

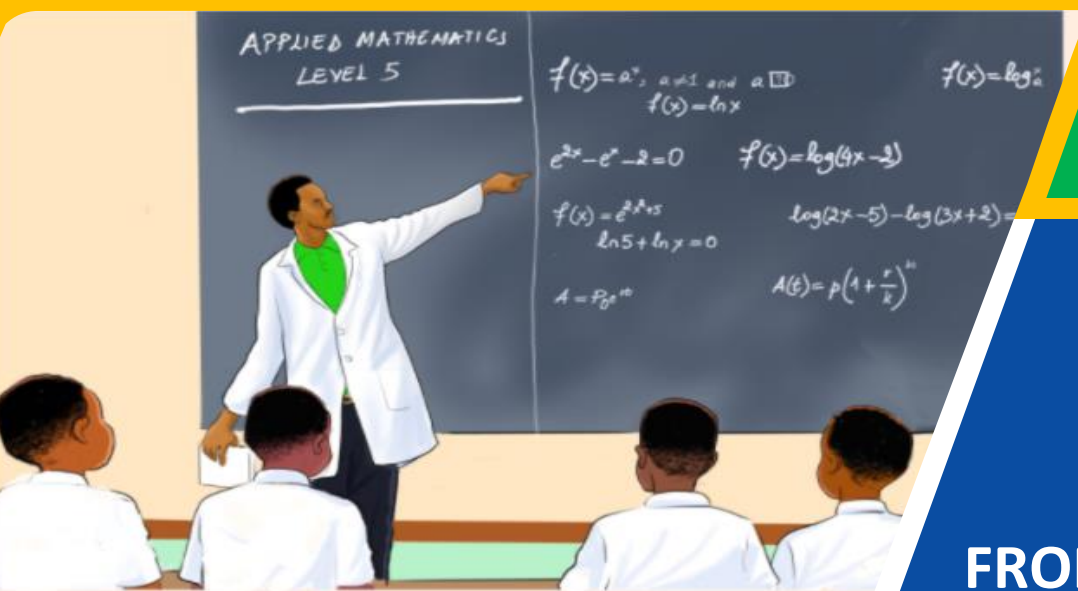


Republic of Rwanda
Ministry of Education



RTB | RWANDA
TVET BOARD

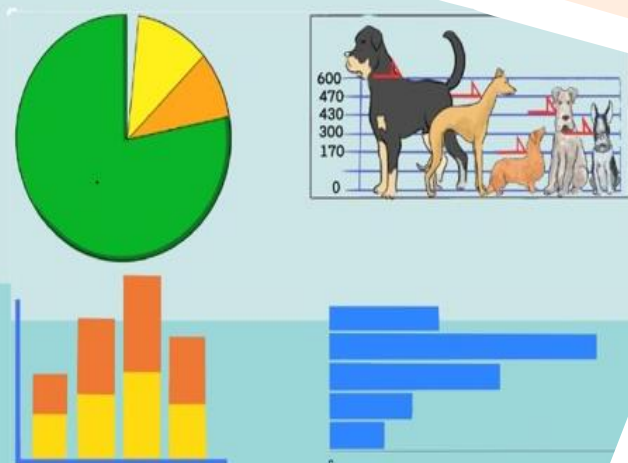
RQF LEVEL 5



FRONT OFFICE AND HOUSEKEEPING OPERATIONS

GENMP501

Applied Mathematics



TRAINEE'S MANUAL

April 2025



Republic of Rwanda
Ministry of Education



RTB | RWANDA
TVET BOARD

APPLY BASIC MATHEMATICAL ANALYSIS, STATISTICS AND PROBABILITY



2025

AUTHOR'S NOTE PAGE (COPYRIGHT)

The competent development body of this manual is **Rwanda TVET Board** © reproduced with permission.

All rights reserved.

- This work was produced initially with the Rwanda TVET Board, with the support from the European Union (EU).
- This work has copyright but permission is given to all the Administrative and Academic Staff of the RTB and TVET Schools to make copies by photocopying or other duplicating processes for use at their workplaces.
- This permission does not extend to making copies for use outside the immediate environment for which they are made, nor making copies for hire or resale to third parties.
- The views expressed in this version of the work do not necessarily represent the views of RTB. The competent body does not give a warranty nor accept any liability.
- RTB owns the copyright to the trainee and trainer's manuals. The training providers may reproduce these training manuals in part or in full for training purposes only. Acknowledgment of RTB copyright must be included in any reproductions. Any other use of the manuals must be referred to the RTB.

© **Rwanda TVET Board**

Copies available from:

- *HQs: Rwanda TVET Board-RTB*
- *Web: www.rtb.gov.rw*

KIGALI-RWANDA

Original published version: April 2025.

ACKNOWLEDGEMENTS

Rwanda TVET Board (RTB) would like to recognize all parties who contributed to the development of the trainer's and trainee's manuals for the TVET Certificate V in in Front Office and Housekeeping Operations for the module: **“GENMP501– Apply basic mathematical analysis, statistics and probability.”**

Thanks to the EU for financial support and Ubukerarugendo Imbere Project for technical support on the implementation of this project.

We also wish to acknowledge all trainers, technicians and practitioners for their contribution to this project.

The management of Rwanda TVET Board appreciates the efforts of its staff who coordinated this project.

Finally, RTB would like to extend its profound gratitude to the MCT Global team that technically led the entire assignment.

This training manual was developed:



Under Rwanda TVET Board (RTB) guiding policies and directives



Under European Union financing



Under Ubukerarugendo imbere project implementation, technical support and guidance

COORDINATION TEAM

Aimable Rwamasirabo

Felix Ntahontuye

Eugène Munyanziza

Production Team

Authoring and Review

Jean Baptiste Habiyakare

Fidele Hagenimana

Dorcas Niyonagira

Conception, Adaptation and Editorial works

Jean Marie Vianney Muhire

Vincent Havugimana

Marie Jeanne Musabyimana

Formatting, Graphics, Illustrations and infographics

Jean Claude Asoka Niyonsaba

Paul Semivumbi

Aliane Umutesi

Augustin Habimana

Coordination and Technical support

Ubukerarugendo Imbere Project and RTB

Project Implementation

MCT Global Ltd

TABLE OF CONTENT

AUTHOR'S NOTE PAGE (COPYRIGHT)	i
ACKNOWLEDGEMENTS	ii
TABLE OF CONTENT	v
LIST OF FIGURES	vii
LIST OF ABBREVIATIONS and ACRONYMS	ix
INTRODUCTION	1
UNIT 1: LINEAR AND QUADRATIC EQUATIONS OR INEQUALITIES	3
Topic 1.1: Linear equations and inequalities	10
Topic 1.2: Parameter from a given equation	21
Topic 1.3: Simultaneous linear equations	24
Topic 1.4: Quadratic equation	33
UNIT 2: ALGEBRAIC FUNCTIONS ANALYSIS	53
Topic 2.1: Domain and range of algebraic function	67
Topic 2.2: Symmetry of algebraic function or Parity of numerical function (odd or even)	75
Topic 2.3: Limits of functions	79
Topic 2.4: Derivative of functions.	103
Topic 2.5: Application of derivative	122
Topic 2.6: Sketching curve of algebraic function	138
UNIT 3: EXPONENTIAL AND LOGARITHMIC EQUATIONS	160
Topic 3.1: Properties for exponential expressions	171
Topic 3.2: Properties for logarithmic expression	175
Topic 3.3: Exponential equations	179

Topic 3.4: Logarithmic equations	184
Topic 3.5: Logarithms and exponential applications	193
UNIT 4: NUMERICAL AND GRAPHICAL METHODS TO DISPLAY DATA.....	207
Topic 4.1: Ungrouped quantitative data	214
Topic 4.2: Measures of central tendency	222
Topic 4.3: Measures of dispersion	226
UNIT 5: FUNDAMENTALS OF PROBABILITIES.....	242
Topic 5.1: Counting techniques.....	1
Topic 5.2: Probabilities.....	21
Topic 5.3: Conditional probability	33
REFERENCE	53

LIST OF FIGURES

Figure 1: Linear function	12
Figure 2: linear inequalities.....	19
Figure 3: Parametric curve.....	23
Figure 5. Simuttaneous equations	30
Figure 6. graph of quagratic function	44
Figure 7: Graph of range	73
Figure 8: Even function	76
Figure 9: Odd function	77
Figure 10: Types of asymptotes	92
Figure 11: Vertical and Oblique asymptotes to the curve.....	97
Figure 12: vertical and horizontal asymptotes to the curve.....	101
Figure 13: Gradient of a curve	104
Figure 14: Slope of a curve.....	110
Figure 15: Trigonometric circle	115
Figure 16: Tangent and normal lines	123
Figure 17: Extrema points	129
Figure 18: Minimum and maximum points	130
Figure 19: Increasing function	131
Figure 20: Decreasing function	131
Figure 21: Inflection point.....	134
Figure 22: Concavity point	135
Figure 23: Graph of polynomial with third degree.	140
Figure 24: Graph of rational function	145
Figure 25: Exponential functions	172
Figure 26: Exponential function and its inverse	176
Figure 27: Pie-chart.....	216
Figure 28: Bar-chart	219
Figure 29: Venn diagram.....	2
Figure 30: Tree-diagram.....	10

Figure 31: Arrangement in a circle.....	15
Figure 32: Mutually exclusive of events A and B	30
Figure 33: Inclusive events A and B	30
Figure 34: Probability of Tree diagram	42

LIST OF ABBREVIATIONS and ACRONYMS

CBET:	Competence Base Education and Training
RQF:	Rwanda Qualification Framework
RTB:	Rwanda TVET Board
TVET:	Technical and Vocational Education and Training
LE:	Linear Equations
LI:	Linear Inequalities
PGE:	Parameter from a Given Equation
CM:	Comparison Method
EM:	Elimination Method
DS:	Direct Substitution
CR:	Cramer's Rule
QE:	Quadratic Equation
FM:	Factorization Method
SRP:	Square Root Property
CSQ:	Completing the Square
QF:	Quadratic Formula
AF:	Algebraic Function
LOF:	Limit of Functions
DOF:	Derivatives of Functions
FD:	First Derivative
SD:	Second Derivative
AOD:	Application of Derivatives
EEQ:	Exponential Equations
LEQ:	Logarithmic Equations
ALE:	Applications of Logarithms and Exponentials
UQD:	Ungrouped Quantitative Data
MCT:	Measures of Central Tendency

INTRODUCTION

This trainee's manual encompasses all necessary skills, knowledge and attitudes required to Apply Basic Mathematical Analysis, Statistics and Probability. Students undertaking this module shall be exposed to practical activities that will develop and nurture their competences. The writing process of this training manual embraced competency-based education and training (CBET) philosophy by providing practical opportunities reflecting real life situations.

The trainee's manual is subdivided into units, each unit has got various topics, you will start with a self-assessment exercise to help you rate yourself on the level of skills, knowledge and attitudes about the unit.

A discovery activity is followed to help you discover what you already know about the unit.

After these activities, you will learn more about the topics by doing different activities by reading the required knowledge, techniques, steps, procedures and other requirements under the key facts section, you may also get assistance from the trainer. The activities in this training manual are prepared such that they give opportunities to students to work individually and in groups.

After going through all activities, you shall undertake progressive assessments known as formative and finally conclude with your self-reflection to identify your strengths, weaknesses and areas for improvement.

Do not forget to read the point to remember the section which provides the overall key points and takeaways of the unit.

Module Units:

Unit 1: Linear and quadratic equations

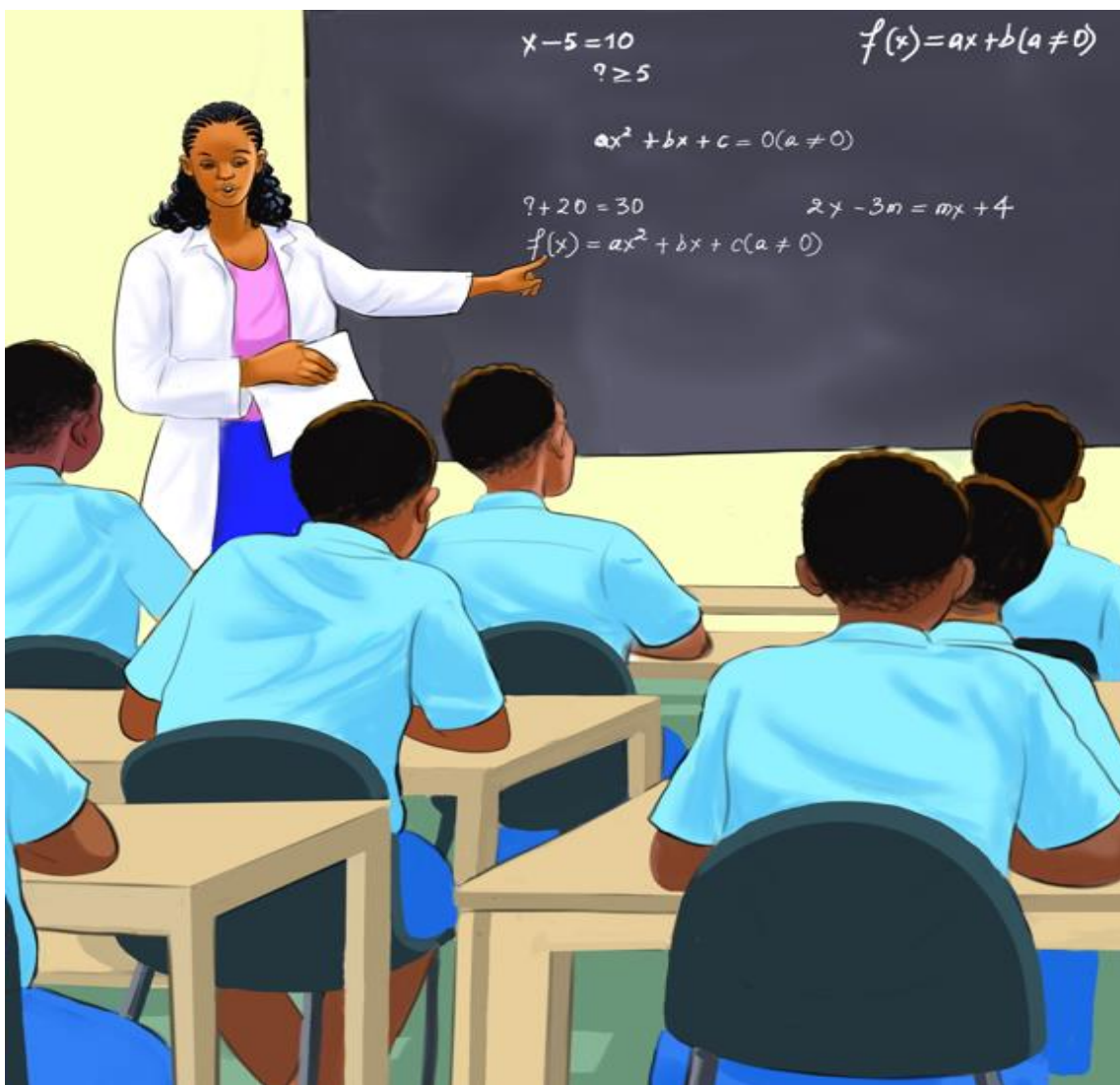
Unit 2: Algebraic functions analysis

Unit 3: Exponential and logarithmic expressions

Unit 4: Numerical and graphical methods to display data

Unit 5: Fundamentals of probabilities

UNIT 1: LINEAR AND QUADRATIC EQUATIONS OR INEQUALITIES



Unit summary

This unit provides you with the knowledge, skills and attitudes required to Solve algebraically or graphically linear and quadratic equations required to Apply Basic Mathematical Analysis, Statistics and Probability. It covers the linear equations and inequalities, parameter from a given equation, simultaneous linear equations, and quadratic equation.

