth and its importance?
  - What is the formula for calculation of population growth?
  - What is population decay?
  - What is the formula for calculation of population decay?
2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.

- There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
- Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
- Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

3. At the end of this unit, you'll assess yourself again.

| My experience                           | I don't have any experience doing this. | I know a little about this. | . I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|-----------------------------------------|-----------------------------------------|-----------------------------|-------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes         |                                         |                             |                                     |                                       |                                          |
| Define compound interest                |                                         |                             |                                     |                                       |                                          |
| Define principal value and future value |                                         |                             |                                     |                                       |                                          |

| My experience                                                                             | I don't have any experience doing this. | I know a little about this. | . I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|-------------------------------------------------------------------------------------------|-----------------------------------------|-----------------------------|-------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes                                                           |                                         |                             |                                     |                                       |                                          |
| Analyse problems deal about compound interest                                             |                                         |                             |                                     |                                       |                                          |
| Pay attention to details while calculating compound interest                              |                                         |                             |                                     |                                       |                                          |
| Analyse correctly the formula for calculating principal value and future value            |                                         |                             |                                     |                                       |                                          |
| Think logically and critically when applying formula of compound interest.                |                                         |                             |                                     |                                       |                                          |
| Apply formulas for calculating any problem related to compound interest                   |                                         |                             |                                     |                                       |                                          |
| Pay attention about one's savings                                                         |                                         |                             |                                     |                                       |                                          |
| Describe different technics to solve problems related to calculation of population growth |                                         |                             |                                     |                                       |                                          |
| Solve problem related to calculation of population growth                                 |                                         |                             |                                     |                                       |                                          |

| My experience                                                                                                                  | I don't have any experience doing this. | I know a little about this. | . I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|--------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------|-----------------------------|-------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes                                                                                                |                                         |                             |                                     |                                       |                                          |
| Describe different technics to solve problems related to calculation of population decay                                       |                                         |                             |                                     |                                       |                                          |
| Solve problems related to calculation of population decay                                                                      |                                         |                             |                                     |                                       |                                          |
| Think logically and critically when applying formula for solving problems related to calculation of population growth or decay |                                         |                             |                                     |                                       |                                          |
| Analyse correctly the difference between rate of growth and rate of decay                                                      |                                         |                             |                                     |                                       |                                          |
| Developmental constructions                                                                                                    |                                         |                             |                                     |                                       |                                          |



## Key Competencies:

| Knowledge                                                                                    | Skills                                                                            | Attitudes                                                                                                                         |
|----------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------|
| 1. Define compound interest                                                                  | 1. Analyse problems deal about compound interest                                  | 1. Pay attention to details while calculating compound interest                                                                   |
| 2. Define principal value and future value                                                   | 2. Analyse correctly the formula for calculating principal value and future value | 2. Think logically and critically when applying formula of compound interest.                                                     |
| 3. Describe different technics to solve problems related to calculation of population growth | 3. Apply formulas for calculating any problem related to compound interest        | 3. Developmental constructions                                                                                                    |
| 4. Describe different technics to solve problems related to calculation of population decay  | 4. Solve problem related to calculation of population growth                      | 4. Pay attention about one's savings                                                                                              |
|                                                                                              | 5. Solve problems related to calculation of population decay                      | 5. Think logically and critically when applying formula for solving problems related to calculation of population growth or decay |
|                                                                                              | 6. Analyse correctly the difference between rate of growth and rate of decay      | 6.                                                                                                                                |



### Discovery activity:

#### Scenario 1:

Imagine you are a Restaurant worker and you want to analyze how the Restaurant will be in 5 years in three aspects, namely 1) Investment 2) A Growing Population of clients need the fish 3) Decay in Population of client do not need the red meats. A Restaurant has an initial amount of 100,000,000 frw which is invested in bank at an annual interest rate of 3%, compounded monthly. A restaurant has an initial population of 2,500 need fish that grows at a rate of 2% per year and of 50,000 clients eating meats decaying at a rate of 3.5% per year.



#### Task 14:

1. What is the appropriate term for the initial amount of money a Restaurant has in the bank?
2. What is the formula used to calculate the final amount of money?
3. How much money will a Restaurant be having after 5 years?
4. What will be the number of clients need the fish in 5 years?
5. What will be the number of clients who eats red meats in 5 years?

## Topic 2.1: Compound interest



### Activity 1: Problem Solving

Three (3) years ago, X bank marketing team reached out to Mr. MUSINGUZI, a waiter at X HOTEL and they explained about a new saving account with compound interest aimed at encouraging people to save money for future. As Mr. MUSINGUZI was planning to open his quick food restaurant in the future, he immediately opened that account and deposited 2,000,000 frw. After three years he went to the bank to withdraw the money from this bank account and was surprised when he was told that, with all interests, his money has grown up to 8,000,000 frw.



### Task 15:

1. What does the compound interest in the above scenario mean?
2. Explain how Mr. MUSINGUZI can discover that power which raised his money.
3. What is the total interest earned over the 3 years?
4. How much interest does Mr. MUSINGUZI earns each year?
5. What was the growth factor  $(1 + r)$  over the 3 years?
6. What would the final amount be if Mr. MUSINGUZI left his money for 5 years instead of 3?

### Key Facts 2.1a: Calculation of Compound interest

- **Description of Compound interest**

To understand what Compound interest, let's first start with knowing what Interest is. Interest is the compensation that one gets for lending a certain asset.

For Instance, suppose that you put some money in a bank account for a year. To reward you for that, it pays you your money plus extra amount, the extra amount is called 'Interest'. There are two types of interest namely 1) Simple interest and 2) Compound interest

#### 1. Simple interest

Simple interest is the interest calculated using the principal only and does not include compounding interest.

#### 2. Compound interest

Compound interest is the interest calculated on the initial principal and also on the accumulated interest of previous periods. Essentially, it allows your money to grow at an increasing rate over time, as the interest earned in each period is added to the principal for future interest calculations. The compounding frequency, which is how often the interest is calculated, can be annual, semi-annually, quarterly, monthly, or even daily, depending on the terms of the investment or loan.

- **Terminologies used in interest calculation**

- 1. Principle value:**

The Principal Value (denoted as **P**) is the initial amount of money that is invested, borrowed, or deposited, before any interest is added. It serves as the foundation upon which interest is calculated in both simple and compound interest scenarios.

Example:

If you invest \$1,000 in a savings account, the \$1,000 is the principal.

If you borrow \$5,000 from a lender, the \$5,000 is the principal.

- 2. Interest rate.**

Interest rate (denoted as  $r$ ) is the amount of interest expressed as proportion of the principal or as percentage.

- 3. Time:**

Time (denoted as  $t$ ) refers to the duration (in years) for which the money is invested, borrowed, or left to grow. It plays a crucial role in determining how much the principal will grow over time, as interest compounds periodically.

- 4. Rate of change:**

The rate of change in the context of compound interest measures how the value of an investment or loan grows over time due to compounding. It quantifies how the accumulated amount (future value) changes as time or other variables (e.g., interest rate, compounding frequency) change.

- 5. Future value:**

The Future Value (FV) is the value of an investment or loan at a specific point in the future, including the accumulated interest or returns. It shows how much money your principal will grow to, given a specific rate of return and time period.

- **Formulas for calculation of interest**

- 1. Simple interest**

The formula for simple interest (SI) is straightforward:  $SI = P * r * t$

Where:

**P** = Principal amount (initial investment)

**r** = Annual interest rate (in decimal form, e.g., 5% = 0.05)

**t** = Time of investment.

## 2. Compound interest

Calculating interest using compound interest methods consists in charging the interest on the principal plus any accumulated interest. Calculating interest using compound interest methods one may need to use the following formulas:

**2.1. For interest compounded annually:**  $A = P \times (1 + r)^t$ ,

Where

**A** = Total amount after interest (principal + interest)

**P** = Principal amount (initial investment)

**r** = Annual interest rate (in decimal form, e.g., 5% = 0.05)

**t** = Time in years

**2. If the interest is compounded more frequently than annually:**

$$A = P \times \left(1 + \frac{r}{n}\right)^{n \times t},$$

Where

**A** = Total amount after interest (principal + interest)

**P** = Principal amount (initial investment)

**r** = Annual interest rate (in decimal form, e.g., 5% = 0.05)

**t** = Time in years

**n** = Number of times interest is compounded per year

### Key Facts 2.1b: Calculation of Compound interest

- **Solution for question 1**

Given data:

i.  $t = 2 \text{ years}$

$$SI = 10,000 \times 0.09 \times 2$$

$$\Rightarrow SI = 1,800$$

$$\text{Amount or Accumulated: } A = P + SI \Rightarrow 10,000 + 1,800 = 11,800 \text{ Frw}$$

ii.  $t = \frac{1}{2} \text{ year} \quad t = 0.5 \text{ year}$

$$SI = 10,000 \times 0.5 \times 0.09 = 450 \text{ Frw}$$

iii.  $r = 9\% = 9/100 = 0.09$

iv.  $P = 10,000 \text{ Frw}$

$$SI = P \times R \times T$$

- **Solution for question 2**

Given data:

$$P = 500,000 \text{ Frw}$$

$$t = 6 \text{ years}$$

$$I = 150,000 \text{ Frw}$$

*i. When compound interest is used.*

$$A = P(1 + r)^t$$

$$A = P + I \Leftrightarrow 500,000 + 150,000 \Rightarrow 650,000 \text{ Frw}$$

$$(1 + r)^t = \frac{A}{P} \Leftrightarrow (1 + r)^6 = \frac{650,000}{500,000} \Leftrightarrow (1 + r)^6 = 1.3$$