Self-Assessment: Unit 1

- Referring to the unit illustration above and a picture of mathematics equations and functions above, answer the following questions:
 - What is a linear equation?
 - What is a linear inequality?
 - What is difference between quadratic equations and quadratic functions?
 - what are the relationship between solving linear equations and solving linear inequalities?
 - which equation with parameter?
 - which equation with inequality sign?
- Fill out the below self-assessment. Think about yourself: do you think you can do this? How well? Read the statements across the top. Assess your level of knowledge, skills and attitudes under this unit.
 - There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills and attitudes acquired during the learning process
 - Think about yourself: do you think you have the knowledge, skills or attitudes to do the task? How well?
 - Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.
- At the end of this unit, you will assess yourself again.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify the components and characteristics of linear equations and inequalities.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe the methods used to solve linear equations and inequalities.					
Recognize the role and significance of parameters in equations.					
Explain the steps involved in isolating and solving for a parameter in an equation					
Explain the concept and techniques for solving simultaneous linear equations					
Distinguish between different methods for solving systems of equations (substitution, elimination, matrix methods).					
Identify the standard form of a quadratic equation and understand its key features (e.g., roots, vertex, and axis of symmetry).					
Explain the different methods used to					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
solve quadratic equations (factoring, quadratic formula, completing the square).					
Solve linear equations and inequalities using various techniques (e.g., substitution, elimination, graphing).					
Interpret solutions to linear equations and inequalities in different real-life contexts.					
Isolate a parameter in an equation using algebraic manipulation					
Apply appropriate methods (e.g., substitution) to determine the value of a parameter					
Solve simultaneous linear equations using appropriate methods (substitution, elimination).					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Evaluate the solution set of simultaneous equations and interpret the relationship between the variables					
Solve quadratic equations by factoring, completing the square, or using the quadratic formula.					
Graph quadratic equations to visualize their solutions and understand the relationship between the coefficients and the parabola's shape					
Think logically when analyzing different forms of linear equations and inequalities.					
Be consistent when solving linear problems.					
Think logically when solving equations with parameters,					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
ensuring clarity and accuracy.					
Be consistent and persistent when working with algebraic expressions involving parameters.					
Be consistent when selecting methods to solve simultaneous equations.					
Collaborate when solving systems of equations, as sharing approaches can enhance problem-solving					
Think logically when solving quadratic equations, encouraging students to embrace challenges.					
Be consistent and persistent when graphing the quadratic functions					



Key Competencies:

Knowledge	Skills	Attitudes
1. Identify the components and characteristics of linear equations and inequalities.	1. Solve linear equations and inequalities using various techniques (e.g., substitution, elimination, graphing).	1. Think logically when analyzing different forms of linear equations and inequalities.
2. Describe the methods used to solve linear equations and inequalities.	2. Interpret solutions to linear equations and inequalities in different real-life contexts.	2. Be consistent when solving linear problems.
3. Recognize the role and significance of parameters in equations	3. Isolate a parameter in an equation using algebraic manipulation	3. Think logically when solving equations with parameters, ensuring clarity and accuracy.
4. Explain the steps involved in isolating and solving for a parameter in an equation	4. Apply appropriate methods (e.g., substitution) to determine the value of a parameter	4. Be consistent and persistent when working with algebraic expressions involving parameters.
5. Explain the concept and techniques for solving simultaneous linear equations	5. Solve simultaneous linear equations using appropriate methods (substitution, elimination).	5. Be consistent when selecting methods to solve simultaneous equations.
6. Distinguish between different methods for solving systems of equations (substitution, elimination, matrix methods).	6. Evaluate the solution set of simultaneous equations and interpret the relationship between the variables	6. Collaborate when solving systems of equations, as sharing approaches can enhance problem-solving
7. Identify the standard form of a quadratic equation and understand its key features (e.g., roots, vertex, axis of symmetry).	7. Solve quadratic equations by factoring, completing the square, or using the quadratic formula.	7. Think logically when solving quadratic equations, encouraging students to embrace challenges.

Knowledge	Skills	Attitudes
8. Explain the different methods used to solve quadratic equations (factoring, quadratic formula, completing the square).	8. Graph quadratic equations to visualize their solutions and understand the relationship between the coefficients and the parabola's shape	8. Be consistent and persistent when graphing the quadratic functions.



Discovery activity:



Task 1

Two friends Sarah and John agree to share the cost of a gift equally and their total cost of the gift is \$48. Sarah decided to buy 5 pens, and paid an additional of \$3 for a notebook on her share and the total bill about the cost of pens and notebook was \$13.

1. How much did each friend get?
2. How much did each pen cost?
3. How can you write the total cost of pens and a notebook in terms of linear equation?

Topic 1.1: Linear equations and inequalities



Activity 1: Problem Solving



Task 2:

If x is the number of pens for a learner, the teacher decides to give him/her two more pens.

- a. What is the number of pens will have a learner with one pen?
- b. Complete the following table of value to indicate the number $y = f(x) = x + 2$ of pens for a learner who had x pens for $x \geq 0$.

X	0	1	2	3	4
---	---	---	---	---	---

Y	2				
(x,y)	(0,2)				

- c. Use the coordinates of points obtained in the table and plot them on the Cartesian plan.
- d. Join all points obtained. What is the form of the graph obtained?
- e. Suppose that instead of writing $f(x) = x + 2$ you write the equation $y = x + 2$. Is this equation a linear equation or a quadratic equation? What is the type of the inequality " $x + 2 \geq 0$ "?

Key Facts 1.1.a: Linear equations and inequalities

- **Algebraic method (AM).**

- ✓ **Definition:** linear function is a function of a straight line which has a form of $y = mx + b$ where $m \neq 0$ and $b \in \mathfrak{R}$, For examples:

1. $y = 2x + 1$

2. $5x = 6 + 3y$

3. $\frac{y}{2} = 3 - x$

Let us look more closely at one example: $y = 2x + 1$ is a linear function. The graph of $y = 2x + 1$ is a straight line.

✚ When x increases, y increases twice as much, hence $2x$

✚ When x is 0, y is already 1. Hence +1 is also needed • So, the graph of $y =$

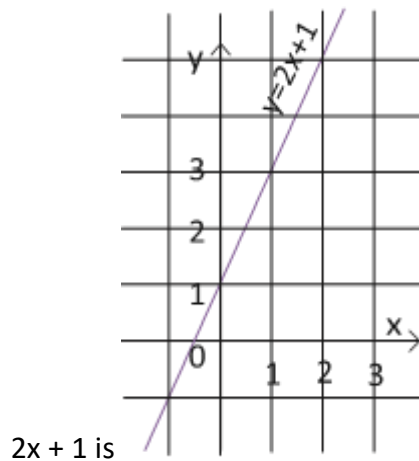


Figure 1: Linear function

The general form of the linear equation with one variable is $ax + b = 0$ Where a , $b \in \mathbb{R}$: and $a \neq 0$

Solving linear equations

- ✚ A linear equation is a polynomial of degree 1.
- ✚ In order to solve for the unknown variable, you must isolate the variable.
- ✚ In the order of operation, multiplication and division are completed before addition and subtraction. the value of x in which the equality is verified is called the root (solution of the equation).

Examples: Find x if $2x + 4 = 10$.

Solution with steps

- ✓ Isolate “ x ” to one side of the equation by subtracting 4 from both sides:

$$2x + 4 - 4 = 10 - 4 \quad 2x = 6$$
- ✓ 2. Divide both sides by 2, $\frac{2x}{2} = \frac{6}{2}$, $x = 3$
- ✓ 3. Check your work with the original equation: $2x + 4 = 10$, $(2 \cdot 3) + 4 = 10$, $6 + 4 = 10$.

Product equation

This is in the form $(ax + b)(cx + d) = 0$.

Since the product of factors is null (zero) either one of them is zero. To solve this

we proceed as follows: $(ax + b)(cx + d) = 0$, $ax + b = 0$ or $cx + d = 0$ $x = \frac{-b}{a}$ or

$x = \frac{-d}{c}$. Fractional equation of the first degree

The general form is $\frac{ax + b}{cx + d} = 0$. We have to fix the existence condition $cx + d \neq 0$

$\frac{ax + b}{cx + d} = 0$
and $\frac{ax + b}{cx + d} \Rightarrow ax + b = 0$. We solve $ax + b$ and we take the value(s) which verify the existence conditions.

- Examples: Solve the following linear equations:

✓ 1. $(4x + 8)(3x + 2) = 0$, solution is $2x + 8 = 0$ or $3x + 2 = 0$

$$\begin{array}{l} 2x + 8 = 0 \\ 2x = -8 \quad 3x + 2 = 0 \\ \text{For } x = \frac{-8}{2} \quad \text{or } 3x = -2 \\ \quad \quad \quad x = -4 \quad \quad x = -\frac{2}{3} \end{array}$$

✓ $\frac{4x - 12}{x - 1} = 0$, solution: for the existence condition $x - 1 \neq 0 \Rightarrow x \neq 1$

Then $\frac{4x - 12}{x - 1} = \frac{0}{1}$ by cross multiplication

$$(4x - 12)(1) = 0(x - 1) \Rightarrow 4x - 12 = 0 \Rightarrow 4x = 12 \Rightarrow x = \frac{12}{4} \Rightarrow x = 3$$

- Exercises 1.1

✓ Solve the following:

(a) $(x + 2)(2x - 1) = 0$ (b) $(5x - 15)(3x - 9) = 0$ (c) $x^2 - 5x = 0$ (d) $\frac{3x - 6}{x + 1} = 0$

(e) $\frac{x - 2}{5x + 3} = 0$ (e) $3(x + 7) = 0$ (f) $\frac{3 - x}{5x - 7} = 0$

✓ Construct the graph of the following linear equations on the Cartesian plan:

a) $f(x) = 2x$ b) $y = -2x + 3$ c) $y = 2x + 1$

Fractional equation of the first degree where we need to find the Lowest Common
(Ndorimana, 2016)



Activity 2: Guided Practice



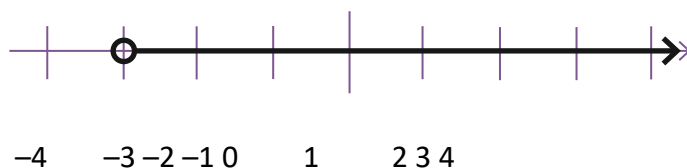
Task 3:

1. The following mathematical statements are always true for only one value of x . For each statement. By collecting the same terms together find out the real value of x :
 - a. $x + 1 = 5$
 - b. $2x - 4 = 0$
 - c. $2x + 1 = -5$
 - d. $x - 4 = 10$
2. Given the two linear equations in 2 unknowns $x - y = 1$ and $x - y = 1$
 - a. For each linear equation, choose any two values of x and use them to find the values of y . This gives you two points in the form of (x, y)
 - b. Plot the obtained points in XY plane and join them to obtain two different lines.
 - c. What is the point of intersection for the two lines?
3. Find at least 5 value(s) of x such that the following statements are true
 - a. i. $x < 10$ ii. $x > 0$ iii. $-3 < x < 10$

Key Facts 1.1. b: Linear equations and inequalities

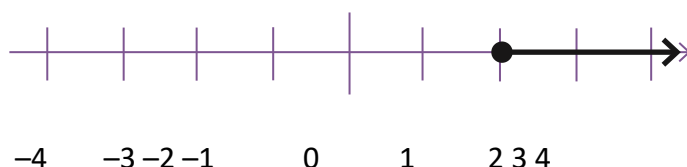
- Graphical method(GM)
 - ✓ Solving linear inequalities by using number line

The graph of a linear inequality in one variable is a number line. We use an unshaded circle for $<$ and $>$ and a shaded circle for $\leq$ and $\geq$. The graph for $x > -3$:



$$S =]-3, +\infty[$$

The graph for $x \geq 2$:



$$S = [2, +\infty[$$

Solve and graph the solution set of: $3(2x + 4) > 4x + 10$. Multiply out the parentheses. Subtract $4x$ from both sides. Subtract 12 from both sides. Divide both sides by 2, but do not change the direction of the inequality sign, since we did not divide by a negative. Unshaded circle at -1 (since x cannot equal -1) and an arrow to the right (because we want values larger than -1)

Solution:

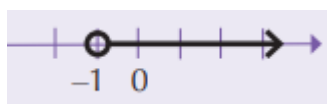
$$3(2x + 4) > 4x + 10$$

$$6x + 12 > 4x + 10$$

$$6x - 4x > 10 - 12$$

$$2x > -2$$

$$x > -1$$

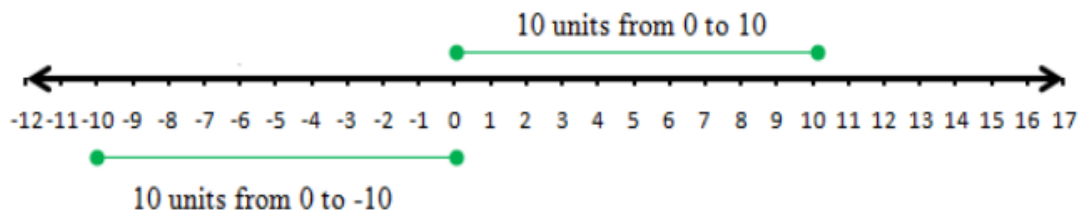


$$S =]-1, +\infty[$$

✓ Linear Inequality with absolute value

The inequality $|x - a| < k$ says that the distance from x to a is less than k so x must lie between $a - k$ and $a + k$ or equivalently a must lie between $x - k$ and $x + k$ if k is a positive number. Recall that absolute value of a number is the number of units

from zero to a number line. That is, $|x| = k$ means k units from zero (k is a positive real number or zero).



For all real number x and $k \geq 0$

a) $|x| = k \Leftrightarrow -k < x < k$

b) $|x - a| < k \Leftrightarrow -k < x - a < +k \Leftrightarrow a - k < x < a + k$

c) $|x| > k \Leftrightarrow x > k, \text{ or } x < -k$

d) $|x - a| > k \Leftrightarrow x > a + k \text{ or } x < a - k$

Examples: 1. $|2x - 2| \geq 4$

2. $|3x - 15| < 3$

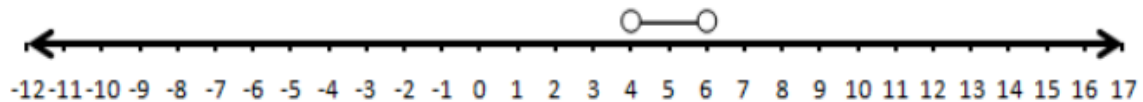
For this question we have two cases:

$$\begin{array}{ll} |2x - 2| \geq 4 & |2x - 2| \geq 4 \\ -(2x - 2) \geq 4 & +(2x - 2) \geq 4 \\ \text{Solutions: 1. } -2x + 2 \geq 4 & \text{and } 2x - 2 \geq 4 \quad S =]-\infty, -1] \cup [3, +\infty[\\ -2x \geq 4 - 2 & 2x \geq 6 \\ -2x \geq 2 & x \geq 3 \\ x \leq -1 & \end{array}$$

Solution 2. There are two ways of performing this question:

By property:

$$\begin{array}{lll}
 |3x-15| < 3 & |3x-15| < 3 & |3x-15| < 3 \\
 -3 < 3x-15 < 3 & -(3x-15) < 3 & +(3x-15) < 3 \\
 -3+15 < 3x < 3+15 & -3x+15 < 3 & 3x-15 < 3 \\
 12 < 3x < 18 & \text{or} & \text{and} \\
 \frac{12}{3} < x < \frac{18}{3} & -3x < 3-15 & 3x < 3+15 \\
 4 < x < 6 & -3x < -12 & 3x < 18 \\
 & x > 4 & x < 6
 \end{array}$$



$$S = [4, 6]$$

Exercises: Solve the following inequality: a. $6(x+4) - 7 - (3x+10) \geq 8(x-1)$

b. $12x + 5 \geq 8(x-1)$ c. $4x + 5 \geq 6x - 1$

- Solving linear function by graphing Cartesian plan.

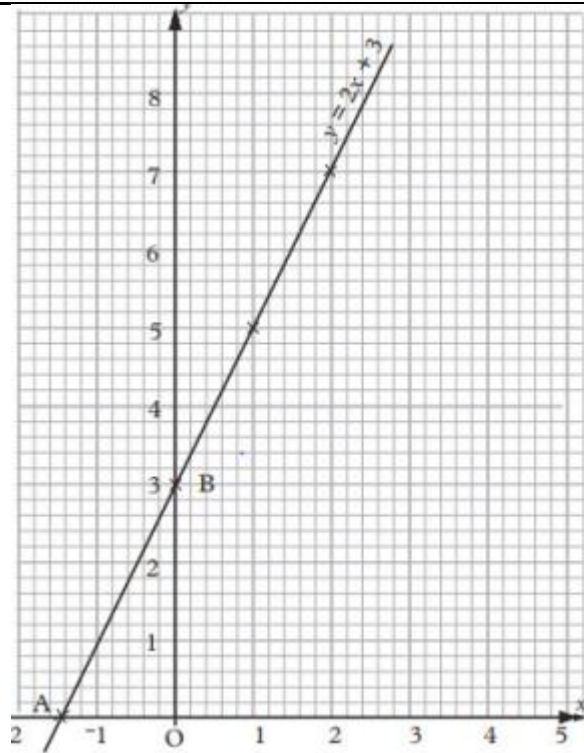
When drawing a graph of a linear function, it is sufficient to plot only two points and these points may be chosen as the x and y intercepts of the graph. In practice, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

If we assign x any value, we can easily calculate the corresponding value of y.

Determine the x-intercept, set $f(x) = 0$ and solve for x and then determine the y intercept, set $x = 0$ to find $f(0)$. Consider the function $y = 2x + 3$.

- ✓ When $x = 0$, $y = 2 \times 0 + 3 = 3$
- ✓ When $x = 1$, $y = 2 \times 1 + 3 = 5$
- ✓ When $x = 2$, $y = 2 \times 2 + 3 = 7$ and so on. For convenience and ease while reading, the calculations are usually tabulated as shown below in the table of values for $y = 2x + 3$:

X	0	1	2	3	4
Y	3	5	7	9	11



For linear inequalities of two variables: The inequality is $y \geq x + 2$.

You can see the line, $y = x + 2$ and the shaded area is where y is greater than or equal to $x + 2$, Shade where there is a solution.

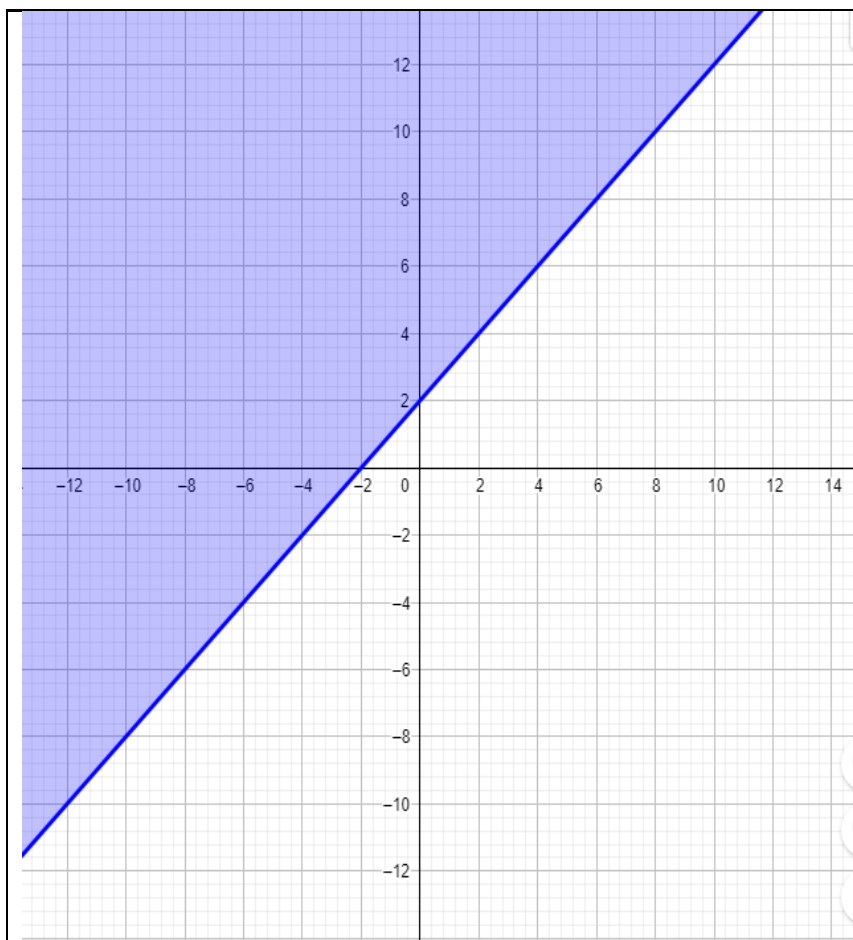


Figure 2.*linear inequalities*

Exercises:

solves and graph the following: 1. $y - x \geq 3$ 2. $\frac{y}{2} + 3 > 2$

(REB, 2023)



Activity 3: Application



Task 4:

- According to the given meaning of equations where x is the number of note books needed by the students, for each statement, find out the real value of x :
 - $x + 1 = 5$
 - $3x + 6 = 0$
 - $15 - x = 7$
 - A company is determining the number of hours; employees can work per week to meet certain conditions. Find at least 5 values of x (hours worked) that satisfy the following conditions: i. $x < 5$ ii. $x > 0$ iii. $-4 < x < 12$
 - A teacher is creating examples of linear inequalities and equalities to help students understand budgeting. Formulate 3 examples of linear inequalities and linear equalities that represent different budgeting scenarios.
- Solve the following linear equations where x is the number of pens needed by the students:
 - $\frac{3x - 5}{5} = \frac{4x + 5}{3}$ b) $\frac{2}{5} + \frac{5x}{3} = \frac{7}{2} - \frac{3x}{10}$ c) $\frac{5x}{3} + 10 = \frac{4x}{7} + \frac{1}{14}$
- A company is determining the minimum number of products x that need to be sold to meet certain profit targets. Solve the following inequalities to find the range of products that need to be sold: a. $2x + 6 \geq x - 5$, b. $\frac{6 + 3x}{12} \geq \frac{3 + 4x}{12}$
- John has 1, 260, 000 Rwandan Francs in an account with his bank. If he deposits 30, 000 Rwanda Francs each week into the account, how many weeks will he need to have more than 1, 820, 000 Rwandan Francs on his account?

Topic 1.2: Parameter from a given equation



Activity 1: Problem Solving



Task 5:

A logistics company manages deliveries using two trucks, Truck A and Truck B. Each truck moves along a straight road, and their positions are described by parametric equations:

- Truck A's position (in kilometres) is given by:

$$X_A(t) = 2t + 5$$

$$Y_A(t) = 3t - 2$$

(where t is the time in hours, and $t \geq 0$).

- Truck B's position (in kilometres) is given by:

$$X_B(t) = -t + 8$$

$$Y_B(t) = 2t - 1$$

- At what time t will the two trucks meet, if at all?
- What is their meeting point (if they meet)?

Key Facts 1.2: Parametric linear equation(PLE)

- Parametric equations (PE)**

There are also a great many curves that we cannot even write down as a single equation in terms of only x and y . So, to deal with some of these problems we introduce parametric equations. Instead of defining y in terms of x like $y = f(x)$ or x in terms of y like $x = h(y)$

we define both x and y in terms of a third variable called a parameter as follows: $x = f(t)$ $y = g(t)$

This third variable is usually denoted by t (but does not have to be). Sometimes we will restrict the values of t that we shall use and at other times we will not. If the coefficients of an equation contain one or several letters (variables) the equation is called parametric and the letters are called real parameters. In this case, we solve

and discuss the equation (for parameters only). Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the parametric curve.

Example:

Solve and discuss the equation $(2 - 3m)x + 1 = m^2(1 - x)$.

$$\begin{aligned} (2 - 3m)x + 1 &= m^2(1 - x) \\ \text{Solution: } 2x - 3mx + 1 &= m^2 - m^2x \Rightarrow x(m^2 - 3m + 2) = m^2 - 1 \\ m^2x - 3mx + 2x &= m^2 - 1 \end{aligned}$$

$$x = \frac{m^2 - 1}{m^2 - 3m + 2} \Rightarrow x = \frac{m + 1}{m - 2}$$

$$x = \frac{(m - 1)(m + 1)}{(m - 1)(m - 2)}$$

The domain of $f(m) = \frac{m + 1}{m - 2}$ is $\mathbb{R} \setminus \{2\}$ and we can say that:

If $m \neq 2$, then the solution is $x = \frac{m + 1}{m - 2}$

If $m = 2$, then there is no solution.

Graphs

Sketching a parametric curve is not always an easy thing to do. Let us look at an example to see one way of sketching a parametric curve. This example will also illustrate why this method is usually not the best.

Example: Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \quad y = 2t - 1$$

solution: At this point our only option for sketching a parametric curve is to pick values of t , substitute them into the parametric equations and then plot the points.

T	X	Y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{2}$	-2
0	0	-1
1	2	1

Here is the sketch of this parametric curve.

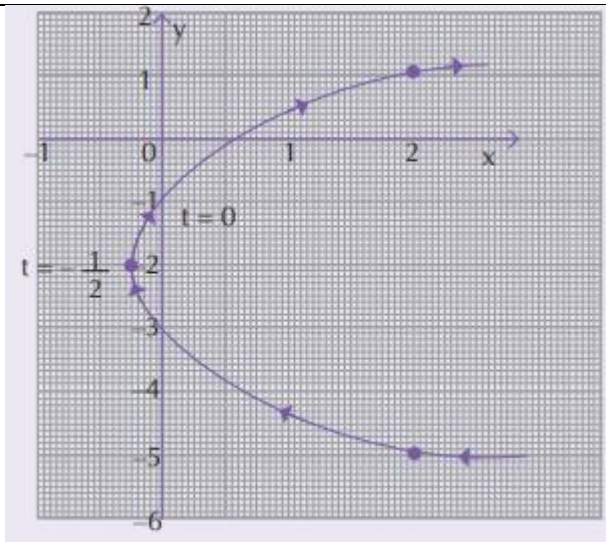


Figure 3: Parametric curve

So, we have a parabola that opens to the right. (Ndorimana, 2016)



Activity 2: Guided Practice



Task 6:

By collecting the same terms together solve and discuss on the equation: $2tx - 3 = 3t - 4x$



Activity 3: Application



Task 7:

1. Imagine you are tracking the movement of a car on a curved path. The car's position at any given time can be described using the following parametric equations:

- The horizontal position of the car, x , is given by:

$$x = t^2 + t$$

- The vertical position of the car, y , is given by:

$$y = 2t - 1$$

Where t represents time in seconds.

Problem:

Eliminate the parameter t from the given equations and find a relationship between x and y that describes the car's path.

2. Imagine you're observing the motion of a point on a Ferris wheel. The horizontal position of the point as it moves in a circular motion is described by the equation:

The horizontal position x is given by

- $x = 5 \cos t$

The vertical position y is given by:

- $y = 2 \sin t$

Where t represents time, and the motion occurs over one full revolution, i.e $0 \leq t \leq 2\pi$

Problem:

Sketch the path traced by the point on the Ferris wheel as it moves. Clearly indicate the direction of motion over the time interval $0 \leq t \leq 2\pi$

Topic 1.3: Simultaneous linear equations



Activity 1: Problem Solving



Task 8:

Three friends, Alex, Bella, and Chris, went to a coffee shop. They ordered the following:

- Alex ordered 2 coffees and 1 sandwich, which cost a total of \$10.
- Bella ordered 1 coffee and 2 sandwiches, which cost a total of \$13.

The cost of a coffee and a sandwich is the same for everyone.

1. How much does one coffee cost?
2. How much does one sandwich cost?
3. How much do you pay if you want three coffees and 4 sandwiches?

Key Facts 1.3.a: Simultaneous linear equations

- **Algebraic method(AM)**

- ✓ **Simultaneous equations in two unknowns**

A linear equation in two variables x and y is an equation of the form $ax + by = c$ where $a \neq 0$, $b \neq 0$ and a, b, c are real numbers.

Let us consider such equation $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, b_1, c_1, a_2, b_2 and c_2 are

constants. We say that we have two simultaneous linear equations in two unknowns or a system of two linear equations in two unknowns.

The pair (x, y) satisfying both equations is the solution of the given equation.

Examples: The solution of $\begin{cases} x + y = 8 \\ x - y = 2 \end{cases}$ is $(5, 3)$

We can solve such systems of linear equations by using one of the following methods:

1. Direct substitution (**DM**)
2. Elimination method (**EM**)
3. Cramm's method (**CM**)
4. Comparison method (**CM**)
5. Graphical method (**GM**)

- ✓ **Substitution**

This method is used when one of the variables is given in terms of the other.

Example:

Find the simultaneous solution of the following pair of equations: $y = 2x - 1$, $y = x + 3$

3. Note that the system can also be written as $\begin{cases} y = 2x - 1 \\ y = x + 3 \end{cases}$, then

$$2x - 1 = x + 3 \Rightarrow 2x - x = 3 + 1 \Rightarrow x = 4 \text{ And so } y = 4 + 3, y = 7$$

So, the simultaneous solution is $x = 4$ and $y = 7$.

- **Elimination method (EM):**

To eliminate one of the variables from either of equations to obtain an equation in just one unknown, make one pair of coefficients of the same variable in both equations negatives of one another by multiplying both sides of an equation by the same number. Upon adding the equations, that unknown will be eliminated.

Example:

Solve the system of equations using elimination method. $\begin{cases} x + y = 2 \\ 2x + 3y = 2 \end{cases}$

$$\text{Solution: } \begin{cases} x + y = 2 \\ 2x + 3y = 2 \end{cases} \begin{array}{l} -2 \\ 1 \end{array} \Rightarrow \begin{array}{l} -2x - 2y = -4 \\ 2x + 3y = 2 \end{array} \Rightarrow y = -2 \Rightarrow x - 2 = 2 \Rightarrow x = 4$$

$$S = \{(x, y)\} \Rightarrow S = \{(4, -2)\}$$

- **Cramer's method (CM)**

To solve the two simultaneous linear equations in two unknowns x and y, Cramer's rule requires to go through the following steps:

1. Arrange the equations to get $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ and calculate the principal

$$\text{determinant } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

If $D = 0$, then the system has no solution or infinitely many solutions; the system is not a Cramer's system. If $D \neq 0$, then the system is a Cramer's system and has unique solution, proceed to the next step:

2. Write down and calculate: $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$ and

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

3. Write down and calculate $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$; the solution set of the

simultaneous equations is, $S = \left\{ \left(\frac{D_x}{D}, \frac{D_y}{D} \right) \right\}$

In the same way, for the three simultaneous linear equations in three unknowns, x, y and z,

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}, \text{ The principal determinant is } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $D = 0$, then the system is not a Cramer's system, it may have zero solution or infinitely many solutions.

If $D \neq 0$, then the system is a Cramer's system and has unique solution; the solution set is,

$$S = \left\{ \left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D} \right) \right\}, \text{ where } D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example: Use Cramer's rule to solve the simultaneous equations:
$$\begin{cases} 3x + 2y - 2z = 4 \\ x + 3y + z = 7 \\ 2x + y - z = 11 \end{cases}$$

$$\text{Solution: } D = \begin{vmatrix} 3 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 4$$

$$D_x = \begin{vmatrix} 4 & 2 & -2 \\ 7 & 3 & 1 \\ 11 & 1 & -1 \end{vmatrix} = 4 \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 7 & 1 \\ 11 & -1 \end{vmatrix} - 2 \begin{vmatrix} 7 & 3 \\ 11 & 1 \end{vmatrix} = 72$$

$$D_y = \begin{vmatrix} 3 & 4 & -2 \\ 1 & 7 & 1 \\ 2 & 11 & -1 \end{vmatrix} = 3 \begin{vmatrix} 7 & 1 \\ 11 & -1 \end{vmatrix} - 4 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 7 \\ 2 & 11 \end{vmatrix} = -36$$

$$D = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 3 & 7 \\ 2 & 1 & 11 \end{vmatrix} = 3 \begin{vmatrix} 3 & 7 \\ 1 & 11 \end{vmatrix} - 2 \begin{vmatrix} 1 & 7 \\ 2 & 11 \end{vmatrix} + 4 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 64$$

$$S = \left\{ \left(\frac{D_x}{D}, \frac{D_y}{D}, \frac{D_z}{D} \right) \right\} \Rightarrow S = \left\{ \left(\frac{72}{4}, \frac{-36}{4}, \frac{64}{4} \right) \right\} \Rightarrow s = \{(18, -9, 16)\}$$

(REB, 2023)

2. Comparison method (CM)

Solving a system of linear equations by equating two same variables

To find the value of unknown from simultaneous equation by equating the same variable in terms of another, we do the following steps:

- i. Find out the value of one variable in first equation
- ii. Find out the value of another variable in second equation
- iii. Equating the obtained two same variables
- iv. Solve the equation to find out the unknown variables
- v. Substitute the obtained value of one unknown in one equation to get the second value.

Example:

- 1) Algebraically, solve the simultaneous linear equation by equating the same variables.

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

Solution: From equation (1) $4x + 5y = 2 \Rightarrow 4x = 2 - 5y \Rightarrow x = \frac{2-5y}{4}$, From equation (2)

$x = -1 - 2y$ Equalize the values of from equation (1) and (2).

$$\begin{array}{lcl} \frac{2-5y}{4} = -1-2y & & x = -1-2y \\ 2-5y = -4-8y & \Rightarrow & 3y = -6 \\ -5y+8y = -4-2 & \Rightarrow & y = -2 \\ & & x = -1-2(-2) \Rightarrow S = \{(3, -2)\} \\ & & x = -1+3 \\ & & x = 3 \end{array}$$

Key Facts 1.3.b: : Simultaneous linear equations

- **Graphical method(GM)**

- ✓ **Solving a system of linear equations by Graphical Method**

One way to solve a system of linear equations is by graphing. The intersection of the graphs represents the point at which the equations have the same -value and the same -value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the solution to the system of equations.

The following steps can be applied in solving system of linear equation graphically:

1. Find at least two points for each equation.
2. Plot the obtained points in XY plane and join these points to obtain the lines.

Two points for each equation give one line.

3. The point of intersection for two lines is the solution for the given system

Example

Solve the following system by graphical method $\begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$

Solution: for $x+y=4$,

X	-3	5
Y	7	-1

For $x-y=2$

X	-3	5
Y	-5	3

Graph

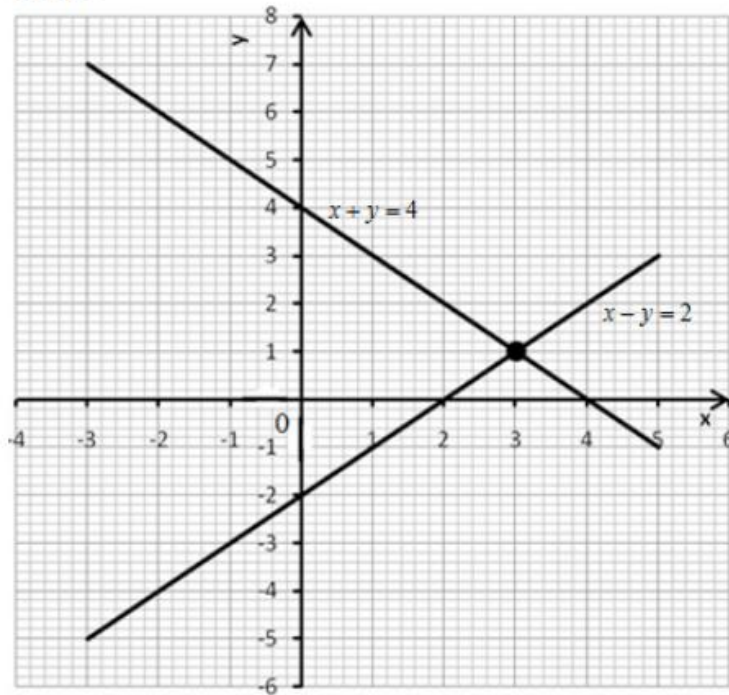


Figure 4. Simuttaneous equations

- ✓ The two lines intersect at point (3,1). Therefore, the solution is $S = \{(3,1)\}$.