**Apply log for both side to get:**

$$\log(1+r)^6 = \log 1.3$$

$$6\log(1+r) = 0.1139$$

$$\log(1+r) = \frac{0.1139}{6}$$

$$\log(1+r) = 0.0189$$

$$1+r = 10^{(0.0189)} \Leftrightarrow 1+r = 1.0439$$

$$\Rightarrow r = 1.0439 - 1 \Leftrightarrow r = 0.0439$$

$$\Rightarrow r = \% \Leftrightarrow 0.0439 \times 100$$

$$\Rightarrow r = 4.4\%$$

**ii. When simple interest is used,**

$$SI = P \times r \times t$$

$$150,000 = 500,000 \times 6 \times r \Leftrightarrow 150,000 = 3,000,000 \times r$$

$$\Rightarrow r = \frac{150,000}{3,000,000} \Leftrightarrow r = 0.05, r = 0.05 \times 100$$

$$\Leftrightarrow r = 5\%$$

• **Solution for question 3**

$$FV = 1000 \times \left(1 + \frac{0.05}{1}\right)^{1 \times 3}$$

$$FV = 1000 \times (1.05)^3$$

$$FV = 1000 \times 1.157625 \Leftrightarrow 1157.63$$

Hence the future value is **\$1,157.63**

• **Solution for question 4**

- $P=5,000$
- $r=0.06$
- $n=12$  (compounded monthly)
- $t=5$  years

$$FV = 5,000 \left(1 + \frac{0.06}{12}\right)^{12 \cdot 5} \Leftrightarrow 5,000(1 + 0.005)^{60}$$

$$\Rightarrow 5,000(1.005)^{60} \Leftrightarrow FV = 6,744.25 \text{ Frw}$$

Hence the future value is **6,744.25 Frw.**



## Activity 2: Guided Practice



### Task 16:

Using the formulas in key fact 2.1a, answer the following questions

2. How much Interest do you get if you put 10,000 FRW in a saving account that pays simple interest at 9% per annum for two (2) years?
  - i. How many interest if you invest in the account in 2 years?
  - ii. How many interest if you invest only the money in the account for half a year?
3. Suppose 500,000Frw principal earns 150,000Frw interest after 6 years.
  - i. If compound interest is used, what was the interest rate?
  - ii. What if the simple interest is used?
4. A restaurant worker saves \$1,000 in a restaurant savings plan that offers an annual interest rate of 5% compounded annually. What will the amount grow to in 3 years?
5. You deposit 5,000 in a bank account earning 6% interest per year, compounded monthly. How much will you have after 5 years?



## Activity 3: Application



### Task 17:

Ubwiza motel needs 1,500,000,000 frw to expand its business by renovating its kitchen and the bar and restaurant as well as increasing number of items in the bar and on its menu which will attracts more clients. They have 300,000,000 and plan to invest it at an annual interest rate of 7.5% compounded annually. How long will it take to reach their goal?

## Topic 2.2: Calculation of population growth



### Activity 1: Problem Solving



#### Task 18:

X Restaurant do rabbits farming intended for their kitchen. Currently the restaurant has an initial population of 1,000 rabbits. If the rabbit population grows at a steady rate of 5% per year. Let's calculate the population at different time intervals and explore the relationship between the components.

- What is the initial population of rabbits in the restaurant?
- What does the growth rate of 5% per year mean?
- How many years will we track the population to observe its growth?
- What will the population be after 1 year? (Use the formula  $P = P_0xe^{rt}$ , where  $e \approx 2.718$ ).
- What happens to the population as time increases?
- If the growth rate decreases to 2% per year, will the population grow faster or slower?
- Why is it important to know the rate of growth and time of growth?

### Key fact 2.2a: Calculation of population growth

- **Description of Population growth**

**Population Growth** refers to the change in the number of individuals in a population over time. It is a critical aspect of demography and is influenced by factors such as birth rates, death rates, immigration, and emigration.

- **Terminologies used in population growth calculation**

1. **Initial Population**

The initial population (denoted as  $P_0$ ) is the starting number of individuals in a population at the beginning of a given time period.

Example:

If a town has 1,000 residents in the year 2000, then 1,000 is the initial population.

## 2. Future Population

The future population (denoted as  $P$ ) is the number of individuals in the population at a specific time in the future. The future population can be estimated based on the growth rate and the time period of interest.

### 3. Rate of Growth.

The rate of growth (denoted as  $r$ ) is the percentage or fraction that indicates how fast the population is increasing (or decreasing) per unit of time. A positive rate ( $r > 0$ ) indicates population growth, while a negative rate ( $r < 0$ ) suggests population decline. Growth rate is often expressed as an annual percentage, e.g., 2% per year ( $r = 0.02$ ).

### 4. Time of Growth (t):

The time of growth (denoted as  $t$ ) is the period over which the population changes, typically measured in years.

Example:

If you want to calculate the population after 10 years,  $t = 10$

- **Formulas for calculation of population growth**

The formula for population growth calculation is  $P = P_0 \times e^{rt}$

Where  $e$  is the base of natural logarithms (approximately 2.718)

- **Formulas for doubling time (T)**

The formula for doubling time is:  $T = \frac{\ln(2)}{r}$

## 2.2b: Calculation of population growth

- **Solution for question1.**

a. In the given example, in 1960,  $P_0 = 10,000$ . Thus, in 1960, we have  $P_{10} = 12,000$  while in 1980, we have  $P_{20}$ . So,

$$P_{10} = 12,000 \Leftrightarrow 12,000 = 10,000 \cdot e^{r \cdot 10}, \text{By applying natural logarithms both side}$$

$$\ln 12,000 = \ln 10,000 \times e^{10r}$$

will get  $\ln 12,000 = \ln 10,000 + \ln e^{10r}$ , here we will need to apply properties

$$\ln 12,000 - \ln 10,000 = \ln e^{10r}$$

of logarithms ,where  $\ln a^x = x \ln a$  ,then we will get

$$\ln 12,000 - \ln 10,000 = 10r \ln e$$

$$\Rightarrow \ln 12,000 = 9.39266$$

$$\Rightarrow \ln 10,000 = 9.21034$$

$$\Rightarrow \ln e = 1, \text{Then}$$

$$9.39266 - 9.21034 = 10r$$

$$\Leftrightarrow 0.18232 = 10r, \Rightarrow r = 0.0182$$

The number of oranges in 1980 is given by  $P_{20} = 10,000 \times e^{0.0182 \times 20}$ .  
 $\Rightarrow P_{20} = 14,390$

The number of oranges in 1980 is **14390**.

- b.** The doubling time for the number of oranges means the time for which  $P_n = 2P_0$

$$\Rightarrow 2P_0 = P_0 \times e^{rt} \Leftrightarrow e^{rt} = 2$$

$$\Rightarrow \ln e^{rt} = \ln 2, rt \ln e = \ln 2$$

$$t = \frac{\ln 2}{r}, t \Rightarrow \frac{\ln 2}{0.0182} \Leftrightarrow t = 38 \text{ years}$$

Hence, the doubling time for the number of oranges is **38 years**.

• **Solution for question 2**

- a. We are given the formula for the population of the island:  $P(t) = 4200 \times 1.04^t$   
 where t is the number of years after January 1, 2016. On January 1, 2016, t=0:

$$P(0) = 4200 \times 1.04^0 = 4200 \times 1 \Rightarrow 4200. \text{ Hence, the population was } \mathbf{4200}$$

- b.** We know that  $P(t) = P_0 \times e^{rt}$ , and we are given that  $P(t) = 4200 \times 1.04^t$  by  
 comparing these two equation we will get,  $P_0 \times e^{rt} = 4200 \times 1.04^t$  this means  
 that  $P_0 = 4200 \Leftrightarrow e^{rt} = 1.04^t$  after re-arrangement one can get  $e^{rt} = 1.04^t$   
 ,apply natural logarithm for both side to get

$$\ln e^{rt} = \ln 1.04^t \Leftrightarrow rt \ln e = t \ln 1.04$$

$$\Rightarrow rt = t \ln 1.04 \Leftrightarrow r = \frac{t \ln 1.04}{t} \Rightarrow t = \ln 1.04 \approx 0.04 \Leftrightarrow r = 4\%$$

- c.** On January 1, 2021,  $t = 2021 - 2016, t = 5$ , then  $P(5) = 4200 \times 1.04^5$ , We can  
 calculate this step by step:

$$P(5) = 4200 \times 1.04^5 \Leftrightarrow 4200 \times 1.21665$$

$$\Rightarrow 5,109.942 \approx 5110$$

So, the population on January 1, 2021, was approximately **5110**.

• **Solution for question3**

**a. Data:**

- i. Initial population=10,000
- ii. Final population=15,000
- iii. Time =5 years
- iv. Asked rate of growth, this is given from the following relation

$P(t) = P_0 \times e^{rt}$ , by applying natural logarithm for both side we will get

$$\ln P(t) = \ln P_0 \times e^{rt} \Rightarrow \ln P(t) = \ln P_0 + \ln e^{rt} \Rightarrow \ln P(t) - \ln P_0 = rt \ln e \Rightarrow \ln \left( \frac{P(t)}{P_0} \right) = rt$$

$$\text{, then } r = \left( \frac{\ln \frac{P(t)}{P_0}}{t} \right) \Rightarrow r = \left( \frac{\ln \frac{15,000}{10,000}}{5} \right) = \frac{\ln 1.5}{5} \approx \frac{0.4055}{5} \approx 0.0811 \approx 8.11\%$$

$$\text{b. } t = \left( \frac{\ln \frac{12,000}{8,000}}{0.04} \right) = \frac{\ln(1.5)}{0.04} \approx \frac{0.4055}{0.04} = 10 \text{ years}$$

**c. Future number of cocks is given by**

$$P(t) = P_0 \times e^{rt} \Rightarrow P(5) = 5,000 \times e^{0.03 \times 5} \approx 5809$$

so after 5 years the number of cocks will be **5809**.