Exercise: 1. At a clearance sale all, CDs are sold for one price and all DVDs are sold for another price. Mugabo bought 3 CDs and 2 DVDs for a total of 1,900 FRW, and Mucyo bought 2 CDs and 5 DVDs for a total of 3,100 FRW. Find the cost of each item.

2. By using all methods solve the simultaneous equation:
$$\begin{cases} 5x + 3y = 12 \\ 7x + 2y = 19 \end{cases}$$
3. Mr. John invested a part of his investment in 10% bond A and a part in 15% bond B. His interest income during the first year is 4,000Frw. If he invests 20% more in 10% bond A and 10% more in 15% bond B, his income during the second year increases by 500Frw. Find his initial investment in bonds A and B using matrix method. (REB, 2023)



Activity 2: Guided Practice



Task 9:

Given simultaneous equation $\begin{cases} 3x + 2y = 4 \\ 5x - 4y = 14 \end{cases}$

- a. complete the following table by finding the value of y for $3x + 2y = 4$

X	-2	0	2	3
Y				

And find the value of y for $5x - 4y = 14$

X	-2	0	2	3
Y				

- b. By using mathematical set sketch the graph of the given equations by joining the coordinates which are above.
- c. Find the intersection point of two simultaneous equations
- d. Find the value of x and y for given function



Activity 3: Application



Task 10:

- Use elimination rule to solve the following simultaneous equations:
 - A store sells two types of fruits: apples and bananas. Let x be the price of one apple and y be the price of one banana. The total cost for 4 apples and 7 bananas is 34 dollars. The total cost for 3 apples and 1 banana is 11 dollars. Solve the system to find the price of each fruit: $\begin{cases} 4x + 7y = 34 \\ 3x - y = 11 \end{cases}$
 - In a game, two players have a total of 1 point together. The difference in their scores is 2 points. Let x be the score of the first player and y be the score of the second player. Solve the system to find their individual scores: $\begin{cases} x + y = 1 \\ x - y = 2 \end{cases}$

2. Use graphical method to solve:

A farmer has a total of 4 animals, some are cows and some are goats. The difference between the number of cows and goats is 2. Let x be the number of cows and y be the number of goats. Solve the system to find the number of each animal:

$$\begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$$

3. Use Cramer's rule to solve the following simultaneous equations:

- a. A bakery sells two types of pastries. Let x be the number of croissants and y be the number of muffins. The total number of pastries sold is 8, and the difference between the number of muffins and croissants is 2. Solve the system to find the

number of each pastry sold:
$$\begin{cases} 4y + 4x = 8 \\ -x + y = 2 \end{cases}$$

- b. A store sells three types of products: A, B, and C. Let x , y , and z be the prices of products A, B, and C respectively. The following equations represent the total cost for different combinations of these products. Solve the system to find the

price of each product:
$$\begin{cases} x - 2y + z = -2 \\ 3x + y - 2z = 7 \\ x + 3y - z = 2 \end{cases}$$

4. Use comparison method to solve the following simultaneous equations:

A store sells two types of drinks: juice and soda. Let x be the price of one juice and y be the price of one soda. The total cost for 2 juices and 1 soda is 3 dollars. The total cost for 6 juices and 3 sodas is 9 dollars. Solve the system to find the price of

each drink:
$$\begin{cases} 2x + y = 3 \\ 6x + 3y = 9 \end{cases}$$

Topic 1.4: Quadratic equation



Activity 1: Problem Solving

A gardener wants to fence a rectangular garden with an area of 100 square meters. The length of the garden is 10 meters longer than its width. Let the width be x meters. Then:

Area=Length times Width $\Rightarrow 100 = (x + 10)x$, this simplifies to: $x^2 + 10x - 100 = 0$.



Task 11:

1. What are the dimensions of the garden? (Solve for x).
2. What is the perimeter of the garden?
3. What is the length of its diagonal?

Key Facts 1.4.a: Quadratic equation

- **Algebraic method(AM)**

Equations of the type $ax^2 + bx + c = 0 (a \neq 0)$ are called quadratic equations.

There are four main ways of solving such equations:

- a. By factorizing or finding square roots
- b. Square root property

- c. By using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- d. By completing the square

- ✓ **Solving Quadratic equations by factorizing or finding square roots**

The method of solving quadratic equations by factorization should only be used if it is readily factorized by inspection. To factorize a quadratic equation, one can use the sum and product of its roots. Let x and y be two real numbers such that $x+y=s$ and $xy = p$, here $y=s-x$ and $x(s-x)=p$ or

$$xs - x^2 = p \text{ then } xs - x^2 - p = 0 \Rightarrow x^2 - xs + p = 0.$$

This equation is said to be quadratic equation and s, p are the sum and product of the two roots respectively.

Quadratic equation or equation of second degree has the form

$ax^2 + bx + c = 0 (a \neq 0)$, where the sum of two roots is $s = -\frac{b}{a}$ and their product is

$$p = \frac{c}{a}.$$

Solving quadratic equations by factorizing

Let us use an example:

✓ **$x^2 - 5x + 6$. To solve $x^2 - 5x + 6 = 0$ we must first factorize $x^2 - 5x + 6$.**

To do this we have to find two numbers with a sum of -5 and a product of 6 . The numbers required are -2 and -3 , so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

a. by using factorization method solve: $5x^2 + 7x - 6 = 0$

$$5x^2 + 7x - 6 = 0$$

Solution: $p = 5 \times -6 = -30$ Two numbers are -3 and 10

$$S = 7$$

$$\Rightarrow 5x^2 - 3x + 10x - 6 = 0$$

$$(5x^2 - 3x) + (10x - 6) = 0 \Rightarrow$$

$$x(5x - 3) + 2(5x - 3) = 0$$

$$\begin{array}{l} x + 2 = 0 \quad 5x - 3 = 0 \\ x = -2 \quad 5x = 3 \end{array} \quad S = \left\{ -2, \frac{3}{5} \right\}$$

$$x = \frac{3}{5}$$

b. by using factorization method solve: $x^2 - 9 = 0$

$$x^2 + 0x - 9 = 0$$

Solution: This question is like writing this $x^2 - 3x + 3x - 9 = 0 \Rightarrow$
 $(x^2 - 3x) + (3x - 9) = 0$

$$x(x - 3) + 3(x - 3) = 0$$

$$(x + 3)(x - 3) = 0 \Rightarrow x + 3 = 0 \Rightarrow x = -3 \quad \text{and} \quad \begin{array}{l} x - 3 = 0 \\ x = 3 \end{array} \Rightarrow S = \{-3, 3\}$$

When the product of two or more numbers is zero, then at least one of them must be zero. So, if $ab = 0$ then $a = 0$ or $b = 0$.

c. Square root property (SRP)

To find square the root property for quadratic equation with the form

$$ax^2 + b = 0, \text{ for } a \neq 0 \text{ and } b \leq 0$$

Example: by using square root property solve $2x^2 - 32 = 0$

$$x^2 = 16$$

$$\text{Solution: } 2x^2 - 32 = 0 \Rightarrow x = \sqrt{16} \Rightarrow S = \{-4, 4\}$$

$$x = \pm 4$$

d. Completing the square (CS)

A quadratic equation in the unknown x is an equation of the form

$$ax^2 + bx + c = 0 (a \neq 0), \text{ where } a, b \text{ and } c \text{ are given real numbers, with } a \neq 0.$$

This may be solved by completing the square or by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$, there are two distinct real roots
- If $b^2 - 4ac = 0$, there is a single real root (which may be convenient to treat as two equal or coincident roots)
- If $b^2 - 4ac < 0$, the equation has no real roots.

We know that the quadratic equation is of the form:

$$ax^2 + bx + c = 0 (a \neq 0), \text{ Let us get a formula to solve } x.$$

$$a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \Rightarrow$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right) = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

Remember that to solve for x: $x = \frac{-b \pm \sqrt{D}}{2a}$

Example: by using completing square Solve the following quadratic equations:

$$2x^2 - 3x - 2 = 0$$

Before solving quadratic equations by completing the square, let's look at some examples of expanding a binomial by squaring it.

- $(x+3)^2 = x^2 + 6x + 9$.
- $(x-5)^2 = x^2 - 10x + 25$

Notice that the constant term k² of the trinomial is the square of half of the coefficient of trinomial's x -term. Thus, to make the expression $x^2 + kx$ a perfect square, you must add $\left(\frac{1}{2}k\right)^2$ to the expression. When completing the square to solve quadratic equation, remember that you must preserve the equality. When you add a constant to one side of the equation, be sure to add the same constant to the other side of equation.

Solution:

$$\begin{aligned}
 2\left(x^2 - \frac{3}{2}x - \frac{2}{2}\right) &= 0 \\
 x^2 - \frac{3}{2}x - 1 &= 0 & \left(x - \frac{3}{4}\right)^2 &= \frac{25}{16} & x_1 &= \frac{3+5}{4} \\
 x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16} - 1 &= 0 & \Rightarrow x - \frac{3}{4} &= \sqrt{\frac{25}{16}} & x_1 &= 2 \\
 \Rightarrow x - \frac{3}{4} &= \pm \sqrt{\frac{25}{16}} & \Rightarrow x_2 &= \frac{3-5}{2} & S &= \{-1, 2\} \\
 \left(x - \frac{3}{4}\right)^2 - \frac{9-16}{16} &= 0 & x &= \frac{3 \pm 5}{4} & x_2 &= \frac{-2}{2} \\
 \left(x - \frac{3}{4}\right)^2 - \frac{25}{16} &= 0 & x &= \frac{3 \pm 5}{4} & x_2 &= -1
 \end{aligned}$$

e. Quadratic formula(QF)

Solving Quadratic equations by the formula

to solve this equation, first we find the discriminant (delta): $D = b^2 - 4ac$

In fact, $x = \frac{-b \pm \sqrt{D}}{2a}$ as we have seen on completing square.

$$x_1 = \frac{-b + \sqrt{D}}{2a} \quad x_2 = \frac{-b - \sqrt{D}}{2a}$$

Example: by using quadratic formula solve:

$$x^2 - 7x + 5 = -5$$

Solution: $x^2 - 7x + 5 + 5 = 0 \Rightarrow a=1, b=-7 \text{ and } c=10 \Rightarrow$
 $x^2 - 7x + 10 = 0$

$$D = b^2 - 4ac \Rightarrow D = (-7)^2 - 4(1)(10) \Rightarrow D = 49 - 40 \Rightarrow D = 9$$

$$x_1 = \frac{-b + \sqrt{D}}{2a} = \frac{-(-7) + \sqrt{9}}{2(1)} = \frac{7+3}{2} = \frac{8}{2} = 4$$

$$x_2 = \frac{-b - \sqrt{D}}{2a} = \frac{-(-7) - 3}{2(1)} = \frac{7-3}{2} = \frac{4}{2} = 2 \quad S = \{2, 4\}$$

f. Quadratic Inequalities in one unknown

The product ab of two factors is positive if and only if (i) $a > 0$ and $b > 0$ or (ii) $a < 0$ and $b < 0$.

Thus $(x - 1)(x + 2) > 0$ if and only if

(i) $x - 1 > 0$ and $x + 2 > 0$ or

(ii) $x - 1 < 0$ and $x + 2 < 0$ i.e., $x > 1$ and $x > -2$ or $x < 1$ and $x < -2$

Sign diagrams

Although this method is sound it is not of much practical use in more complicated problems. A better method which is useful in more complicated problems is the following which uses the "sign diagram" of the product $(x - 1)(x + 2)$.

$$(x - 1)(x + 2) > 0$$

Solution:

X Factors	$-\infty$ $+\infty$ -2 1
X+2	----- 0+++++
x-1	----- 0+++++ ++
(x+2)(x-1)	+++++0-----0+++++ +

$$S =]-\infty, -2[\cup]1, +\infty[$$

Note: The sign diagram of a linear function $ax + b$ is summarized here. The

critical value is $x = \frac{-b}{a}$ (it is the value of x when $ax + b = 0$).

X Factors	$-\infty$ $+\infty$ $-\frac{b}{a}$
Ax+b	Opposite sign of A 0 Same sign as A

For any interval, the sign of the product of two linear functions is the product of the signs of its factors.

Note: The summary of sign diagram for $ax^2 + bx + c$:

If $\Delta > 0$, there are two critical values, x_1 and x_2

X Factors	$-\infty$	x1	x2
	$+\infty$		
Ax²+bx+c	Same sign as A	0 opposite sign of A	0 Same sign as A

If $\Delta = 0$, there is one critical value x_1 and the quadratic has the same sign as a .

If $\Delta < 0$, there is no critical value and the quadratic has the same sign as a .

Inequalities depending on the quotient of two linear factors

The quotient $\frac{b}{a}$ of two factors is positive if and only if (i) $a > 0$ and $b > 0$ or (ii) $a < 0$ and $b < 0$.

These are the same conditions under which the product ab is positive.

Therefore, solving inequalities such as $\frac{x+3}{x-1} > 0$ is the same as solving $(x+3)(x-1) > 0$.

Example: solve $\frac{2x-4}{x+3} \geq 0$

Solution:

X	$-\infty$	-3	2	+
Factors	∞			
X+3	----- 0+++++			
2x-4	----- -0+++++			
$\frac{-2x+4}{x+3}$	+++++		----- 0+++++	

$$S =]-\infty, -3[\cup [2, +\infty[$$

g. Solving general inequalities

The techniques illustrated in the previous pages can be used to solve complicated inequalities. There are also other techniques which may be used in special cases.

Example: Solve $\frac{(x-1)(x^2+1)}{x+2} \geq 0$

X	$-\infty$	-2	1	$+\infty$
Factors				
X-1	-	----- 0+++++		
X+2	-	----- 0+++++		
X ² +1	+++++			
$\frac{(x-1)(x^2+1)}{x+2}$	+++++		----- 0+++++	

$$S =]-\infty, -2[\cup [1, +\infty[\text{ (Ndorimana, 2016)}$$

h. Parametric quadratic equations in one unknown

If at least one of the coefficients a, b and c depend on the real parameter which is not determined, the root of the parametric quadratic equation depends on the values attributed to that parameter.

Examples: Find the values of k for which the equation $x^2 + (k + 1)x + 1 = 0$ has: (a) two distinct real roots (b) no real roots. (c) repeated root

Solution: $a=1$, $b= k+1$ and $c=1$

$$D = b^2 - 4ac$$

$$D = (K+1)^2 - 4(1)(1)$$

$$D = k^2 + 2K + 1 - 4$$

$$D = k^2 + 2K - 3 \quad D = (K-1)(K+3)$$

- If $b^2 - 4ac > 0$, there are two distinct real roots
- If $b^2 - 4ac < 0$, the equation has no real roots
- If $b^2 - 4ac = 0$, there is a single real root (which may be convenient to treat as two equal or coincident roots)

K	$-\infty$	-3	1	$+\infty$
Factors				
K+3	- - - - - 0 + + + + +			
K-1	- - - - - 0 + + + + +			
(K+3)(K-1)	+ + + + + 0 - - - - - 0 + + + + +			

- $S =]-\infty, -3[\cup]1, +\infty[$
- $S =]-3, 1[$
- $K_1 = -3$ and $K_2 = 1$ (Ndorimana, 2016)



Activity 2: Guided Practice



Task 12:

By using the factorization method where the two numbers you multiply to get -6 and their summation is -1 are -3 and 2, solve the following quadratic equation:

$$2x^2 - x - 3 = 0$$

Key Facts 1.4.b: Quadratic equation

- **Graphical method(GM)**

A polynomial in which the highest power of the variable is 2 is called a quadratic function. The expression form $y = ax^2 + bx + c = 0 (a \neq 0)$, where a, b and c are given real numbers, with $a \neq 0$, is called a quadratic function of x or a function of the second degree (highest power of x is two).

Table of values are used to determine the coordinates that are used to draw the graph of a quadratic function. To get the table of values, we need to have the domain (values of an independent variable) and then the domain is replaced in each quadratic function to find range (values of dependent variables). The values obtained are useful for plotting the graph of a quadratic function. All quadratic function graphs are parabolic in nature.

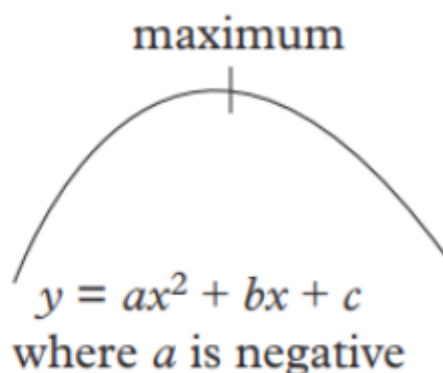
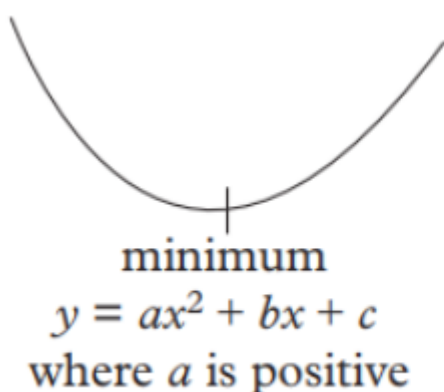
Any quadratic function has a graph which is symmetrical about a line which is parallel to the y-axis i.e., a line $x = h$ where h is constant value. This line is called axis of symmetry.