• **Solution for question 4**

**Given data:**

- i.  $P_0=10,000$  (population in 2000),
- ii.  $P(t)=12,000$  (population in 2010),
- iii.  $t=10$  years (from 2000 to 2010).

**a. Estimating population in 2020**

using formula:  $P(t) = P_0 e^{rt}$ , we can find the rate of growth, hence

$$12,000 = 10,000 e^{10r} \text{ Divide both sides by 10,000 to get:}$$

$1.2 = e^{10r}$  take the natural logarithm (ln) of both sides:

$$\ln(1.2) = \ln(e^{10r}) \Rightarrow \ln(1.2) = 10r, \text{ solve for } r$$

$$r = \frac{\ln(1.2)}{10} \approx \frac{0.1823}{10} = 0.01823 (\text{per year}), \text{ so population in 2020 (t=20) will be:}$$

$r=0.01823$ ,  $P_0=10,000$ , and  $t=20$

$P(20) = 10000e^{0.01823 \times 20} \approx 10000 \times 1.439 \approx 14390$ , The estimated population in 2020 is **14390**.

The formula for doubling time  $T$  in exponential growth is:

$$T = \frac{\ln(2)}{r} = \frac{\ln(2)}{0.01823} \approx \frac{0.693}{0.01823} \approx 38.02, \text{ The doubling time for the restaurant}$$

business' population is approximately **38 years**.



## Activity 2: Guided Practice



### Task 19:

Using the formulas in **key facts 2.2a**, answer the following questions:

2. A Farmer had 10,000 orange trees in 1960 and 12,000 in 1970.
  - a. Assuming an exponential growth model, estimate the number of orange trees in 1980.
  - b. What is the doubling time for the number of orange trees?
3. The population,  $P$ , of an island  $t$  years after January 1<sup>st</sup> 2016 is given by this formula
$$P = 4200 \times (1.04)^t$$
  - a. What was the population of the island on January 1<sup>st</sup> 2016?
  - b. What is the constant rate?
  - c. Work out the population of the island on January 1<sup>st</sup> 2021.
4. If the number of population take lunch in restaurant grew from 10,000 to 15,000 in 5 years,
  - a. Calculate the rate of this growth.
  - b. If the current population is 8,000, growing at 4% annually, how long to reach 12,000?
5. The restaurant business in town were 10,000 in 2000 and 12,000 in 2010.
  - a. Assuming an exponential growth model, estimate the population in 2020.
  - b. What is the doubling time for the restaurant business population?



### Activity 3: Application



#### Task 20:

Answer the following questions

1. The population of a city is growing at a rate of 5% per year. If the current population is 10,000, what will the population be in 10 years?
2. A restaurant has a current annual clients of 3,000, and the number of clients is increasing at a rate of 5% per year. How long will it take for the number of clients to double?
3. The number of population take the lunch in restaurant increased from 5,000 to 8,000 in 5 years. What is the annual growth rate?
4. According to United Nation data, the world population at the beginning of 1975 was approximately 4 billion and growing at rate of about 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2030.
5. The population of a country grows according to the law  $P = Ae^{0.06t}$  where P million is the population at time t years and A is a constant. Given that at time t=0, the population is 27.3 million, calculate the population when
  - a) t=10
  - b) t=15
  - c) t=25
6. The population of a country grows according to the law  $P = 12e^{kt}$  where p million is the population at time t years and k is the constant .Given that when t=7, P=15. Find the time for which the population will be a) 20 million b) 30 million c) 35 million
7. A city in Texas had a population of 75,000 in 1970 and a population of 200,000 in 1995. The growth between the years 1970 and 1995 followed an exponential pattern of the form  $f(t) = A \times e^{\alpha t}$ 
  - a. Find the values of A and  $\alpha$
  - b. Using the given model, estimate the population for the year 2010

## Topic 2.3: Population decay



### Activity 1: Problem Solving



#### Task 21:

During an experiment, a scientist notices that the number of bacteria halves every second. If there were  $2.3 \times 10^{30}$  bacteria at the start of the experiment, how many bacteria were left after 5 seconds? Give your answer in standard form correct to two significant figures.

### Key Facts 2.3a: Calculation of population decay

- **Description of population decay**

**Population decay** refers to a process where the size of a population decreases over time. This phenomenon is commonly modeled using **exponential decay**, which assumes that the rate of change of the population is proportional to its current size. This can be due to various factors, such as resource depletion, disease, natural disasters, or environmental changes.

- **Terminologies used in calculation of Population Decay**

1. **Decay Rate:**

The decay constant (denoted as **k**) determines how quickly the population decreases. A larger **k** corresponds to a faster rate of decay.

2. **Half-Life:**

The half-life (denoted as  $T_{\frac{1}{2}}$ ) is the time it takes for the population to reduce to half its original size.

3. **Doubling time**

Doubling Time (denoted as **T**) refers to the period it takes for a population (or any other quantity) to double in size at a constant growth rate. It is a useful concept in demography, economics, and environmental studies to understand how quickly a population or resource is growing.

4. **Exponential Behavior:**

The decrease is rapid initially and slows over time, never truly reaching zero but approaching it asymptotically.

- **Formulas for calculation of population decay**

1. **Decay Rate:**

A population whose rate of decrease is proportional to the size of the population at any time obeys a law of the forms  $P = Ae^{-kt}$  or  $P = P_0e^{-kt}$ . The negative sign on exponent indicates that the population is decreasing. This is known as exponential decay. If a quantity has an exponential growth model, then the time required for it to double in size is called the doubling time. Similarly, if a quantity has an exponential decay model, then the time required for it to reduce in value by half is called the halving time. For radioactive elements, halving time is called half-life.

2. **Half-Life**

The formula of half-life is :  $T_{\frac{1}{2}} = \frac{\ln 2}{k}$

3. **Doubling time**

The formula of half-life is:  $T = \frac{\ln 2}{k}$

### Key Facts 2.3b: Calculation of population decay

- **Solution for question 1**

Use the exponential decay formula:  $P = P_0e^{-kt}$

Where:

- $P_0=10,000$  (initial population)
- $k=0.08$  (decay rate)
- $t=5$  years

Substitute into the formula:  $P(5) = 10,000 \times e^{-0.08 \times 5} \Leftrightarrow P(5) = 10,000 \times e^{-0.4}$  Using  $e^{-0.4} \approx 0.67032$

$P(5) = 10,000 \times 0.67032 = 6,703.2$ , so after 5 years, the population will be approximately **6,703 individuals**.

- **Solution for question 2**

Use the exponential decay formula:  $P(t) = P_0 e^{-rt}$  Rearrange to solve for t let apply natural logarithm in both side to get  $\ln P(t) = \ln P_0 e^{-rt}$  with the help of basic rules of logarithms we will get  $\ln P(t) = \ln P_0 + \ln e^{-rt} \Leftrightarrow \ln P(t) - \ln P_0 = -rt \ln e$  remember that natural logarithm of e is equal to 1, so the time will be given by :

$$t = \frac{\ln\left(\frac{P(t)}{P_0}\right)}{-r} =$$

Where:

- $P_0=50,000$  (initial population)
- $P(t)=20,000$  (final population)
- $r=0.06$  (decay rate)

Substitute values in formula to get  $t = \frac{\ln\left(\frac{20,000}{50,000}\right)}{-0.06} \Leftrightarrow \frac{\ln(0.4)}{-0.06}, t \approx 15.27$  . It will take approximately **15.27 years** for the population to decay to 20,000.

- **Solution for question3**

To solve this problem, we use the **exponential decay formula**:  $N(t) = N_0 e^{-kt}$

Where:

- $N(t)$  is the number of cells after time t (in minutes),
- $N_0=5,000,000$  is the initial number of cells,
- $k$  is the decay constant,
- $t$  is the time in minutes.

The problem states that **45% of the cells are dying every minute**, meaning 55% of the cells remain after each minute. This corresponds to a decay factor of 0.55 per minute.

The decay factor is related to the decay constant  $k$ :  $e^{-k} = 0.55$  Take the natural logarithm in both side to get  $-k = \ln(0.55) \Leftrightarrow -\ln 0.55 \approx 0.5978$

Substitute  $N_0=5,000,000$  and  $k \approx 0.5978$  in exponential decay formula.

$1,000 = 5,000,000 \times e^{-0.5978t}$  . Divide both sides by 5,000,000 to get

$$\frac{1,000}{5,000,000} = e^{-0.5978t} \Rightarrow 0.0002 = e^{-0.5978t}$$

Take the natural logarithm of both sides:  $\ln(0.0002) = -0.5978t$ .

$$t = \frac{\ln(0.0002)}{-0.5978} \approx 14.25$$

So it will take approximately **14.3 minutes** for the cell population to drop below 1,000.



### Activity 2: Guided Practice

Using the formulas in key fact 2.3a, answer the following questions



#### Task 22:

1. A population of bacteria starts with 10,000 individuals and decreases at a decay rate of 8% per year. What will the population be after 5 years?
2. If a population of 50,000 decreases by 6% per year, how long will it take for the population to decay to 20,000?
3. If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 1,000 cells?



### Activity 3: Application



#### Task 23:

Answer the following questions.

1. A city's population is decreasing exponentially by 4% per year. If the current population is 100,000, how long will it take for the population to halve?
2. In a certain experiment, the number of bacteria reduces by a quarter each second. If the number of bacteria initially was X, write a formula that can be used to calculate the number of bacteria, V, remaining after t seconds
3. The population of a particular town on July 1, 2011 was 20,000. If the population decreases at an average annual rate of 1.4%, how long will it take for the population to reach 15,300?

4. A certain town has an initial population of 40,000 people. If the population is decreasing by 2% per year, how long will it take for the population to fall below 30,000?
5. A population of a certain species of fish in a lake is initially 5,000. If the population decreases at a rate of 3% per year, how many years will it take for the population to reduce to 2,000?
6. A certain species of plant in a controlled environment has a population of 500. If the population decreases at a rate of 4% per month, how many plants will remain after 6 months?



### Formative Assessment

#### 1. Multiple Choice questions

- A. The formula for compound interest is given by  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ . What does the variable  $t$  represent?
  - a. Time in years
  - b. Interest rate
  - c. Principal amount
  - d. Number of times interest is compounded
- B. Population grows according to the formula  $P(t) = P_0 e^{kt}$ . If  $k > 0$ , what does it indicate?
  - a. Population decay
  - b. Population stability
  - c. Population growth
  - d. None of the above
- C. If the half-life of a substance is 5 years, what is the value of the decay constant  $k$  (rounded to 3 decimal places)?
  - a. 0.693
  - b. -0.693
  - c. 0.138
  - d. -0.138
- D. Which of the following equations is used to calculate continuous compound interest?
  - a.  $A = P(1 + r)^t$

b.  $A = P\left(1 + \frac{r}{n}\right)^{nt}$

c.  $A = Pe^{rt}$

d.  $A = P + Prt$

2. Differentiate Initial population to Future population
3. Differentiate simple to Compound Interest.
4. A \$1000 deposit is made at a bank that pays 12% compounded weekly. How much will you have on your account at the end of 10 years?
5. The restaurant business company had a population of 10,000 of restaurants in 2000 and 12,000 in 2010. a) Assuming an exponential growth model, estimate the population in 2020. b) What is the doubling time for the company's population?
6. A principal amount of \$10,000 is invested at an annual interest rate of 5%, compounded quarterly. Calculate the amount after 3 years. (Use  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ )
7. A radioactive substance has a half-life of 10 years. How much of a 100g sample remains after 30 years? Use  $P(t) = P_0e^{-kt}$  and calculate  $k = \frac{\ln(2)}{\text{half-life}}$
8. A population of bacteria decreases from 100,000 to 25,000 in 8 hours. Calculate the decay constant  $k$  and the time it takes for the population to reduce to 12,500.
9. A loan of \$50,000 is taken out with an annual interest rate of 6%, compounded continuously. Calculate the total amount to be repaid after 10 years. (Use  $A = Pe^{rt}$ )
10. The population of a country is 1 million, and it is observed to double every 35 years. Derive the exponential growth equation for this population and predict the population after 70 years.
11. Solve for  $t$  in the compound interest equation:  $2000 = 1000(1 + 0.05)^t$
12. A population decreases by 3% annually. If the current population is 12,000, how long will it take for the population to reduce to 8,000? Use the formula ( $P(t) = P_0e^{-rt}$ )



## Points to Remember

- Exponential and logarithmic functions are used in financial calculations like compound interest as well as modeling population growth and radioactive decay.
- Compound interest calculations is used for saving and loans in banks. It is better to calculate the interest before you ask for a loan or invest your money.
- A quantity is said to have an exponential growth model if at each instant of time its rate of increase (decrease) is proportional to the amount of the quantity present. Whereas the decay is when the rate is for decreasing.
- Work on a variety of problems to get comfortable with these concepts.



## Self-Reflection

1. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

| My experience                                                | I don't have any experience doing this. | I know a little about this. | I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|--------------------------------------------------------------|-----------------------------------------|-----------------------------|-----------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes                              |                                         |                             |                                   |                                       |                                          |
| Define properly compound interest                            |                                         |                             |                                   |                                       |                                          |
| Define principle value and future value                      |                                         |                             |                                   |                                       |                                          |
| Analyse problems deal about compound interest                |                                         |                             |                                   |                                       |                                          |
| Pay attention to details while calculating compound interest |                                         |                             |                                   |                                       |                                          |
| Analyse correctly the formula for calculating                |                                         |                             |                                   |                                       |                                          |

| My experience                                                                             | I don't have any experience doing this. | I know a little about this. | I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|-------------------------------------------------------------------------------------------|-----------------------------------------|-----------------------------|-----------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes                                                           |                                         |                             |                                   |                                       |                                          |
| principle value and future value                                                          |                                         |                             |                                   |                                       |                                          |
| Think logically and critically when applying formula of compound interest.                |                                         |                             |                                   |                                       |                                          |
| Apply formulas for calculating any problem related to compound interest                   |                                         |                             |                                   |                                       |                                          |
| Pay attention about one's savings                                                         |                                         |                             |                                   |                                       |                                          |
| Describe different technics to solve problems related to calculation of population growth |                                         |                             |                                   |                                       |                                          |
| Solve problem related to calculation of population growth                                 |                                         |                             |                                   |                                       |                                          |
| Describe different technics to solve problems related to calculation of population decay  |                                         |                             |                                   |                                       |                                          |
| Solve problems related to calculation of population decay                                 |                                         |                             |                                   |                                       |                                          |
| Think logically and critically when                                                       |                                         |                             |                                   |                                       |                                          |

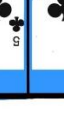
| My experience                                                                              | I don't have any experience doing this. | I know a little about this. | I have some experience doing this | I have a lot of experience with this. | I am confident in my ability to do this. |
|--------------------------------------------------------------------------------------------|-----------------------------------------|-----------------------------|-----------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills and attitudes                                                            |                                         |                             |                                   |                                       |                                          |
| applying formula for solving problems related to calculation of population growth or decay |                                         |                             |                                   |                                       |                                          |
| Analyse correctly the difference between rate of growth and rate of decay                  |                                         |                             |                                   |                                       |                                          |
| Developmental constructions                                                                |                                         |                             |                                   |                                       |                                          |

2. Fill in the table above and share results with the trainer for further guidance.

| Areas of strength | Areas for improvement | Actions to be taken to improve |
|-------------------|-----------------------|--------------------------------|
| 1.                | 1.                    | 1.                             |
| 2.                | 2.                    | 2.                             |
| 3.                | 3.                    | 3.                             |

## UNIT 3: APPLY FUNDAMENTALS OF PROBABILITY



| Playing Cards    | Ace                                                                                 | 2                                                                                   | 3                                                                                   | 4                                                                                   | 5                                                                                   | 6                                                                                   | 7                                                                                    | 8                                                                                     | 9                                                                                     | 10                                                                                    | Jack                                                                                  | Queen                                                                                 | King                                                                                  |
|------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------|
| <b>Spades:</b>   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| <b>Diamonds:</b> |    |    |    |    |    |    |    |    |    |    |    |    |    |
| <b>Hearts:</b>   |  |  |  |  |  |  |  |  |  |  |  |  |  |
| <b>Clubs:</b>    |  |  |  |  |  |  |  |  |  |  |  |  |  |

### Unit summary

This unit provides you with the knowledge, skills and attitudes required to apply fundamentals of probability. It covers application of counting techniques, computation of probabilities and calculation of the conditional probability.

## Self-Assessment: Unit3

1. Answer the following questions:

- Define probability, events, sample spaces, and outcomes.
- What are some real life examples of probability?
- How to calculate probability?
- Ivan rolls a fair dice, with sides labelled A,B,C,D,E and F. What is the probability that the dice lands on a vowel?
- Max tested a coin to see whether it was fair. The table shows the results of his coin toss experiment

| Head | Tail |
|------|------|
| 26   | 41   |

What is the relative frequency of the coin landing on heads?

2. Fill in and complete the self-assessment table below to assess your level of knowledge, skills, and attitudes under this unit.

- There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills, and attitudes acquisition during the learning process.
- Think about yourself; do you think you have the knowledge, skills, or attitudes to do the task? How well?
- Read the statements across the top, put a check-in a curriculum that best represents your level of knowledge, skills, and attitudes.