For any quadratic function $f(x) = ax^2 + bx + c = 0 (a \neq 0)$, whose axis of symmetry is the line $x = -\frac{b}{2a}$, we can get the y-coordinate of the vertex by substituting the x-

coordinate of axis of symmetry. The vertex becomes, $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ or $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

. The vertex of a quadratic function is the point where the function crosses its axis of symmetry. If the coefficient of the x^2 term is positive, the vertex will be the lowest point

on the graph, the point at the bottom of the U-shape. If the coefficient of the term x^2 is negative, the vertex will be the highest point on the graph, the point at the top of the shape. The shapes are as below:



The intercepts with axes are the points where a quadratic function cuts the axes. There are two intercepts i.e. x-intercept and y-intercept. x-intercept for any quadratic function is calculated by letting $y = 0$ and y- intercept is calculated by letting $x = 0$

Graph of a quadratic function

The graph of a quadratic function can be sketched without table of values if the following are known.

- The vertex
- The x-intercepts
- The y-intercept

Example: Find the vertex and axis of symmetry of the parabolic curve $y = 2x^2 - 8x + 6$

Solutions: The coefficients are $a = 2$, $b = -8$ and $c = 6$

$$h = -\frac{b}{2a} = -\frac{-8}{2(2)} = 2$$

- The x-coordinate of the vertex is
- The y-coordinate of the vertex is obtained by substituting the x-coordinate of the vertex to the quadratic function. We get $y = 2(2)^2 - 8(2) + 6 = -2$
- The vertex is $(2, -2)$ and the axis of symmetry is $x = 2$.
- When $x = 0$, $y = 2(0)^2 - 8(0) + 6 = 6$.
- The y-intercept is $(0, 6)$

When $y = 0$, $0 = 2x^2 - 8x + 6$, we therefore solve the quadratic equation for the values of x and we find the x -intercepts are $(1,0)$ or $(3,0)$

The graph is as below:

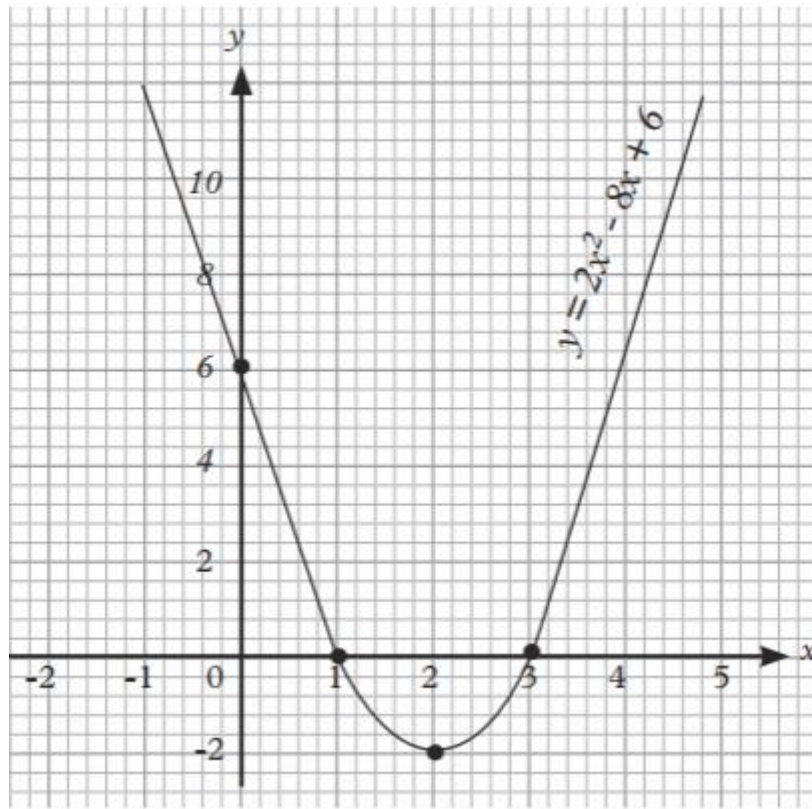


Figure 5. graph of quadratic function

Exercises: Find the vertex and axis of symmetry of the parabolic curve and the additional points and make the graph of: a. $y = x^2 + x + 1$ b. $y = -2x^2 + 3x + 4$ (REB, 2022)



Activity 3: Application



Task 13:

1. Solve in the set of real numbers the following equations using the quadratic formula:
 - a. A ball is thrown into the air, and its height x in meters after t seconds is given by the equation $x^2 - 12x + 11 = 0$. Solve the equation to find the time at which the ball hits the ground.

- b. The area of a rectangular garden is given by the equation $x^2 + 2x = 35$, where x is the length of the garden in meters. Solve the equation to find the dimensions of the garden.
2. Solve the following inequalities:
- a. A company produces two products. The profit from selling x units of the first product and y units of the second product is given by the inequality $(1-x)(2x+1) > 0$. Determine the range of x for which the company makes a profit.
- b. The cost C in dollars of producing x items is given by the inequality $-2x^2 + x + 3 \leq 0$. Determine the number of items that can be produced without exceeding the budget.
- c. The number of hours' x a worker can work in a week is given by the inequality $x(x+4) > x-4$. Determine the range of hours the worker can work to meet the company's requirements.
3. Solve the following inequalities:

- a. A company's profit margin is given by the inequality $\frac{2x-1}{x+2} > 4$, where x represents the number of units sold. Determine the range of units that need to be sold to achieve a profit margin greater than 4.

- b. The efficiency of a machine is given by the inequality $\frac{2x-1}{x+2} \geq 4$, where x represents the hours of operation. Determine the range of hours the machine should operate to maintain an efficiency of at least 4.

- c. The performance rating of an employee is given by the inequality $\frac{2x-1}{x+2} > 2$, where x represents the number of tasks completed. Determine the range of tasks the employee needs to complete to achieve a performance rating greater than 2.

4.

- a. A ball is thrown, and its height is modeled by the equation $(m-3)x^2 - 8x + 4 = 0$, where m is a parameter affecting the throw and x is the time in seconds. Find the range of values of m for which the equation has:

- i. Two distinct real roots, meaning the ball reaches two different heights at two different times.
 - ii. No real roots, meaning the ball never reaches a certain height.
 - iii. One double root, meaning the ball reaches a maximum height at one specific time.
- b. The cost of producing items is modeled by the $2x^2 - 5x + 3m - 1 = 0$, where m is a cost parameter and x is the number of items produced. Find the range of values of m for which the equation has:
- c. Two distinct real roots, meaning there are two different production levels with the same cost.
 - d. No real roots, meaning there is no production level that matches a certain cost.
 - e. One double root, meaning there is a unique production level that matches a specific cost.



Formative Assessment

1. Without using table of values, state the vertices, intercepts, axes of symmetry, and sketch the graphs of: $y = 2x^2 + 5x - 1$
2. Find the values of m for which the equation $x^2 + (m + 3)x + 4m = 0$ has:
 - a. Two distinct real roots ii) No real roots. iii) Repeated roots ($x_1 = x_2$)
 - b. find the range of values of k for which the equation $x^2 - 2(k + 1)x + k^2 = 0$ has: i) Two distinct real roots ii) No real roots iii) Repeated roots ($x_1 = x_2$).
3. Use direct substitution to solve the following simultaneous equation:
$$\begin{cases} x^2 + y^2 = 221 \\ xy = 110 \end{cases}$$
4. Use Cramer's rule to solve the following simultaneous equations:
 - a.
$$\begin{cases} 4x + y = 34 \\ 3x - 2y = 11 \end{cases}$$
 - b.
$$\begin{cases} x - 2y + z = -2 \\ 3x + y - 2z = 7 \\ x + 3y - z = 2 \end{cases}$$

5. Suppose your cell phone plan is 3000 FRW per month plus 20 FRW per minute. Your bill is 7 000 FRW. Use the equation $3000 + 20x = 7000$, to find out how many minutes are on your bill.

6. Solve the following equations and inequalities in set of real numbers

a. $x + 5 = 2x - 8$

b. $2x - 8 \geq 0$ c. $\frac{3x}{4} - \frac{1}{5} = \frac{6x}{2} - \frac{3}{5}$ d. $(3x + 7)(x - 2) < 0$ e. $\frac{x + 3}{x - 2} < 7$ f. $6x^2 - 5x + 1 \geq 0$

g. $x^2 - 10x + 1 = 0$ h. $\frac{x^3 + x^2 - 9x - 9}{2x - 2} \geq 0$ j. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

7. The width of a rectangle is 20 meters. What must the length be if the perimeter is at least 180 meters?

$$\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y \geq 1 \\ x \leq 4 \end{cases}$$

8. Solve the system of inequalities by graphing:
9. A company produces three products every day. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total combined production of the first and the third product is twice that of the second product. Determine the production level of each product using Cramer's rule.

10. The function $g(x) = x^3 + px^2 - 5x + q$ has a factor $x - 2$ and has a value of 5 when $x + 3 = 0$, find p and q.

11. given $p(x) = x^3 - 3x + 2$

a. Determine the numbers c and d such that $p(x) = (x - 1)(x^2 + cx + d)$

b. Factorize $p(x)$ completely /3marksiii. Solve for x the equation $p(x) = 0$

12. Given that α and β are roots of the equation $5x^2 + 2x + 1 = 0$, calculate the value of E

if $E = \frac{5}{\alpha^2} + \frac{5}{\beta^2} + 32$

13. Solve $\sqrt{5x - 24} - \sqrt{x} = 0$



Points to Remember

- Linear function and linear equation
- Construct the graph of linear functions
- Linear inequalities in one unknown
- Solving linear inequalities by using number line.
- Linear inequalities with absolute value
- Solving linear function by Cartesian plan
- Solving linear inequalities by Cartesian plan
- Parametric linear equation and parametric linear functions
- Solving two simultaneous linear equations with 2 unknowns
- Solving three simultaneous linear equation with 3 unknowns
- Algebraic methods of solving linear simultaneous equations.
- Solving quadratic equations.
- Solving quadratic inequalities
- Solving quadratic parametric equations
- Graphical method of solving quadratic inequalities.



Self-Reflection

1. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify the components and characteristics of linear equations and inequalities.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe the methods used to solve linear equations and inequalities.					
Recognize the role and significance of parameters in equations.					
Explain the steps involved in isolating and solving for a parameter in an equation					
Explain the concept and techniques for solving simultaneous linear equations					
Distinguish between different methods for solving systems of equations (substitution, elimination, matrix methods).					
Identify the standard form of a quadratic equation and understand its key features (e.g., roots, vertex, axis of symmetry).					
Explain the different methods used to solve					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
quadratic equations (factoring, quadratic formula, completing the square).					
Solve linear equations and inequalities using various techniques (e.g., substitution, elimination, graphing).					
Interpret solutions to linear equations and inequalities in different real-life contexts.					
Isolate a parameter in an equation using algebraic manipulation					
Apply appropriate methods (e.g., substitution) to determine the value of a parameter					
Solve simultaneous linear equations using appropriate methods (substitution, elimination).					
Evaluate the solution set of simultaneous equations and interpret					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
the relationship between the variables					
Solve quadratic equations by factoring, completing the square, or using the quadratic formula.					
Graph quadratic equations to visualize their solutions and understand the relationship between the coefficients and the parabola's shape					
Think logically when analyzing different forms of linear equations and inequalities.					
Be consistent when solving linear problems.					
Think logically when solving equations with parameters, ensuring clarity and accuracy.					
Be consistent and persistent when working with algebraic expressions involving parameters.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Be consistent when selecting methods to solve simultaneous equations.					
Collaborate when solving systems of equations, as sharing approaches can enhance problem-solving					
Think logically when solving quadratic equations, encouraging students to embrace challenges.					
Be consistent and persistent when graphing the quadratic functions.					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	1.	1.
2.	2.	2.
3.	3.	3.

UNIT 2: ALGEBRAIC FUNCTIONS ANALYSIS



Unit summary

This unit provides you with the knowledge, skills and attitudes required to Analyze and graph algebraic functions required to Apply Basic Mathematical Analysis, Statistics and Probability. It covers the domain and range of algebraic function, symmetry of algebraic function, limit of function, and derivative of functions, Application of derivative and curve of algebraic function.

Self-Assessment: Unit 2

1. Look at the unit illustration in the Manuals and together discuss:
 - a. What is the domain and range of the function?
 - b. What is the parity of the functions, and how to find it?
 - c. What is the formula with the form of limit of function?
 - d. What are the indeterminate forms above, how to remove them by using limit of function?
 - e. What is the notation of differentiation?
 - f. Which formulas used in application of differentiation?
 - g. Which graph with extrema points and inflection point?
 - h. How to find the extrema points and inflection points?
2. Fill out the below self-assessment. Think about yourself: do you think you can do this? How well? Read the statements across the top. Assess your level of knowledge, skills and attitudes under this unit.
3. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills and attitudes acquired during the learning process
4. Think about yourself: do you think you have the knowledge, skills or attitudes to do the task? How well?
5. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.
6. At the end of this unit, you will assess yourself again.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify a function as a rule and recognize rules that are not functions.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Determine the domain and range of a function					
Construct composition of functions					
Find whether a function is even, odd, or neither					
Find whether a function is even, odd, or neither					
Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.					
Evaluate the limit of a function and extend this concept to determine the asymptotes of the given function.					
Evaluate derivatives of functions using the definition of derivative					
Define and evaluate from first principles the gradient at a point					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Distinguish between techniques of differentiation to use in an appropriate context					
Apply the formula for the slope of the tangent line using the derivative at the given point					
Recall the relationship between the tangent and normal lines (normal line is perpendicular to the tangent).					
Interpret the derivative as the rate of change of position (velocity) in kinematics					
Explain the application of L'Hopital's Rule to resolve indeterminate forms in limits					
Define maximum and minimum points using the first and second derivative tests.					
Recall how the sign of the first derivative indicates increasing or decreasing behaviour					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Explain the definition of an inflection point as where concavity changes.					
Identify concave up and concave down intervals based on the sign of the second derivative					
Contrast the purpose of a variation table to summarize critical points and intervals of increase/decrease					
Identify additional key points (e.g., intercepts, turning points) that help in sketching the graph					
Recall the behaviour of polynomial functions based on degree and leading coefficient.					
Perform operations on functions.					
Apply different properties of functions to model and solve related problems in various practical contexts					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Analyse, model and solve problems involving linear or quadratic functions and interpret the results					
Calculate limits of certain elementary functions					
Develop introductory calculus reasoning					
Apply informal methods to explore the concept of a limit including one sided limits.					
Use the concepts of limits to determine the asymptotes to the rational and polynomial functions.					
Use first principles to determine the gradient of the tangent line to a curve at a point					
Apply the concepts and techniques of differentiation to model, analyse and solve rates or optimization problems in different situations					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify various differentiation techniques, such as product differentiation, market differentiation, and pricing differentiation					
Compute the slope and equation of the tangent line at a specific point on the curve.					
Determine the equation of the normal line by using the negative reciprocal of the tangent slope.					
Use the derivative to calculate velocity and acceleration from a position-time function					
Apply L'Hopital's Rule to evaluate limits involving $0/0$ or ∞/∞ forms					
Identify critical points and use the first or second derivative to determine maxima or minima.					
Identify critical points and use the first or second derivative to					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
determine maxima or minima.					
Determine the inflection point by finding where the second derivative changes sign.					
Classify intervals as concave up or concave down by analysing the second derivative.					
Construct a variation table to organize information about a function's behaviour					
Plot additional points to refine the graph and ensure accuracy in function representation.					
Sketch the graph of a polynomial by identifying key features like roots, turning points, and end behaviour.					
Sketch the graph of a rational function by determining vertical and horizontal asymptotes and analysing intercepts					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Be consistent and persistent when performing operations on functions.					
Show concern on patience, mutual respect and tolerance when applying properties of functions.					
Show concern on patience, mutual respect and tolerance when solving problems about polynomial, rational and irrational functions					
Show concern on the importance, the use and determination of limit of functions					
Appreciate the use of parity of functions.					
Appreciate the use of gradient as a measure of rate of change (economics)					
Be consistent and persistent when using the concepts of limits to determine the asymptotes to the rational and polynomial functions					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Think logically when determining the gradient of the tangent line to a curve at a point					
Be consistent and persistent when applying techniques of differentiation to model, analyse and solve rates or optimization problems in different situations					
Show interest in derivatives to help in the understanding optimization problem					
Think logically when calculating the tangent line equation.					
Think logically on the consideration of perpendicular slopes when working with normal lines.					
Foster curiosity about how calculus connects to real-life motion and change.					
Be consistent and persistent when faced with challenging limit					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
problems and the use of appropriate rules					
Think logically when determining extreme values of a function.					
Show interest in systematic analysis to accurately determine function behaviour over intervals.					
Think logically when determining the inflection point by finding where the second derivative changes sign					
Instil precision and clarity when analysing the concavity of functions.					
Think logically when using variation tables.					
Be consistent and persistent when selecting and plotting critical points					
Think logically and step-by-step analysis when graphing polynomial functions.					
Think logically when analysing and sketching rational functions.					



Key Competencies:

Knowledge	Skills	Attitudes
1. Identify a function as a rule and recognize rules that are not functions.	1. Perform operations on functions.	1. Be consistent and persistent when performing operations on functions.
2. Determine the domain and range of a function	2. Apply different properties of functions to model and solve related problems in various practical contexts	2. Show concern on patience, mutual respect and tolerance when applying properties of functions.
3. Construct composition of functions	3. Analyse, model and solve problems involving linear or quadratic functions and interpret the results	3. Show concern on patience, mutual respect and tolerance when solving problems about polynomial, rational and irrational functions
4. Find whether a function is even, odd, or neither	4. Calculate limits of certain elementary functions	4. Show concern on the importance, the use and determination of limit of functions
5. Find whether a function is even, odd, or neither	5. Develop introductory calculus reasoning	5. Appreciate the use of parity of functions.
6. Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.	6. Apply informal methods to explore the concept of a limit including one sided limits.	6. Appreciate the use of gradient as a measure of rate of change (economics)
7. Evaluate the limit of a function and extend this concept to determine the asymptotes of the given function.	7. Use the concepts of limits to determine the asymptotes to the rational and polynomial functions.	7. Be consistent and persistent when using the concepts of limits to determine the asymptotes to the rational and polynomial functions

Knowledge	Skills	Attitudes
8. Evaluate derivatives of functions using the definition of derivative	8. Use first principles to determine the gradient of the tangent line to a curve at a point	8. Think logically when determining the gradient of the tangent line to a curve at a point
9. Define and evaluate from first principles the gradient at a point	9. Apply the concepts and techniques of differentiation to model, analyse and solve rates or optimization problems in different situations	9. Be consistent and persistent when applying techniques of differentiation to model, analyse and solve rates or optimization problems in different situations
10. Distinguish between techniques of differentiation to use in an appropriate context	10. Identify various differentiation techniques, such as product differentiation, market differentiation, and pricing differentiation	10. Show interest in derivatives to help in the understanding optimization problem
11. Apply the formula for the slope of the tangent line using the derivative at the given point	11. Compute the slope and equation of the tangent line at a specific point on the curve.	11. Think logically when calculating the tangent line equation.
12. Recall the relationship between the tangent and normal lines (normal line is perpendicular to the tangent).	12. Determine the equation of the normal line by using the negative reciprocal of the tangent slope.	12. Think logically on the consideration of perpendicular slopes when working with normal lines.
13. Interpret the derivative as the rate of change of position (velocity) in kinematics	13. Use the derivative to calculate velocity and acceleration from a position-time function	13. Foster curiosity about how calculus connects to real-life motion and change.
14. Explain the application of L'Hopital's Rule to	14. Apply L'Hopital's Rule to evaluate limits	14. Be consistent and persistent when faced

Knowledge	Skills	Attitudes
resolve indeterminate forms in limits	involving $0/0$ or ∞/∞ forms	with challenging limit problems and the use of appropriate rules
15. Define maximum and minimum points using the first and second derivative tests.	15. Identify critical points and use the first or second derivative to determine maxima or minima.	15. Think logically when determining extreme values of a function.
16. Recall how the sign of the first derivative indicates increasing or decreasing behaviour	16. Identify critical points and use the first or second derivative to determine maxima or minima.	16. Show interest in systematic analysis to accurately determine function behaviour over intervals.
17. Explain the definition of an inflection point as where concavity changes.	17. Determine the inflection point by finding where the second derivative changes sign.	17. Think logically when determining the inflection point by finding where the second derivative changes sign
18. Identify concave up and concave down intervals based on the sign of the second derivative	18. Classify intervals as concave up or concave down by analysing the second derivative.	18. Instil precision and clarity when analysing the concavity of functions.
19. Contrast the purpose of a variation table to summarize critical points and intervals of increase/decrease	19. Construct a variation table to organize information about a function's behaviour	19. Think logically when using variation tables.
20. Identify additional key points (e.g., intercepts, turning points) that help in sketching the graph	20. Plot additional points to refine the graph and ensure accuracy in function representation.	20. Be consistent and persistent when selecting and plotting critical points
21. Recall the behaviour of polynomial functions based on degree and leading coefficient.	21. Sketch the graph of a polynomial by identifying key features like roots,	21. Think logically and step-by-step analysis when graphing polynomial functions.

Knowledge	Skills	Attitudes
	turning points, and end behaviour.	
22. Recognize asymptotic behaviour and discontinuities in rational functions	22. Sketch the graph of a rational function by determining vertical and horizontal asymptotes and analysing intercepts	22. Think logically when analysing and sketching rational functions.



Discovery activity:



Task 14

A ball is thrown vertically into the air so that it reaches a height of $y = 19.6t - 4.9t^2$ meters in t seconds.

- Find the velocity and acceleration of the ball at time t seconds.
- Find the time taken for the ball to reach its highest point.
- How high did the ball rise?
- At what time(s) would the ball be at half its maximum height? Topic 2.1: Domain and range of algebraic function

Topic 2.1: Domain and range of algebraic function



Activity 1: Problem Solving



Task 15:

The polynomial function $f(x) = x^2 - x + 6$ is provided.

- Explain how you would determine the domain of this polynomial function.
- Determine the range of $f(x) = x^2 - x + 6$. Show your calculations.

Key Facts 2.1: Domain and range of algebraic function

- **Existence condition**

- ✓ **Numerical functions**

Consider two sets A and B. A function from A into B is a rule which assigns a unique element of A exactly one element of B. We write the function from A to B as $f: A \rightarrow B$. Most of time we do not indicate the two sets and we simply write $y = f(x)$. The sets A and B are called the domain and range of f, respectively. The domain of f is denoted by $dom(f)$.

Let $f(x) = \frac{5x}{x+1}$. Find the following: a. $f(2)$ b. $f(2x-7)$ c. $f(x+h)$ d. $f(x^2)$ e.

$$f\left(\frac{1}{x^2}\right)$$

Solutions:

$$\text{a. } f(2) = \frac{5(2)}{2+1} = \frac{10}{3} \quad \text{b. } f(2x-7) = \frac{2(2x-7)}{(2x-7)+1} = \frac{4x-14}{2x-6}$$

$$\text{c. } f(x+h) = \frac{2(x+h)}{(x+h)+1} = \frac{2x+2h}{x+h+1} \quad \text{d. } f(x^2) = \frac{2(x^2)}{x^2+1} = \frac{2x^2}{x^2+1}$$

$$\text{e. } f(x) = \frac{2\left(\frac{1}{x}\right)^2}{\left(\frac{1}{x}\right)^2+1} = \frac{\frac{2}{x^2}}{\frac{1}{x^2}+1} = \frac{\frac{2}{x^2}}{\frac{1+x^2}{x^2}} = \frac{2}{1+x^2}$$

- ✓ **Domain and range of a function:**

To know the properties of domain of functions first we have to know the types of functions:

- ✚ **Types of functions**

- **A constant function:** A function that assigns the same value to every member of its domain is called a constant function. This is $f(x) = c$, where c is a given real number.

Example: The function given by $f(x) = 4$ is constant.

○ **Polynomial Function**

A function that is expressible as the sum of finitely many monomials in x is called polynomial in x .

Example: $f(x) = x^3 + 4x + 4$ is a polynomial function. Also

$f(x) = (x - 2)^3$ is a polynomial function in x because it is expressible as a sum of monomials. In general, f is a polynomial function in x if it is expressible in the form $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, $a_n \neq 0$, where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n$ are real constants.

✓ **A polynomial function is called:**

✚ Monomial if it has the form cx^n , where c is constant and n is a non-negative integer.

Example: $f(x) = 2x^2$

✚ linear if it has the form $a_0 + a_1x$ where $a_1 \neq 0$, with degree 1

Example: $f(x) = 2x + 1$

✚ Quadratic if it has the form $a_0 + a_1x + a_2x^2$, $a_2 \neq 0$ with degree 2

Example: $f(x) = 2x^2 - 3x + 4$

✚ Cubic if it has the form $a_0 + a_1x + a_2x^2 + a_3x^3$ ($a_3 \neq 0$), with degree 3

Example: $f(x) = 3 - 2x - x^3$

• **n th degree polynomial** if it has the form $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ ($a_n \neq 0$), with degree n

○ **Rational function:** A function that is expressible as ratio of two polynomials is called rational function. It has the form

$$f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_mx^m} \text{ where } n \text{ and } m \text{ are nonnegative}$$

integers and $a_0, a_1, a_2, \dots, a_n, b_0, b_1, b_2, \dots, b_m$ are real constants.

Example: $f(x) = \frac{x^2 + 4}{x - 1}$ and $g(x) = \frac{1}{3x - 5}$ are rational functions 59

- **Irrational function:** A function that is expressed as root extractions is called irrational function. It has the form $\sqrt[n]{f(x)}$ (, where $f(x)$ is a polynomial and n is a positive integer greater or equal to 2.

Example: a) $f(x) = \frac{\sqrt{x^2 + 4}}{\sqrt[3]{x - 1}}$ b) $g(x) = \sqrt{\frac{x}{x - 5}}$ are irrational functions

Exercises:

Classify the following functions into polynomial, rational and irrational function:

$$f(x) = \frac{1}{x} \quad 2. \quad f(x) = x^3 + 2x^2 - 2 \quad 3. \quad f(x) = \frac{x + 2}{x - 1} \quad 4. \quad f(x) = \sqrt{2x + 1} \quad (\text{REB, 2022})$$

- **Properties of Domain of functions:**

Case 1: The given function is a polynomial: linear, quadratic, cubic, etc. Given that $f(x)$ is polynomial, and then the domain of definition is the set of real numbers.

That is $\text{Dom}f = \mathbb{R}$

Example: The domain of the function $f(x) = x^3 - 3x + 2$ is $\mathbb{R}$ since it is a polynomial.

Case 2: The given function is a rational function Given that $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$

and $h(x)$ are polynomials, and then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is

$$\text{Dom}f = \{x \in \mathbb{R} : h(x) \neq 0\}$$

Example: Given $f(x) = \frac{2x - 1}{3x + 6}$, find the domain of definition.

Solution:

Condition: $3x + 6 \neq 0 \Rightarrow 3x \neq -6 \Rightarrow x \neq \frac{-6}{3} \Rightarrow x \neq -2$ $Domf = \mathbb{R} / \{-2\}$ or

$$Domf =]-\infty, -2[\cup]-2, +\infty[$$

Case 3: The given function is an irrational function Given that $f(x) = \sqrt[n]{g(x)}$, where $g(x)$ is a polynomial, there are two cases

a. If n is odd number, then the domain is the set of real numbers. That is

$$Domf = \mathbb{R}$$

b. If n is even number, then the domain is the set of all values of x such $g(x)$ is positive or zero. That is $Domf = \{x \in \mathbb{R} : g(x) \geq 0\}$.

Example: Given $f(x) = \sqrt{x^2 - 1}$, find domain of definition Solution Condition:

$x^2 - 1 \geq 0$. We need to construct sign table to see where $x^2 - 1$ is positive

$$x^2 = 1 \Rightarrow x = \sqrt{1} \Rightarrow x = \pm 1.$$

Factors X	$-\infty$	-1	1	$+\infty$
$x+1$	-----	0	+	+++++
$x-1$	-----	0	+	+++++
$x^2 - 1$	+++++	0	-----	+++++

Thus, $domf =]-\infty, -1] \cup [1, +\infty[$

a. For $f(x) = \frac{g(x)}{\sqrt[n]{h(x)}}$ if n is even, condition is $h(x) > 0$ and if n is odd $h(x) \neq 0$

Example:

$$\text{i) } \frac{2x+3}{\sqrt[4]{-2x+4}} \quad \text{ii) } f(x) = \frac{5-5x}{\sqrt[3]{x^2-x-6}}$$

Solutions:

$$\text{i. } -2x+4 > 0 \Rightarrow -2x > -4 \Rightarrow x < \frac{-4}{-2} \Rightarrow x < 2 \quad x =]-\infty, 2[$$

$$\begin{array}{llll}
 x^2 - x - 6 \neq 0 & (x+2)(x-3) \neq 0 & & \\
 \text{ii. Condition: } x^2 - 3x + 2x - 6 \neq 0 & \Rightarrow x+2 \neq 0 & \Rightarrow x-3 \neq 0 & \\
 x(x-3) + 2(x-3) \neq 0 & x \neq -2 & x \neq 3 &
 \end{array}$$

$$\text{Dom}f = \mathbb{R} \setminus \{-2, 3\} \quad (\text{Ndorimana, 2016})$$

- **Range(R)**

Let $f: A \rightarrow B$ be a function. The range of f , denoted by $\text{Im}(f)$ is the image of A under f , that is, $\text{Im}(f) = F(A)$. The range consists of all possible values the function f can have.

Steps to finding range of function

To find the range of function f described by formula where the domain is taken to be the natural domain:

1. Put $y = f(x)$.
2. Solve x in terms of y
3. The range of f is the set of all real numbers y such that x can be solved.

Examples: For each of the following functions, find the range. 1. $f(x) = x + 5$

$$f(x) = \sqrt{6-3x}$$

Solutions:

$$1. f(x) = x + 5 \Rightarrow y = x + 5 \Rightarrow y - 5 = x \Rightarrow x = y - 5, \text{ here the range is } \mathbb{R}$$

$$\begin{array}{l}
 f(x) = \sqrt{6-3x} \Rightarrow y = \sqrt{6-3x} \Rightarrow y^2 = (\sqrt{6-3x})^2 \Rightarrow y^2 = 6-3x \Rightarrow y^2 - 6 = -3x \\
 \text{a. } 3x = 6 - y^2 \Rightarrow x = \frac{6 - y^2}{3} \Rightarrow \text{Range} = \mathbb{R}
 \end{array}$$

b. Determine the domain and range of the given function: Solution

$$y = \frac{x^2 + x - 2}{x^2 - x - 2}. \text{ The domain is a set of all the values that } x \text{ is allowed to take.}$$

The only problem we have with this function is that we need to be careful not to divide by zero. So the only values that x cannot take on are those which would cause division by zero. So we set the denominator equal to zero and solve

$$\begin{array}{l}
 \text{For domain } x^2 - x - 2 = 0 \Rightarrow x^2 - 2x + 1x - 2 = 0 \Rightarrow x(x-2) + 1(x-2) = 0 \\
 (x+1)(x-2) = 0 \Rightarrow x+1 = 0 \Rightarrow x = -1 \Rightarrow x-2 = 0 \Rightarrow x = 2
 \end{array}$$

Then the domain is "all x not equal to -1 or 2 "

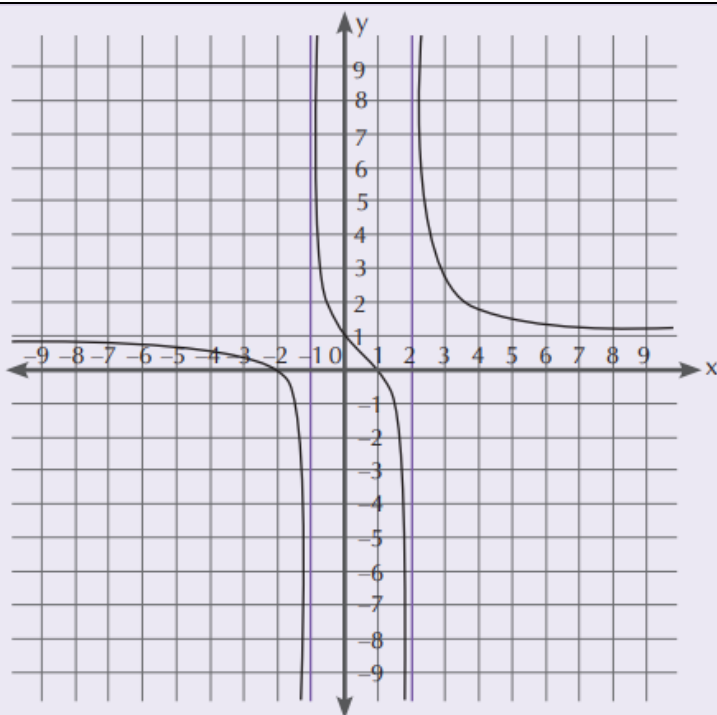


Figure 6: Graph of range

As we can see from, the graph “covers” all y-values (that is, the graph will go as low as I like, and will also go as high as I like). Since the graph will eventually cover all possible values of y, then the range is “all real numbers”.

Exercises:

1. Determine the domain of the functions below:

a. $F(x) = \sqrt[5]{x^2 - 4}$

b. $F(x) = 6x^3 - 2x^2 + 4x - 6$ c)

c. $F(x) = \sqrt{x^2 - 9}$ d)

d. $f(x) = \frac{2x + 2}{x^2 - x - 6}$

e. $f(x) = \sqrt{x + 7} - \sqrt{x^2 + 2x - 15}$

f. $g(x) = \frac{\sqrt{x}}{4x - 8}$

g. $f(x) = \sqrt{\frac{x + 1}{x - 1}}$

h. $g(x) = \frac{\sqrt{4x - 12}}{x^2 - 8x + 7}$

2.i. Find the range of the function $f(x) = x^2 + 3x + 1$ for domain = $\{-2, -1, 0, 1, 2\}$

ii. Find the range of the following functions:

a. $f(x) = \frac{2x+3}{x-1}$

b. $f(x) = \frac{2x+1}{3x-4}$

c. $f(x) = 2x^2 - 8$

d. $g(x) = 2x + 4$

e. $f(x) = \frac{5}{x}$

f. $g(x) = x^2 + 4x - 5$ (Shampiona, 2005)



Activity 2: Guided Practice



Task 16:

2. Domain and range of $g(x) = \frac{8}{2x-4}$ is $2x-4 \neq 0 \Rightarrow x \neq 2 \Rightarrow \text{dom}f = \mathbb{R} \setminus \{2\}$

$$g(x) = \frac{8}{2x-4} \Rightarrow y = \frac{8}{2x-4} \Rightarrow y(2x-4) = 8 \Rightarrow 2xy - 4y = 8 \Rightarrow 2xy = 8 + 4y \Rightarrow x = \frac{8+4y}{2y}$$

3. Here condition for range is $2y \neq 0 \Rightarrow \text{Range} = \{ \forall y \in \mathbb{R} : y \neq 0 \} \text{ range} = \mathbb{R} \setminus \{0\}$

4. Find the domain and range of $g(x) = \frac{3x-2}{2x-4}$



Activity: Application



Task 17:

a. A company's profit function is given by $f(x) = \frac{2x+3}{x-1}$, where x represents the

number of units sold. Find the domain and range of this function to determine the feasible number of units and corresponding profit levels.