3. At the end of this unit, you'll assess yourself again.

| Experience                                            | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                      |                                          |                             |                                    |                                       |                                          |
| Define correctly probability and its associated terms |                                          |                             |                                    |                                       |                                          |
| Determine the number of permutations and              |                                          |                             |                                    |                                       |                                          |

| Experience                                                                                            | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-------------------------------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                                      |                                          |                             |                                    |                                       |                                          |
| combinations of “n” items, “r” taken at a time.                                                       |                                          |                             |                                    |                                       |                                          |
| Pay attention to details while calculating probability                                                |                                          |                             |                                    |                                       |                                          |
| Describe the application of probability to different fields including tourism, social sciences etc... |                                          |                             |                                    |                                       |                                          |
| Use counting techniques to solve related problems.                                                    |                                          |                             |                                    |                                       |                                          |
| Approach problems systematically and logically.                                                       |                                          |                             |                                    |                                       |                                          |
| Distinguish properly between combination and permutation                                              |                                          |                             |                                    |                                       |                                          |
| Use properties of combinations                                                                        |                                          |                             |                                    |                                       |                                          |
| Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios.  |                                          |                             |                                    |                                       |                                          |
| Define the combinatorial Analysis.                                                                    |                                          |                             |                                    |                                       |                                          |
| Analyse problems to determine relevant probabilities.                                                 |                                          |                             |                                    |                                       |                                          |

| Experience                                                                                                    | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|---------------------------------------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                                              |                                          |                             |                                    |                                       |                                          |
| Be patient while solving complex probability problems.                                                        |                                          |                             |                                    |                                       |                                          |
| Describe different formulas used to solve problems related to calculation of probability                      |                                          |                             |                                    |                                       |                                          |
| Apply concepts to solve real-world problems and scenarios.                                                    |                                          |                             |                                    |                                       |                                          |
| Describe uses of Unions, Intersections and Complements in probability contexts.                               |                                          |                             |                                    |                                       |                                          |
| Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment. |                                          |                             |                                    |                                       |                                          |
| Interpret data to make probabilistic inferences                                                               |                                          |                             |                                    |                                       |                                          |



### Key Competencies:

| Knowledge                                                | Skills                                                                                      | Attitudes                                                 |
|----------------------------------------------------------|---------------------------------------------------------------------------------------------|-----------------------------------------------------------|
| 1. Define correctly probability and its associated terms | 1. Determine the number of permutations and combinations of “n” items, “r” taken at a time. | 1. Pay attention to details while calculating probability |

| Knowledge                                                                                                        | Skills                                                        | Attitudes                                                                                               |
|------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|
| 2. Describe the application of probability to different fields including tourism, social sciences etc...         | 2. Use counting techniques to solve related problems.         | 2. Approach problems systematically and logically.                                                      |
| 3. Distinguish properly between combination and permutation                                                      | 3. Use properties of combinations                             | 3. Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios. |
| 4. Define the combinatorial Analysis.                                                                            | 4. Analyse problems to determine relevant probabilities.      | 4. Be patient while solving complex probability problems.                                               |
| 5. Describe different formulas used to solve problems related to calculation of probability                      | 5. Apply concepts to solve real-world problems and scenarios. |                                                                                                         |
| 6. Describe uses of Unions, Intersections and Complements in probability contexts.                               | 6. Interpret data to make probabilistic inferences            |                                                                                                         |
| 7. Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment. |                                                               |                                                                                                         |



### Discovery activity:

#### Scenario 1:

A restaurant in a taste food franchise has determined that the chance a customer will order a soft drink is 0.91. The probability that a customer will order a hamburger is 0.68. The probability that a customer will order French fries is 0.49.



#### Task 24:

**Read the following scenario and respond the corresponding questions**

1. What is the meaning of term probability?
2. If a customer places an order, what is the probability that the order will include a soft drink and no fries if these two events are independent
3. The restaurant has also determined that if a customer orders a hamburger, the probability the customer will order fries is 0.82. Determine the probability that the order will include a hamburger and fries.

## Topic 3.1: Counting techniques in probability



### Activity 1: Problem Solving



#### Task 25:

Answer the following questions:

1. What is probability?
2. What is the formula for probability calculation?
3. What are the branches of probability?
4. What are the Techniques for counting the number of elements in a sample space?
5. Provide formula for each technique.

### Key Facts 3.1a: Counting techniques in probability

- **Introduction to Probability**

- ✓ **Definition**

Probability is a branch of mathematics that deals with the likelihood or chance of different outcomes occurring in a random experiment. It provides a framework for analyzing situations where outcomes are uncertain, allowing us to make predictions and informed decisions based on quantitative reasoning. It is the chance that something will happen or how likely it is that some event will happen.

For example, when the weather forecast says there's a 60% chance of rain, that's a probability.

- ✓ **Basic formula of probability**

The probability of an event **A**, denoted **P(A)**, is a measure of how likely the event is to occur.

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes in sample (S)}}$$

- ✓ **Branches of Probability**

There are 4 main branches of probability:

1. **Theoretical probability:** Also known as classical probability. This is the probability of an event occurring in an experiment where all outcomes are equally likely.

**Example:** When rolling a fair die, what is the probability of rolling each number?



**Solution**

The probability is  $1/6$ . This is because that a die has six sides with numbers from 1 to 6. As each number is represented only once, the probability is  $1/6$ .

- 2. Empirical probability:** Also known as experimental probability. This refers to a probability that is based on historical data of a conducted experiment. In other words empirical probability is an event occurring based on historical data.

$$\text{Empirical probability} = \frac{\text{frequency of event}}{\text{total frequency}}$$

**Example:** A bag contains 5 red marbles, 3 blue marbles, and 2 green marbles. You draw one marble from the bag, record its color, and then replace it back into the bag. This process is repeated 50 times, and the results are as follows: Red: 28 times, Blue: 15 times and Green: 7 times. What is the experimental probability of drawing each color?

**Solution**

- 1.** What is the experimental probability of drawing a red marble?

Experimental Probability = (Number of times red was drawn) ÷ (Total number of trials)

$$\Rightarrow \frac{28}{50} = 0.56 \approx 56\%$$

- 2.** What is the experimental probability of drawing a blue marble?

Experimental Probability = (Number of times blue was drawn) ÷ (Total number of trials)

- 3.** What is the experimental probability of drawing a green marble?

Experimental Probability = (Number of times green was drawn) ÷ (Total number of trials)

$$\Rightarrow \frac{7}{50} = 0.14 \approx 14\%$$

In the above example, the theoretical probability of drawing each color is:

- Red: Theoretical probability:  $\frac{5}{10} = 0.5 \approx 50\%$

- Blue: Theoretical probability:  $\frac{3}{10} = 0.3 \approx 30\%$

- Green: Theoretical probability:  $\frac{2}{10} = 0.2 \approx 20\%$

**3. Subjective probability:** Is likelihood event based on their personal judgment or experience, rather than on calculations or observations. This is also called a personal probability.

**Example:** Anna is planning to go hiking tomorrow. She checks the weather forecast, which says there's a 40% chance of rain. However, Anna notices that the sky is clear today and believes the chance of rain tomorrow is much lower. Based on her experience, she estimates there's only a 10% chance of rain. What is Anna's subjective probability of rain tomorrow, and how does it differ from objective probability?

#### **Solution**

Anna's subjective probability is **10%**. This is based on her personal judgment and experience, rather than statistical or objective data. And The objective probability is **40%**, as provided by the weather forecast, which is based on meteorological data and statistical analysis.

**4. Axiomatic probability:** Is a mathematical theory that describes the likelihood of an event by establishing a set of axioms that apply to all probability approaches.

Probability is a set function  $P(E)$  that assigns to every event  $E$  a number called the probability of  $E$  such that:

- The probability of an event is greater than or equal to zero.  $P(E) \geq 0$
- The probability of the sample space is one.  $P(U) = 1$

- **Techniques for counting the number of elements in a sample space.**

There are five (5) techniques for Counting the number of elements in a sample space namely: 1) Venn diagram, 2) Tree diagram, 3) Multiplication principle, 4) Permutations and 5) Combination

#### **1. Venn diagram**

This is a way of representing sets in a closed curve. The universal set, written  $U$  or  $\epsilon$  is a set containing all the sub-sets considered in the problem.

In probability theory, the universal set is called the **sample space** and its subsets are called the **events**.

**The complement** of a set say  $A$  is the set written  $A'$  or  $\bar{A}$  of an element in the universal set but not in the set  $A$ .

**Intersection of sets**  $A$  and  $B$  is denoted by  $A \cap B$ . This represents the set of all elements common to both  $A$  and  $B$ .

**Union of set**  $A$  and  $B$  is denoted by  $A \cup B$ . This represents the set of all elements belonging to  $A$  or  $B$  or both  $A$  and  $B$ .

**Disjoint sets** are sets with no elements in common. If  $A$  and  $B$  are disjoint sets, then  $A \cap B = \emptyset$ . Also  $A$  and  $B$  are said to be **mutually exclusive**.

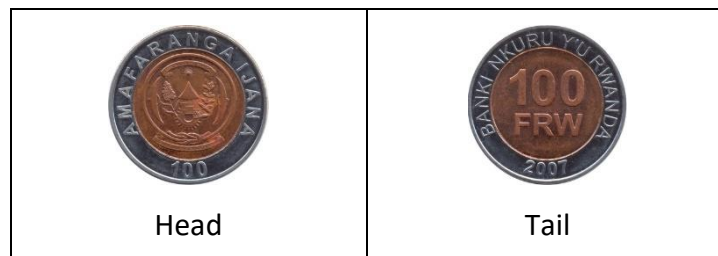
## 2. Tree diagram

A tree diagram is a diagram with a structure of branching connecting lines representing a relationship. It can be used to find the number of possible outcomes of experiments where each experiment occurs in a finite number of ways in succession.

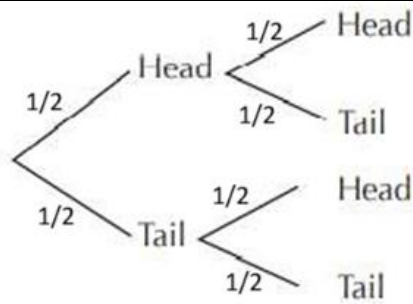
The outcome is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring.

For each trial, the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes,  $A$  and  $B$ , of each trial.

For example, when you toss a coin, the outcome is either head or tail.



A second toss would also give head or tail. We represent this as:



### 3. Multiplication rule

This is also known as product rule or product principle or multiplication principle. Suppose that an experiment (operation) is to be performed in two successive ways. If the operation is composed of  $k$  successive steps which may be performed in  $n_1, n_2, \dots, n_k$  distinct ways, respectively, then the operation may be performed in  $n_1 \times n_2 \times \dots \times n_k$  distinct ways.

### 4. Permutations

Permutation is an ordered arrangement of the items in a set.

The number of arrangements of  $r$  objects, taken from  $n$  unlike objects, can be considered as the number of ways of filling  $r$  places in order by the  $n$  given objects and is given by:

$$P_r^n = \frac{n!}{(n-r)!} = n(n-1)(n-2) \dots (n-r+1)$$

Notes:

- $!$  is factorial notation
- A different notation can be used of **permutation with no repetition**:  $P_r^n = P(n, r)$  Condition  $n \geq r$  and the number of arrangements of  $r$  objects, taken from  $n$  unlike objects where each of which may be repeated any number of times is  $P_r^n = n^r$ .
- The number of permutations of  $n$  object with  $n_1$  identical objects of type 1,  $n_2$  identical objects of type 2,....., and  $n_k$  identical objects of  $k$  type is  $P = \frac{n!}{n_1!n_2! \dots n_k!}$
- Number of permutations of  $n$  objects, repetition not allowed is  $P = n!$

### 5. Combination

Combination is a selection of objects where the order is not taken into account. In other words, the order in which the objects are arranged is not important.

When considering the number of combinations of  $r$  objects, the order in which they are placed is not important.

In general, the number of combinations of  $r$  objects from  $n$  unlike objects is

$$\binom{n}{r} \text{ or } C(n, r) = \frac{n!}{r!(n-r)!}$$

Condition  $n \geq r$

Note: Other notations that can be used are  ${}^nCr$ ,  $C^n_r$  or  ${}_nC_r$ .

**Some properties are:**

1.  $\binom{n}{r} = \binom{n}{n-r}$
2.  $\binom{n}{n} = \binom{n}{0}$
3.  $\binom{n}{1} = n$

Sometimes we are given a condition that must be taken into account in a combination.

Note:  $C_r^n = \frac{P_r^n}{r!}$ , where  $P_r^n$  is the permutation of  $n$  objects taken  $r$  at a time.

### Key Facts 3.1b: Application of techniques for Counting the number of elements in a sample space.

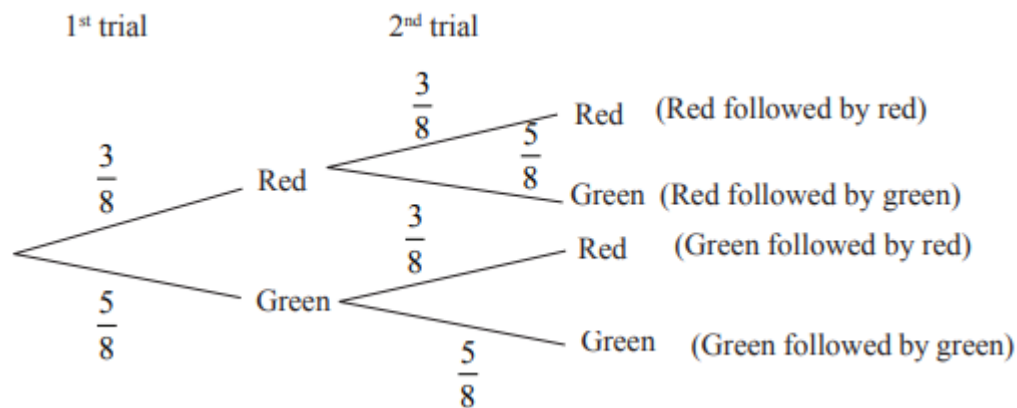
- **Solution for question 1**

Given  $U = \{1, 2, 3, 4, 5, 6, 7\}$      $A = \{1, 3, 5, 7\}$      $B = \{4, 5, 6, 7\}$     and  $C = \{2, 3, 5, 7\}$

- a.  $A \cup B = \{1, 3, 4, 5, 6, 7\} \Rightarrow$  Elements in  $A$  + Elements in  $B$
- b.  $A \cap C = \{3, 5, 7\} \Rightarrow$  common elements in  $A$  and in  $C$
- c.  $B - C = \{4, 6\} \Rightarrow$  Elements of  $B$  that are not in  $C$
- d.  $A \cup B' = A + (U - B) = \{1, 3, 5, 7\} + \{1, 2, 3\} = \{1, 2, 3, 5, 7\}$  where  $B'$  is a complement set of  $B$
- e.  $C' - A' = (U - C) - (U - A) = \{1, 4, 6\} - \{2, 4, 6\} = \{1\}$
- f.  $A' \cap C = \{2\}$

- **Solution for question 2**

Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is  $3 \div 8$  and probability of choosing a green ball is  $5 \div 8$  and since after the 1st trial, the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial. Draw a tree diagram showing the probabilities of each outcome of the two trials.



a.  $P(\text{Red followed by green}) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

$$\text{b. } P(\text{Red and green in any order}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{15}{32}$$

c.  $P(\text{both of the same colors}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{5}{8} = \frac{17}{32}$

- **Solution for question 3**

The number of all possible different clothing for Habimana is  $4 \times 3 \times 2 = 24$ .

Therefore, The maximum number of days Habimana does not need to repeat his clothing is 24.

- **Solution for question 4**

The first place can be filled in 5 ways, since any contestant can come first. When the first place has been filled, there are 4 more contestants to choose from for the second place. Hence the second place can be filled in  $5 \times 4$  ways. Finally, for each of these ways, the third place can be filled by any of the remaining 3 contestants, and the total number of ways is  $5 \times 4 \times 3 = 60$

- **Solution for question 5**

There are  $P_4^{15} = \frac{15!}{(15-4)!} = \frac{15 \times 14 \times 13 \times 12 \times 11!}{11!} = 15 \times 14 \times 13 \times 12 = 32760$  different line ups

- **Solution for question 6**

Total number  $n=5$  , number of digits taken each time  $r=3$

If there is repletion of element  $P_r^n = n^r = 5^3 = 125$

- **Solution for question 7**

Since there are three **m**, two **a** and one **l** in the word mammal. Total number of letters is **6**. We have for the number of ways we can arrange the letters in the following ways:

$$P = \frac{n!}{n_1!n_2!n_3!} = \frac{6!}{3! \times 2! \times 1!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 2! \times 1!} = \frac{6 \times 5 \times 4}{2 \times 1} = 60$$

- **Solution for question 8**

The number of committees is  $(^{12}_3) = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} = 220$

- **Solution for question 9**

The number of ways of choosing the men is  $(^{12}_3) = \frac{12!}{3!(12-3)!} = \frac{12 \times 11 \times 10 \times 9!}{3! \times 9!} = 220$

The number of ways of choosing the women is

$$(^8_2) = \frac{8!}{2!(8-2)!} = \frac{8 \times 7 \times 6!}{2! \times 6!} = 28$$

Therefore, the total number of ways of choosing the committee (given by the product rule) is

$$220 \times 28 = 6160$$



## Activity 2: Guided Practice



### Task 26:

Using the techniques for counting the number of elements in a sample space, answer the following questions:

1. Let  $U = \{1,2,3,4,5,6,7\}$ ,  $A = \{1,3,5,7\}$ ,  $B = \{4,5,6,7\}$  and  $C = \{2,3,5,7\}$ . List the elements in each of the following sets:

a.  $A \cup B$

d.  $A \cup B'$

b.  $A \cap C$

e.  $C' - A'$

c.  $B - C$

f.  $A' \cap C$

2. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Using three diagram, find the probability that the ball drawn will be:
  - a. red followed by green,
  - b. red and green in any order,
  - c. of the same color.
3. Habimana has 4 shirts, 3 pair of trousers and 2 pairs of shoes. He chooses a shirt, a pair of trousers and a pair of shoes to wear every day. Using multiplication rule, find the maximum number of days he does not need to repeat his clothing.
4. Using multiplication rule, find how many ways the first three places (with no ties) can be filled in a race with 5 contestants.
5. Using permutation, how many ways can 4 students from a group of 15 be lined up for a photograph?
6. Using permutation, How many three digit numbers can be formed with the digits: 1,2,3,4,5 with repletion allowed?
7. In how many ways can six letters of the word “**mammal**” be arranged in a row using permutation?
8. Using combination technique, how many different committees of 3 people can be chosen from a group of 12 people?
9. A committee of 5 is to be chosen from 12 men and 8 women. In how many ways can this be done if there are to be 3 men and 2 women on the committee using conditional combination?



### Activity 3: Application

You are hosting a dinner party and want to create a menu. You have the following choices:

- Appetizers: 3 options (Salad, Soup, Breadsticks)
- Main Courses: 4 options (Chicken, Steak, Fish, Vegetarian)
- Desserts: 2 options (Cake, Ice Cream). You want to select one item from each category.



### Task 27:

**From the scenario above, perform the following tasks:**

1. How many different menu combinations can you create?
2. If you randomly select a menu, what is the probability that it includes Breadsticks, Steak, and Ice Cream?

## Topic 3.2: Probabilities for an event



### Activity 1: Problem Solving



#### Task 28:

Answer the following questions:

1. What do you understand by term event in probability?
2. What are the types of events in probability?
3. How many Food technologists like at least one of the activities?
4. What is the probability that a randomly selected Food scientist likes exactly two activities?

### Key fact 3.2a: Description of formulas for computation of probabilities for an event

- **Definition of key terms**

1. **Sample space** is a set of all possible outcomes that may occur in a particular experiment is denoted by **S**.

For example:

- a. When a coin is tossed:  $S = \{H, T\}$ .
  - b. When two coins are tossed:  $S = \{HT, TH, HH, TT\}$ .
  - c. When a die is thrown:  $S = \{1, 2, 3, 4, 5, 6\}$
2. **Possible outcomes** are all likely results of an experiment.
  3. **Event**: It is a set consisting of possible outcomes of an experiment with the desired qualities.