- b. The efficiency of a machine is modeled by the function $f(x) = \frac{2x+1}{3x-4}$, where x represents the hours of operation. Find the domain and range of this function to understand the operational hours and efficiency levels.
- c. The height of a projectile is given by the function $f(x) = 2x^2 - 8$, where x represents the time in seconds. Find the domain and range of this function to determine the time period and height range of the projectile.
- d. The cost of producing items is given by the linear function $g(x) = 2x + 4$, where x represents the number of items produced. Find the domain and range of this function to understand the production levels and corresponding costs.
- e. The rate of a chemical reaction is given by the function $f(x) = \frac{5}{x}$, where x represents the concentration of a reactant. Find the domain and range of this function to determine the feasible concentration levels and reaction rates.
- f. The revenue generated from selling products is modeled by the function $g(x) = x^2 + 4x - 5$, where x represents the number of products sold. Find the domain and range of this function to understand the sales levels and corresponding revenue.

Topic 2.2: Symmetry of algebraic function or Parity of numerical function (odd or even)



Activity 1: Problem Solving



Task 18:

Consider the rational function $g(x) = \frac{x^3 - x}{x^2 + 2}$.

- a. State the procedure to determine the parity of a function involving fractions.
- b. Test $g(-x)$ and compare it to $g(x)$ and $-g(x)$. Simplify your expressions step by step.
- c. Based on your analysis, is $g(x)$ even, odd, or neither? Explain how the numerator and denominator contribute to your conclusion.

Key Facts 2.2: Symmetry of algebraic function or Parity of numerical function (odd or even)

- Domain and range of algebraic function

- ✓ Even function (EF):

✚ function $f(x)$ is said to be even if the following conditions are satisfied

- $\forall x \in \text{Dom}f, -x \in \text{Dom}f$,
- $f(x) = f(-x)$ The graph of such function is symmetric about the vertical axis. i.e. $x = 0$

Example: a. The function $f(x) = 2x^4 + 4$ is an even function since

$$\forall x \in \text{Dom}f, -x \in \text{Dom}f = \mathbb{R} \text{ and } f(-x) = 2(-x)^4 + 4 \Rightarrow f(-x) = 2x^4 + 4$$

$$\Rightarrow f(x) = f(-x)$$

✚ $f(x) = x^2$ is an even function since $\forall x \in \text{Dom}f, -x \in \text{Dom}f$,

$$f(-x) = (-x)^2 \Rightarrow f(-x) = x^2.$$

Here is the graph of $f(x) = x^2$

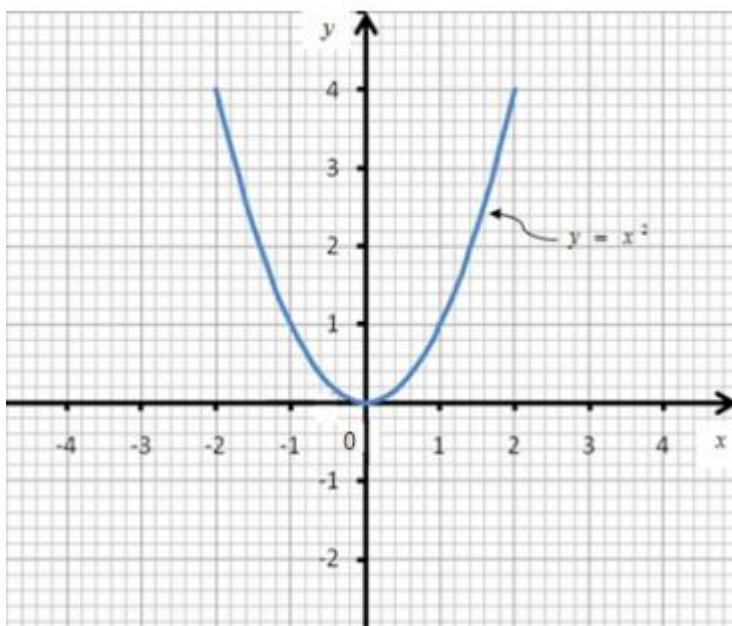


Figure 7: Even function

- **Odd function A function**

✚ Odd function A function $f(x)$ is said to be odd if the following conditions are satisfied

- $\forall x \in \text{dom}f = \mathbb{R}, -x \in \text{Dom}f = \mathbb{R}$,
- $f(-x) = -f(x)$

The graph of such function looks the same when rotated through half a revolution about 0. This is called rotational symmetry

Example: 1. () $f(x) = x^3$ is an odd function since $\forall x \in \text{Dom}f, -x \in \text{Dom}f$, and $f(-x) = (-x)^3 = -x^3 = -f(x)$

Here is the graph of $f(x) = x^3$

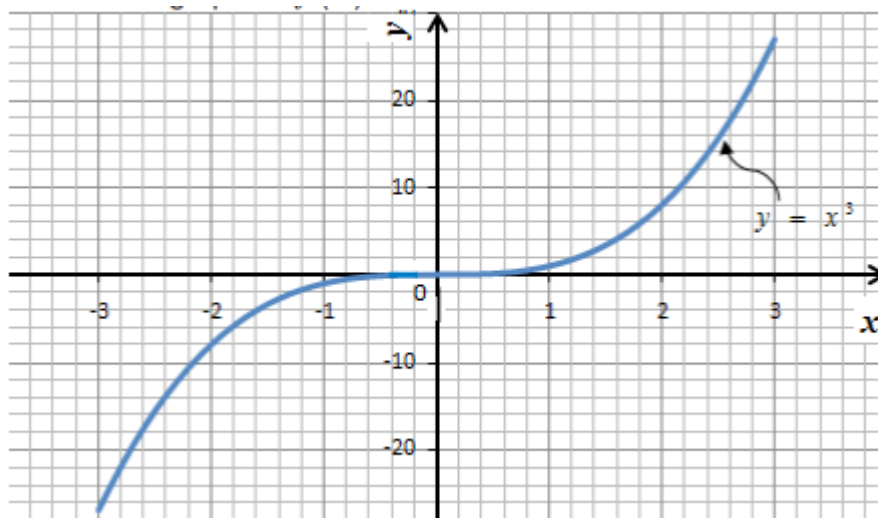


Figure 8: Odd function

Exercises: Determine whether function is odd or even

- a. $f(x) = 6x^5 - 4x^3 - 2x$ b. $f(x) = \frac{2x^4 - 3x^2 + 5}{x^2 + 25}$ c. $f(x) = 2x^5 + 6x^2 + 2x - 10$
- d. $f(x) = \frac{2x^3 + 3x}{3x^4 - 2x^2 + 2}$ (REB, 2022)



Activity 2: Guided Practice



Task 19:

Determine whether function is odd or even

$$f(x) = 6x^5 - 4x^3 - 2x$$

answer:

$$f(-x) = 6(-x)^5 - 4(-x)^3 - 2(-x)$$

$$f(-x) = -6x^5 + 4x^3 + 2x \quad f(-x) \neq f(x) \text{ is not even function}$$

$$f(-x) = -(6x^5 - 4x^3 - 2x)$$

$$f(-x) = -f(x) \text{ it is an odd function}$$

Refer to the corrected example on this activity, determine whether function is odd or even

$$f(x) = 6x^{15} - 4x^5 - 2x$$



Activity 3: Application



Task 20:

1. The function $f(x) = 6x^7 - 4x^3 - 2x$ models the energy consumption of a machine, where x represents the time in hours. Determine whether the energy consumption function is odd or even to understand its symmetry properties.
2. The function $f(x) = \frac{2x^4 - 3x^2 + 5}{x^3 + 25}$ represents the efficiency of a cooling system, where x represents the temperature in degrees Celsius. Determine whether the efficiency function is odd or even to analyze its behavior under temperature changes.
3. The function $f(x) = x^5 - 26x^2 + 2x - 10$ models the profit of a company, where x represents the number of products sold. Determine whether the profit function is odd or even to understand its symmetry and predict profit trends.
4. The function $f(x) = \frac{2x^3 + 3x}{3x^4 - 2x^2 + 2}$ represents the speed of a vehicle, where x represents the time in seconds. Determine whether the speed function is odd or even to analyze its symmetry and predict vehicle performance.

Topic 2.3: Limits of functions



Activity 1: Problem Solving



Task 21:

The braking distance $d(v)$ of a car, in meters, is modeled by the function

$$d(v) = \frac{v^2}{20} \text{ where } v \text{ is the speed of the car in kilometers per hour (km/h).}$$

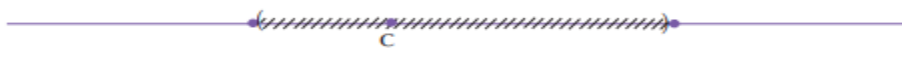
- Calculate $\lim_{v \rightarrow 0} d(v)$. What does this limit represent in the context of car braking?
- Interpret $\lim_{x \rightarrow \infty} d(v)$. Is there a practical implication of this limit for highway safety?
- Discuss how understanding limits can help in designing speed regulations or car safety systems.

Key Facts 2.3: Limits of functions

- **Finite limits**

Neighborhood of a real number

By a neighborhood of a real number c we mean an interval which contains c as an interior point.



A neighborhood of c

On the real line, a neighborhood of a real number is an open interval $(a - \delta, a + \delta)$ where $\delta > 0$, with its center at a . Therefore, we can say that a neighborhood of the real number is any interval that contains a real number a and some point below and above it. We can add: Let x be a real number. A neighborhood of x is a set N such that for some $\epsilon > 0$ and for all y , if $|x - y| < \epsilon$ then $y \in N$.

Limit of a variable at left and right hand side:

Let $f(x) = 2x + 2$ and compute $f(x)$ as x takes values closer to 1. First consider values of x approaching 1 from the left ($x < 1$) then consider x approaching 1 from the right ($x > 1$).

and x approaching 1 from the right ($x > 1$) gives us:

x	$f(x)$
0.8	3.6
0.9	3.8
0.95	3.9
0.99	3.98
0.999	3.998
0.9999	3.9998
0.99999	3.99998

x	$f(x)$
1.2	4.4
1.05	4.1
1.01	4.02
1.001	4.002
1.0001	4.0002
1.00001	4.00002
1.000001	4.000002

In both cases as x approaches 1, $f(x)$ approaches 4. Intuitively, we say that \

$$\lim_{x \rightarrow 1} f(x) = 4$$

Note: We are talking about the values that $f(x)$ takes when x gets closer to 1 and not $f(1)$. In fact, we may talk about the limit of $f(x)$ as x approaches a even when $f(a)$ is undefined.

Suppose the function $y = f(x)$ is a numerical function with independent variable x and dependent variable y where the value y depends on the variable of x . If the values of x can be made as closer to a value a as we please, then this can be written $x \rightarrow a$. This is read as 'x tends to a' or 'x approaches a'. Since the variable y depends on the variable x , we can say that as x tends to a , y will also tend to a certain value that we have to determine. The obtained value of y as x tends to a is the what we call the limiting value of $f(x)$.

Suppose that $f(x) = 2x + 1$. Let us find the limiting value of $f(x)$ as x tends to 3. From the left of 3

X	2.9	2.99	2.999	2.9999
F(X)	6.8	6.98	6.998	6.9998

We say that as x approaches 3 from the left, $f(x)$ approaches 7 from below. From the right

X	3.1	3.01	3.001	3.0001	3.00001
F(X)	7.2	7.02	7.002	7.0002	7.00002

In this case we say that as x approaches 3 from the right, $f(x)$ approaches 7 from above

From either side

X	2.9	2.99	2.999	2.9999	3	3.1	3.01	3.001	3.0001	3.00001
F(X)	6.8	6.98	6.998	6.9998	L	7.2	7.02	7.002	7.0002	7.00002

In summary, we can see that as x approaches 3 from either direction, $f(x)$

approaches a value of 7, and we write $\lim_{x \rightarrow a} f(x) = l$ or $\lim_{x \rightarrow 3} 2x + 1 = 2(3) + 1 = 7$

We find limit by direct substitution.

Finite limit: we find finite limit by direct substitution

Examples: 1. $\lim_{x \rightarrow 2} 2x + 1 = 2(2) + 1 = 4 + 1 = 5$

2. Since a constant function

$f(x) = k$ has the same value k everywhere, it follows that at each point $\lim_{x \rightarrow a} k = k$.

For example, $\lim_{x \rightarrow 4} 5 = 5$

1. $\lim_{x \rightarrow -5} x = -5$, $\lim_{x \rightarrow 0} x = 0$

2. $\lim_{x \rightarrow 2} \sqrt{x^2 - 2x + 1} = \sqrt{2^2 - 2(2) + 1} = \sqrt{4 - 4 + 1} = \sqrt{1} = 1$

3. $\lim_{x \rightarrow 3} \frac{\sqrt{2x+1}}{\sqrt[3]{3x-1}} = \frac{\sqrt{2(3)+1}}{\sqrt[3]{3(3)-1}} = \frac{\sqrt{6+1}}{\sqrt[3]{9-1}} = \frac{\sqrt{7}}{\sqrt[3]{8}} = \frac{\sqrt{7}}{2}$

✓ Right-hand sided limit:

We say that $\lim_{x \rightarrow a^+} f(x)$ is the right hand side limit, and we can make $f(x)$ as

close to L as we want for all x sufficiently close to a and $x > 0$ without actually letting x be a .

✓ Left-hand side limit

We say that $\lim_{x \rightarrow a^-} f(x)$ is the left hand side limit if we can make $f(x)$ as close to

L as we want for all x sufficiently close to a and $x < 0$, without actually letting x be a . For the right-hand sided limit, we have $\lim_{x \rightarrow a^+} f(x)$ (note that the "+")

which means that we will only look at $x > a$. Likewise for the left-handed limit

we have $\lim_{x \rightarrow a^-} f(x)$ (note the "-") which means that we will only be looking at

$x < 0$. So when calculating a limit, it is important to know whether it is a one-sided limit or not.

a. If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right-

hand side, we write that $\lim_{x \rightarrow a^+} f(x) = L_1$ and we read "the limit of $f(x)$ as

x approaches x_0 from the right side equals to L_1 . If the value of $f(x)$

approaches L_2 as x approaches $\lim_{x \rightarrow a^-} f(x) = L_2$ from the left-hand side, we

write that $\lim_{x \rightarrow a^-} f(x) = L_2$ and we read it as “the limit of $f(x)$ as x

approaches x_0 from the left equals L_2 If the limit from the left side is the

same as the limit from the right side, say $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$, then we

write $\lim_{x \rightarrow x_0} f(x) = L$ and we read “the limit of $f(x)$ as x approaches x_0

equals L . Note that: $\lim_{x \rightarrow x_0} f(x)$ exists if and only if $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$

Always recall that the value of a limit at a point does not actually depend upon the value of the function at that point. The value of the limit of a function at a point only depends on the behaviors of the values of the function around, or in the neighborhood of the given point. Therefore, even if the function is not defined at the point, the limit can still exist.

Example: If, $f(x) = \frac{|x+2|}{x+3}$, then find the $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$

$$f(x) = \frac{|x+2|}{x+3} \Rightarrow \lim_{x \rightarrow 2^-} \frac{-(x+2)}{x+3} \quad \lim_{x \rightarrow 2^+} \frac{+(x+2)}{x+3} \Rightarrow \lim_{x \rightarrow 2} \frac{x+2}{x+3} \Rightarrow$$

$$\text{Solution: } \lim_{x \rightarrow 2} \frac{-x-2}{x+3} = \frac{-2-2}{2+3} = -\frac{4}{5} \quad \frac{2+2}{2+3} = \frac{4}{5}$$

Since a $\lim_{x \rightarrow 2^+} f(x) \neq \lim_{x \rightarrow 2^-} f(x)$, then $\lim_{x \rightarrow 2} f(x)$ does not exist.

Formally,

If $f(x)$ is defined for all x near a , except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a , but not equal to a , we say that the function f approaches the limit L as x approaches a , and we write the $\lim_{x \rightarrow a} f(x) = L$.

Practically, to find the limit of a function $f(x)$ as x approaches a , first we evaluate the function at $x = a$ that is we find, if possible $f(a)$ substitute that value a in the function and see what happen. The limit can exist or not.

✓ Infinite limits

Let f be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a , but not equal to a .