#### Types of event

1. **Simple events**

It is one that can only happen in one way in other words, it has a single outcome. If we consider for example of tossing a coin: we get one outcome that is a head or a tail.



**Certain event** is an event that will definitely take place.

The following are examples of certain (sure) events.

- (a) If you put a shirt in water, it will get wet.
- (b) If you put your exercise book in fire, it will burn.

 **Impossible event** is an event that cannot take place.

The following are examples of impossible events.

- (a) A dice cannot give a 9.
- (b) A cassava plant will produce potatoes.

 **Uncertain event** is an event that may or may not take place.

The following are examples of uncertain events.

- a. It will rain tomorrow.
- b. The national football team will win their next match.
- c. Mala will win the elections.
- d. I will eat meat tomorrow

## 2. Complementary event

If E is an event, then E' is the event which occurs when E does not occur.

Event E and E' are said to be complementary events. Theorem  $P(E') = 1 - P(E)$  or  $P(E) = 1 - P(E')$

## 3. Probability of an event under equally likely events

The probability of an event E, denoted by P(E), is a measure of the possibility of the event occurring as the result of an experiment.

The probability of an event E, denoted by P(E), is a measure of the possibility of the event occurring as the result of an experiment. If the sample space S is finite and the possible outcomes E are equally likely, then the probability of the event E is equal to  $\frac{|E|}{|S|} = \frac{n(E)}{n(S)}$

where |E| and |S| denote the number of elements in E and S respectively.

$n(E)$ =Number of favourable outcomes and  $n(S)$ = Number of total outcomes in sample space.

Suppose that an experiment has only a finite number of equally likely outcomes. If E is an event, then  $0 \leq P(E) \leq 1$ .

#### 4. Inclusive events

Inclusive events are events that can happen at the same time. To find the probability of an inclusive event we first add the probabilities of the individual events and then subtract the probability of the two events happening at the same time. Inclusive events are called also non- mutually exclusive.

If A and B are events from a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

#### 5. Mutually exclusive events

A and B are said to be mutually exclusive if the events A and B are disjoint i.e. A and B cannot occur at the same time.

For mutually exclusive events,  $A \cap B = \emptyset$ .  $P(A \cap B) = P(\emptyset) = 0$ ; and the addition law reduces to  $P(A \cup B) = P(A) + P(B)$ .

### Key fact 3.2b: Calculation of probabilities

- **Question 1**

The first question is a Complementary event , with this

- a.  $P(5) = \frac{1}{6}$
- b.  $P(\text{not obtaining } 5) = 1 - P(5) = 1 - \frac{1}{6} = \frac{5}{6}$
- c.  $P(3 \text{ or } 4) = P(3) + P(4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$
- d.  $P(\text{not obtaining } 3 \text{ or } 4) = 1 - P(3 \text{ or } 4) = 1 - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$

- **Question 2**

This question has the equally likely event. Since two of the eleven letters are "A", the probability of choosing a letter "A" is  $\frac{2}{11}$ .

- **Question 3**

Question 3 is and an inclusive event, thus

$$P(A) = P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = P(\text{an spade}) = \frac{13}{52} = \frac{1}{4}$$

$$P(A \cap B) = P(\text{the ace of spades}) = \frac{1}{52}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

• **Question 4**

This is the Mutually exclusive events. Therefore

$$P(A \cup B) \text{ or } P(\text{red or blue}) = P(A) + P(B) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$$

• **Question 5**

Let the first machine be **M1** and the second machine be **M2** , then  $P(M_1) = 80\% = 0.8$  ,  $P(M_2) = 60\% = 0.6$  and  $P(M_1 \cup M_2) = 92\% = 0.92$

Now,

$$\begin{aligned} P(M_1 \cup M_2) &= P(M_1) + P(M_2) - P(M_1 \cap M_2) \\ \Leftrightarrow P(M_1 \cap M_2) &= P(M_1) + P(M_2) - P(M_1 \cup M_2) \\ \Rightarrow 0.8 + 0.6 - 0.92 &= 0.48 = P(M_1) \times P(M_2) \end{aligned}$$

Thus, the two machines operate independently



## Activity 2: Guided Practice



### Task 29:

**Answer the following questions:**

1. An ordinary die of 6 sides is rolled once. With the aid of definition of complement event, answer the following questions.  
Determine the probability of:
  - a. Obtaining 5
  - b. Not obtaining 5
  - c. Obtaining 3 or 4
  - d. Not obtaining 3 or 4
2. A letter is chosen from the letters of the word "MATHEMATICS". What is the probability that the letter chosen is an "A"?
3. A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find  $P(A)$ ,  $P(B)$ ,  $P(A \cap B)$  and  $P(A \cup B)$ .
4. A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find  $P(A)$ ,  $P(B)$  and  $P(A \cup B)$ .

5. A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?



### Activity 3: Application

A Food Technologist want to prepare the three types of fruits on a table. Those fruits are: 3 oranges, 2 bananas and 5 pineapples.



### Task 30:

**Read the scenario and answer the following questions:**

1. How many tables like at least one of the three fruits?
2. What is the probability that a randomly selected table has:
  - a. Oranges only?
  - b. Oranges or bananas?
  - c. Bananas and pineapples?

## Topic 3.3: The conditional probability



### Activity 1: Problem Solving



#### Task 31:

A box contains 3 red pens, 4 green pens and 5 blue pens. One pen is taken from the box and is not replaced. Another pen is taken from the box. Let A be the event “the first pen is red” and B be the event “the second pen is blue”. Is the occurrence of event B affected by the occurrence of event A? Explain.

### Key Facts 3.3a: Calculation of the conditional probability

#### 1. Conditional probability

Conditional probability is the probability of an event occurring given that another event has already occurred. It quantifies how the occurrence of one event affects the likelihood of another event.

#### Formula of conditional probability

- Mathematically, the conditional probability of event A given event B (denoted as  $P(A|B)$ ) is equal to  $P(A \cap B) = \frac{P(A \cap B)}{P(B)}$ , where  $P(B) > 0$ .

- The probability of an event B given event A (denoted as  $P(B/A)$ ) is equal to

$$P(B/A) = \frac{P(B \cap A)}{P(A)}.$$

- Here:

$P(A \cap B)$ : Is the probability that both events A and B occur.

$P(B \cap A)$ : Is the probability that both events B and A occur

$P(A)$ : Is the probability that event A occurs.

$P(B)$ : Is the probability that event B occurs

- The following figure explains well the conditional probability

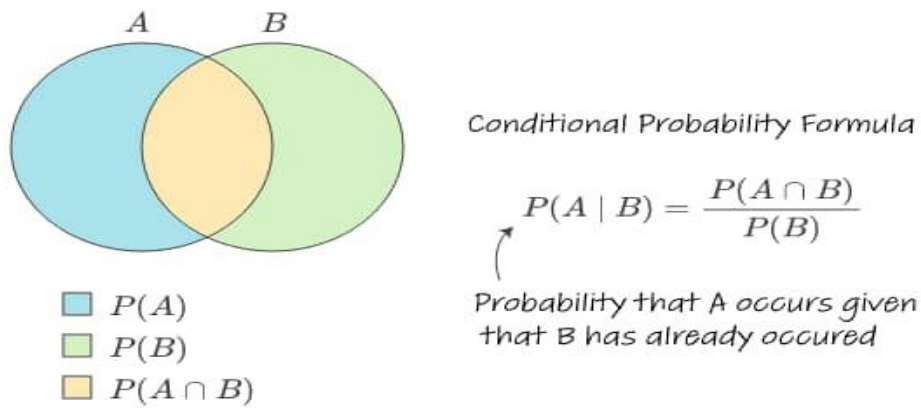


Figure 2: Conditional Probability

## 2. Independent events

Independent events are two or more events whose occurrences do not influence each other. In other words, the occurrence of one event has no effect on the probability of the other event occurring. In other words, in probability, we say two events are independent if knowing one event occurred doesn't change the probability of the other event.

✚ **Multiplication rule1:** Two events A and B are independent if:

$$P(A \text{ and } B) \text{ or } P(A \cap B) = P(A) \cdot P(B)$$

## 3. Dependent events

Dependent events are events where the occurrence of one event affects the probability of the other event occurring. In other words, the outcome of one event influences or changes the likelihood of another event.

✚ **Multiplication rule2:** For dependent events A and B:

$P(A \text{ and } B) \text{ or } P(A \cap B) = P(A) \times P(B | A)$ , where:

- $P(A \cap B)$ : The probability that both AAA and BBB occur.
- $P(B|A)$ : The conditional probability of B occurring given A has occurred.
- If  $P(B|A) \neq P(B)$ , A and B are dependent events.

## 4. Bayes' Theorem

Bayes' Theorem is a mathematical formula used to calculate conditional probabilities. It states that the probability of an event A occurring, given that another event B has occurred, is proportional to the probability of B occurring given A,

multiplied by the prior probability of A. This is expressed mathematically as:

$$P(A \cap B) = \frac{P(B \cap A) \times P(A)}{P(B)} \text{ where,}$$

✚ P(A|B): The conditional probability of A Given B.

✚ P(B|A): The conditional probability of B given A.

✚ P(A): The probability of A before considering B.

✚ P(B): The total probability of B.

### Key Facts 3.3b: Calculation of the conditional probability

- **Solution for question 1**

Let A= event that first candy is pink

and B= event that second candy is pink

$$P(A) = \frac{3}{10}$$

Since the candies are taken out with replacement, this implies that the given events A and B are independent.

$$P(A/B) = P(B) = \frac{3}{10}$$

Hence by the multiplication rule we get,  $P(A \text{ and } B) = P(A) \times P(B)$

$$= \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$$

- **Solution for question 2**

Total number of marbles =7

Number of yellow marbles =4

Number of red marbles=3

The probability of getting a yellow marble in first draw is:  $P(\text{Yellow}) = \frac{4}{7}$

In second draw, there are 6 marbles left in the bag. The probability of getting a red marble in second draw is  $P(\text{red}) = \frac{3}{6} = \frac{1}{2}$

Both events are dependent.

Then the probability of both events together is given by:

$$P(\text{yellow and red}) = P(\text{yellow}) \times P(\text{red}) = \frac{4}{7} \times \frac{1}{2} = \frac{4}{14} = \frac{2}{7}$$

- **Solution for question 3**

Let A be the event that the number is a 5 and B the event that the number is greater than 3. Then  $A \cap B = \{5\}$  and  $B = \{4, 5, 6\}$ .

$$\text{We require } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{6} \times \frac{6}{3} = \frac{1}{3}$$

- **Solution for question 4**

$$P(\text{fire}) = 1\%, P(\text{smoke}|\text{fire}) = 90\%$$

$$\text{By Bayes' Theorem, } P(\text{Fire}|\text{Smoke}) = \frac{P(\text{Fire}) \times P(\text{Smoke}|\text{Fire})}{P(\text{smoke})}$$

$$= \frac{1\% \times 90\%}{10\%} = 9\%$$

So the probability of dangerous Fire when there is Smoke is 9%.



## Activity 2: Guided Practice



### Task 32:

**Answer the following questions:**

1. A bag contains 3 pink candies and 7 green candies. Two candies are taken out from the bags with replacement. Find the probability that both candies are pink.
2. There are four yellow and three red marbles in a bag. Two marbles are drawn one by one. What is the probability that a yellow marble is drawn in first attempt and red is drawn in second attempt?
3. A die is tossed. Find the probability that the number obtained is a 5 given that the number is greater than 3.

4. In a fully operational Hotel Kitchen, dangerous fires are rare, they are at 1% rate but smoke is fairly common 10% due to barbecues, and 90% of dangerous fires makes smoke. What is the probability of dangerous fire when there is smoke?



### Activity 3: Application

Musa, A Hotel manager has denim pants with three colors, Grey, Blue and Navy blue. He has also four shirts: White, black, green, and pink. Musa want to choose one pant and one shirt at random.



#### Task 33:

From the scenario above, perform the following tasks:

1. Are pants and shirts independent events? Explain your answer.
2. What is the probability that he will choose the Navy-blue pants and the black shirt?



### Formative Assessment

1. Answer true or false:
  - a. A complement set of set A is given by all elements in universal set minus all elements of set A.
  - b.  $C_r^n = \frac{P_r^n}{r!}$
2. A restaurant consists 60 chairs and 10 tables.
  - a. How many ways the restaurant workers arrange 5 chairs from 60 chairs.
  - b. How many ways the restaurant worker arrange 6 tables from 10 tables?
3. The X NGO offers 500 Liters of packaged milk and 100 of 50 Kg Sacks of Maize flour to 3 selected Early Childhood Centers (EDCs). What is the probability that all 3 selected EDCs to be given packaged milk and maize flour?
4. Player X is known to successfully convert penalty kicks 80% of the time. During a pre-match, briefing, the coach said that if there are two penalty kicks given during the match, Player X should take them.
  - a. What is the probability that Player X converts both penalties?
  - b. What is the probability that Player X converts at least one penalty?
  - c. What is the probability that Player X misses both penalties?

5. Find the number of different ways of arranging 4 letters of 26-letter alphabet with repetition.
6. A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.
7. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:
  - a. Three boys being chosen.
  - b. Exactly two boys and a girl being chosen.
  - c. Exactly two girls and a boy being chosen.
  - d. Three girls being chosen.
8. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colors noted. The first disc is not replaced before the second is selected. Find the probability that the discs will be:
  - a. both red
  - b. of different colors
  - c. the same colors.
9. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be:
  - a. all red
  - b. all blue
  - c. one of each color.
10. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.
11. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.
12. At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?

13. Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

|                      | Speeding violation<br>in the last year | No speeding<br>violation in the last<br>year | Total |
|----------------------|----------------------------------------|----------------------------------------------|-------|
| Car phone user       | 25                                     | 280                                          | 305   |
| Not a car phone User | 45                                     | 405                                          | 450   |
| Total                | 70                                     | 675                                          | 755   |

Calculate the following probabilities using the table:

- P (person is a car phone user).
- P (person had no violation in the last year).
- P(person had no violation in the last year AND was a car phone user).
- P(person is a car phone user OR person had no violation in the last year).
- P(person is a car phone user GIVEN person had a violation in the last year).
- P(person had no violation last year GIVEN person was not a car phone user).

14. Suppose that machines  $M_1, M_2, M_3$  and produce respectively 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine  $M_3$  ?

15. Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

16. In a certain college, 5% of the men and 1% of the women are taller than 180 cm. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 cm, what is the probability that this student is a woman?

17. A certain federal agency employs three consulting firms (A, B and C) with probabilities 0.4, 0.35, 0.25, respectively. From past experience, it is known that the probabilities of cost overrun for the firms are 0.05, 0.03, and 0.15 respectively. Suppose a cost overrun is experienced by the agency.
- What is the probability that the consulting firm involved is company C?
  - What is the probability that it is company A?
18. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?
19. A nationwide survey found that 72% of people in the United States like pizza. If 3 people are selected at random, what is the probability that all the three like pizza?
20. A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?
21. A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight?
22. At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking Technology?
23. Two discs are selected one at a time without replacement from a box containing 5 red and 3 blue discs. Find the probability that
- the discs are of the same color
  - if the discs are of the same color, both are red.



### Points to remember

- Probability measures the likelihood of an event occurring, ranging from 0 (impossible) to 1 (certain).
- Sample Space (S): All possible outcomes of an experiment.
- In probability, each counting technique is applied based on given condition

- The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.
- The probability of event A occurring given that event B has occurred is a conditional probability
- Bayes' Theorem is useful for finding the probability of an event based on prior knowledge of conditions that might be related to the event.
- Probability has many applications, you can use it to analyze data and making predictions in your business, to analyse market trends and investment risks, etc...



### Self-Reflection

1. Read the statements across the top, put a check-in a column that best represents your level of knowledge, skills, and attitudes.

| Experience                                                                                            | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|-------------------------------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                                      |                                          |                             |                                    |                                       |                                          |
| Define correctly probability and its associated terms                                                 |                                          |                             |                                    |                                       |                                          |
| Determine the number of permutations and combinations of "n" items, "r" taken at a time.              |                                          |                             |                                    |                                       |                                          |
| Pay attention to details while calculating probability                                                |                                          |                             |                                    |                                       |                                          |
| Describe the application of probability to different fields including tourism, social sciences etc... |                                          |                             |                                    |                                       |                                          |

| Experience                                                                                           | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|------------------------------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                                     |                                          |                             |                                    |                                       |                                          |
| Use counting techniques to solve related problems.                                                   |                                          |                             |                                    |                                       |                                          |
| Approach problems systematically and logically.                                                      |                                          |                             |                                    |                                       |                                          |
| Distinguish properly between combination and permutation                                             |                                          |                             |                                    |                                       |                                          |
| Use properties of combinations                                                                       |                                          |                             |                                    |                                       |                                          |
| Recognize the ethical implications of probabilistic reasoning especial in decision making scenarios. |                                          |                             |                                    |                                       |                                          |
| Define the combinatorial Analysis.                                                                   |                                          |                             |                                    |                                       |                                          |
| Analyse problems to determine relevant probabilities.                                                |                                          |                             |                                    |                                       |                                          |
| Be patient while solving complex probability problems.                                               |                                          |                             |                                    |                                       |                                          |
| Describe different formulas used to solve problems related to calculation of probability             |                                          |                             |                                    |                                       |                                          |
| Apply concepts to solve real-world problems and scenarios.                                           |                                          |                             |                                    |                                       |                                          |
| Describe uses of Unions, Intersections and                                                           |                                          |                             |                                    |                                       |                                          |

| Experience                                                                                                    | I do not have any experience doing this. | I know a little about this. | I have some experience doing this. | I have a lot of experience with this. | I am confident in my ability to do this. |
|---------------------------------------------------------------------------------------------------------------|------------------------------------------|-----------------------------|------------------------------------|---------------------------------------|------------------------------------------|
| Knowledge, skills, and attitudes                                                                              |                                          |                             |                                    |                                       |                                          |
| Complements in probability contexts.                                                                          |                                          |                             |                                    |                                       |                                          |
| Recognise whether repetition is allowed or not. And if order matters or not in performing a given experiment. |                                          |                             |                                    |                                       |                                          |
| Interpret data to make probabilistic inferences                                                               |                                          |                             |                                    |                                       |                                          |

2. Fill in the table above and share results with the trainer for further guidance.

| Areas of strength | Areas for improvement | Actions to be taken to improve |
|-------------------|-----------------------|--------------------------------|
| 1.                | 1.                    | 1.                             |
| 2.                | 2.                    | 2.                             |
| 3.                | 3.                    | 3.                             |

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$\Re$  : Real number

$\Sigma$ : Summation symbol

**e**: Euler's number (~2.718)

**exp(x)**: Exponential function ( $e^x$ )

**ln**: Natural logarithm (logarithm to base e)

**log**: Logarithm (commonly base 10 if unspecified)

**n!** :Factorial of n



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