The Mathematical statement $\lim_{x \rightarrow a} f(x) = +\infty$, shows that the values of $f(x)$

increases without bound while $\lim_{x \rightarrow a} f(x) = -\infty$, shows that the values of $f(x)$

decreases without bound. (REB, 2022)

• Limits at infinity

Let f be a function defined on some interval, (a, ∞) . Then $\lim_{x \rightarrow \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Let f be a function defined on some interval, $(-\infty, a)$. Then $\lim_{x \rightarrow -\infty} f(x) = L$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative. Consider for example a polynomial of the 3rd degree in x .

$$\lim_{x \rightarrow \infty} ax^3 + bx^2 + cx + d \Rightarrow \lim_{x \rightarrow \infty} x^3 \left(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} \right) \Rightarrow a \lim_{x \rightarrow \infty} x^3$$

This reasoning is valuable for a polynomial with any degree

Operations with infinity and Limits at infinity.

For any real number c , we have: $(+\infty) + c = c + (+\infty) = +\infty$
 $(-\infty) + c = c + (-\infty) = -\infty$ One can convince

himself/herself by considering the quantity of water in a lake as infinity, compared to the quantity of water in a spoon. If you add a spoon of water to the water of a lake, there is no change.

$(+\infty) + (+\infty) = +\infty$
 $(-\infty) + (-\infty) = -\infty$: adding two infinities of the same sign yields to infinity of that common sign.

$(+\infty) + (-\infty) = (-\infty) + (+\infty)$: adding two infinities of different signs yields to an indeterminate case (or indeterminate form): we need to remove the indetermination to discover the true value hidden by the indetermination.

ii) Subtraction We have: $(+\infty) - (-\infty) = +\infty$
 $(-\infty) - (+\infty) = -\infty$

$(+\infty) - (+\infty), (-\infty) - (-\infty)$: subtracting infinity from infinity of the same sign yields to an indeterminate case

Multiplication

We have: $(+\infty)(+\infty) = +\infty, (+\infty)(-\infty) = (-\infty)(+\infty) = -\infty$
 $(-\infty)(-\infty) = +\infty$

$$a(+\infty) = \begin{cases} +\infty, a > 0 \\ -\infty, a < 0 \end{cases}; \quad a(-\infty) = \begin{cases} -\infty, a > 0 \\ +\infty, a < 0 \end{cases}$$

But $\begin{cases} 0(+\infty) = (+\infty)0 \\ 0(-\infty) = (-\infty)0 \end{cases}$ are indeterminate cases

We have: For any real number a , $\frac{\infty}{a} = \infty$: the sign is to be determined, note that

∞ stands

for $+\infty$ or $-\infty$; in particular, $\frac{\infty}{0} = \infty$; For any real number a , $\frac{a}{\infty} = 0$; in particular,

$\frac{0}{\infty} = 0$ But $\frac{\infty}{\infty}$ is an indeterminate case.

Notice: The following ; $\left(\infty - \infty, 0(\infty), \frac{\infty}{\infty}, \frac{0}{0}, 0^0, 1^\infty, \infty^0 \right)$ are indeterminate cases in

limits calculation.

- **Limit of polynomial functions**

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0 = \lim_{x \rightarrow \infty} a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \dots + \frac{a_0}{a_n x^n} \right)$$

$$= \lim_{x \rightarrow \infty} a_n x^n$$

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \dots + b_0 = \lim_{x \rightarrow \infty} b_m x^m \left(1 + \frac{b_{m-1}}{b_m x} + \frac{b_{m-2}}{b_m x^2} + \dots + \frac{b_0}{b_m x^m} \right)$$

$$= \lim_{x \rightarrow \infty} b_m x^m$$

And then we have 3 cases:

- If $m=n$, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$
- If $n > m$, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{a_n x^{n-m}}{b_m} = \infty$, the sign of infinity is to be determined.
- If $n < m$, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{a_n}{b_m x^{m-n}} = 0$,

Basic properties of limit at infinity:

- $\lim_{x \rightarrow \pm\infty} x = \pm\infty$, $\lim_{x \rightarrow \pm\infty} x^n = +\infty$ for n which is even number
- $\lim_{x \rightarrow -\infty} x^n = -\infty$, for n which is odd number
- $\lim_{x \rightarrow \pm\infty} ax = \pm\infty$, depending on the value of a . if $\lim_{x \rightarrow -\infty} ax = +\infty$, for $a < 0$ and $\lim_{x \rightarrow -\infty} ax = -\infty$, for $a > 0$
- $\lim_{x \rightarrow +\infty} ax = -\infty$, for $a < 0$ and $\lim_{x \rightarrow +\infty} ax = +\infty$, for $a > 0$
- $\lim_{x \rightarrow \pm\infty} \frac{a}{x} = \frac{a}{\pm\infty} = 0$, for a which is constant
- $\lim_{x \rightarrow 0} \frac{a}{x} = \frac{a}{0} = \pm\infty$, for a which is constant
- $\lim_{x \rightarrow a} k = k$; where a and k

Examples: Evaluate the following limits

d. $\lim_{x \rightarrow +\infty} -6 = -6$ b. $\lim_{x \rightarrow \pm\infty} \frac{1}{x}$ and $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$

e. $\lim_{x \rightarrow -\infty} -3x^3 + 2x^2 - 3x + 2$

$$\lim_{x \rightarrow -\infty} x^3 \left(-3 + \frac{2}{x} - \frac{3}{x^2} + \frac{2}{x^3} \right) = \lim_{x \rightarrow -\infty} x^3 \lim_{x \rightarrow -\infty} \left(-3 + \frac{2}{x} - \frac{3}{x^2} + \frac{2}{x^3} \right)$$

Solution: a. $\lim_{x \rightarrow -\infty} x^3 \left(-3 + \frac{2}{-\infty} - \frac{3}{(-\infty)^2} + \frac{2}{(-\infty)^3} \right) \Rightarrow \lim_{x \rightarrow -\infty} x^3 (-3 + 0 + 0 + 0)$

$$\lim_{x \rightarrow -\infty} x^3 (-3) = \lim_{x \rightarrow -\infty} -3x^3 = -3(-\infty)^3 = -3(-\infty) = +\infty$$

It is not necessary to find all steps; you can only take the monomial of highest degree:

$$\lim_{x \rightarrow -\infty} -3x^3 + 2x^2 - 3x + 2 = \lim_{x \rightarrow -\infty} -3x^3 = -3(-\infty)^3 = -3(-\infty) = +\infty$$

Exercises: Evaluate the following limits $\lim_{x \rightarrow -\infty} 3 + 3x^4 - 3x^2 + 4x$ (Ndorimana, 2016)

- **Remove of indeterminate cases:**

Suppose that $f(x) \rightarrow 0$ and $g(x)$ as $x \rightarrow a$. Then the limit of the quotient $\frac{f(x)}{g(x)}$ as x

$\rightarrow a$ is said to give an indeterminate form, sometimes denoted by $\frac{0}{0}$. It may be

that the limit of $\frac{f(x)}{g(x)}$ can be found by some methods such as factor method,

rationalization method, hospital rule, etc...

Similarly, if $f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$, then the limit of $\frac{h(x)}{g(x)}$ gives an

indeterminate form, denoted by $\frac{\infty}{\infty}$. Also, if $f(x) \rightarrow 0$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$, then

the limit of the product $f(x)g(x)$ gives an indeterminate form $0 \times \infty$.

Suppose that:

✓ $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$. The limit of the product $g(x)f(x)$ has the indeterminate form, $0 \times \infty$ at $x=a$. To evaluate this limit, we change the limit

into one of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way: $f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \frac{g(x)}{\frac{1}{f(x)}}$

✓ If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = +\infty$; then $\lim_{x \rightarrow a} [f(x) - g(x)]$ has the indeterminate form

$\infty - \infty$. To evaluate this limit, we perform the algebraic manipulations by

converting the limit into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If $f(x)$ or $g(x)$ is expressed as a

fraction, we find the common denominator.

✓ When we are computing the limits of irrational functions, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

- **Method of factors:**

Limits of rational functions involving indeterminate forms Examples:

Indeterminate form with the form $\frac{0}{0}$

✓ $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}$, this is the indeterminate form (I.F)

By factoring the numerator and cancelling, we move out this I.F

$$\begin{aligned} x^2 - 4 \\ x^2 + 0x - 4 &\Rightarrow \frac{x(x-2) + 2(x-2)}{(x+2)(x-2)} \Rightarrow (x+2)(x-2) = x^2 - 4 \\ x^2 - 2x + 2x - 4 \end{aligned}$$

By replacing the value of $x^2 - 4$, $\lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \rightarrow 2} x + 2 = 2 + 2 = 4$

✓ $\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} = \frac{2 - 2}{\sqrt{2} - \sqrt{2}} = \frac{0}{0}$ (IF)

We eliminate the indeterminate form by rationalization:

$$\lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x} + \sqrt{2})}{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})} \Rightarrow (a-b)(a+b) = a^2 - b^2 \Rightarrow \lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x} + \sqrt{2})}{((\sqrt{x})^2 - (\sqrt{2})^2)}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(\sqrt{x} + \sqrt{2})}{(x-2)} = \lim_{x \rightarrow 2} \sqrt{x} + \sqrt{2} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

$$\checkmark \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{2x-2}} = \frac{0}{0} (I.F) \Rightarrow \lim_{x \rightarrow 1} \frac{(x-1)\sqrt{2x-2}}{(\sqrt{2x-2})(\sqrt{2x-2})} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{2x-2})}{2x-2}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)\sqrt{2x-2}}{2(x-1)} = \lim_{x \rightarrow 1} \frac{\sqrt{2x-2}}{2} = \frac{\sqrt{2(1)-2}}{2} = \frac{\sqrt{0}}{2} = \frac{0}{2} = 0$$

$$\checkmark \lim_{x \rightarrow 1} \frac{\sqrt[3]{x}-1}{x-1} = \frac{0}{0} (I.F)$$

$$\lim_{x \rightarrow 1} \frac{(\sqrt[3]{x}-1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)}{(x-1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)} = \lim_{x \rightarrow 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1} = \frac{1}{1+1+1} = \frac{1}{3}$$

Exercises:

a. $\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$

b. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ c) $\lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x} - \sqrt{3}}$

Indeterminate form with the form $\frac{\infty}{\infty}$

a. $\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{\infty}{\infty} (I.F)$

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} \Rightarrow \frac{\lim_{x \rightarrow \infty} 3x^2 - x - 2}{\lim_{x \rightarrow \infty} 5x^2 + 4x + 1} \Rightarrow \frac{\lim_{x \rightarrow \infty} x^2 \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \rightarrow \infty} x^2 \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \Rightarrow$$

$$\frac{\lim_{x \rightarrow \infty} 3 - \frac{1}{x} - \frac{2}{x^2}}{\lim_{x \rightarrow \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)} \Rightarrow$$

$$\frac{3 - \frac{1}{\infty} - \frac{2}{(\infty)^2}}{5 + \frac{4}{\infty} + \frac{1}{(\infty)^2}} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5} \quad \text{It is not necessary to find all steps; you can only}$$

take the monomial of highest degree: for $\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \rightarrow \infty} \frac{3x^2}{5x^2} = \frac{3}{5}$

b. $\lim_{x \rightarrow \infty} \frac{x^2 - x - 2}{2x^5 + 4x^3 + 1x - 5} = \frac{\infty}{\infty} (I.F) \Rightarrow$

$$\lim_{x \rightarrow \infty} \frac{x^2}{2x^5} = \lim_{x \rightarrow \infty} \frac{1}{2x^3} = \frac{1}{2(\infty)^2} = \frac{1}{2(\infty)} = \frac{1}{\infty} = 0$$

Exercises :

a. $\lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 5}{x - 2}$

b. $\lim_{x \rightarrow \infty} \frac{2x^2 + 6}{3x^2 - 4x + 2}$

c. $\lim_{x \rightarrow +\infty} \frac{2x^2 - 3x + 5}{2 + x^2 - x^5}$

d. $\lim_{x \rightarrow +\infty} \frac{4 - 3x + 5x^5}{x^2 - 2x - 3}$

e. $\lim_{x \rightarrow +\infty} \frac{x^5 + 2x^2 - 6}{1 - 2x^2 - 4x^5}$ f. $\lim_{x \rightarrow \infty} \frac{x^2 + 2x + 1}{x^2 + 3x + 2}$

To remove indeterminate forms of: $\infty - \infty$

For examples:

a. $\lim_{x \rightarrow \infty} \sqrt{4x^2 - 2x + 2} - 2x$

b. $\lim_{x \rightarrow \pm\infty} \frac{-3x}{\sqrt{9x^2 - 3x + 6}}$

$$c. \lim_{x \rightarrow 2} \frac{4}{x-2} - \frac{2}{x^2-4}$$

$$d. \lim_{x \rightarrow \infty} \sqrt{x^2 + x + 2} - \sqrt{x^2 - x + 3}$$

$$e. \lim_{x \rightarrow 1} \frac{1}{1-x} - \frac{3}{1-x^3} \quad f. \lim_{x \rightarrow \pm\infty} \frac{\sqrt{4x^2 - 11x - 4}}{x}$$

Solutions: a. $\lim_{x \rightarrow \infty} \sqrt{4x^2 - 2x + 2} - 2x = \infty - \infty (I.F)$

By rationalizing:

$$\lim_{x \rightarrow \pm\infty} \frac{(\sqrt{4x^2 - 2x + 2} - 2x)(\sqrt{4x^2 - 2x + 2} + 2x)}{1(\sqrt{4x^2 - 2x + 2} + 2x)} = \lim_{x \rightarrow \pm\infty} \frac{4x^2 - 2x + 2 - 4x^2}{\sqrt{4x^2 - 2x + 2} + 2x} \text{ by taking the}$$

monomial with highest power.

$$\lim_{x \rightarrow \pm\infty} \frac{-2x + 2}{\sqrt{4x^2 + 2x}} = \lim_{x \rightarrow \pm\infty} \frac{-2x + 2}{|2x| + 2x} \Rightarrow |2x| = \begin{cases} -2x; & x < 0 \\ +2x; & x > 0 \end{cases} \text{ we use limit at } -\infty \text{ for } x < 0;$$

and $+\infty$ at $x > 0$

$$\lim_{x \rightarrow -\infty} \frac{-2x + 2}{-2x + 2x} = \frac{-2x}{0x} = \frac{-2}{0} = -\infty \text{ for } \lim_{x \rightarrow +\infty} \frac{-2x + 2}{2x + 2x} = \frac{-2x}{4x} = -\frac{1}{2}$$

$$b. \lim_{x \rightarrow \pm\infty} \frac{-3x}{\sqrt{9x^2 - 3x + 6}} = \frac{\pm\infty}{\sqrt{\infty - \infty}} (I.F) \text{ By taking highest power}$$

$$\lim_{x \rightarrow \pm\infty} \frac{-3x}{\sqrt{9x^2}} = \lim_{x \rightarrow \pm\infty} \frac{-3x}{|3x|} \text{ for } \lim_{x \rightarrow -\infty} \frac{-3x}{-3x} = 1 \text{ and } \lim_{x \rightarrow -\infty} \frac{-3x}{3x} = -1$$

$$c. \lim_{x \rightarrow 2} \frac{4}{x-2} - \frac{2}{x^2-4} = \infty - \infty (IF)$$

$$\lim_{x \rightarrow 2} \frac{4}{x-2} - \frac{2}{x^2-4} = \lim_{x \rightarrow 2} \frac{4(x+2) - 2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{4x+6-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{4x+4}{x^2-4} = \frac{4(2)+4}{2^2-4} = \frac{12}{0} = +\infty$$

To remove indeterminate form of the form $(0 * \infty)$

Examples: a. $\lim_{x \rightarrow 0^+} x^3 x^{-1}$ b. $\lim_{x \rightarrow \infty} x \cdot \left(\frac{1}{x}\right)$ c. $\lim_{x \rightarrow \infty} x \cdot (x)^{-4}$ d. $\lim_{x \rightarrow \infty} x \cdot \left(\frac{3}{x}\right)$

Solutions: a. $\lim_{x \rightarrow 0^+} x^3 x^{-1} = (0)^3 (0)^{-1} = 0 * \infty (I.F)$ then we remove it

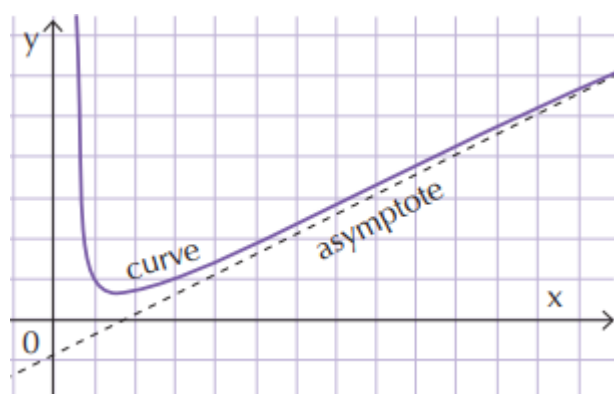
$$\lim_{x \rightarrow 0^+} \frac{x^3}{x} = \lim_{x \rightarrow 0} x^2 = 0^2 = 0$$

$$\text{b. } \lim_{x \rightarrow \infty} x \cdot \left(\frac{1}{x}\right) = \infty \cdot 0 (I.F) \Rightarrow \lim_{x \rightarrow \infty} \frac{x}{x} = 1$$

We will see other many examples about indeterminate form after studying hospital rule, exponential functions, trigonometric functions and logarithmic functions.

- **Determination of asymptotes**

An asymptote is a line that a curve approaches, as it heads towards infinity:



Types of asymptote

There are three types of asymptotes: horizontal, vertical and oblique:

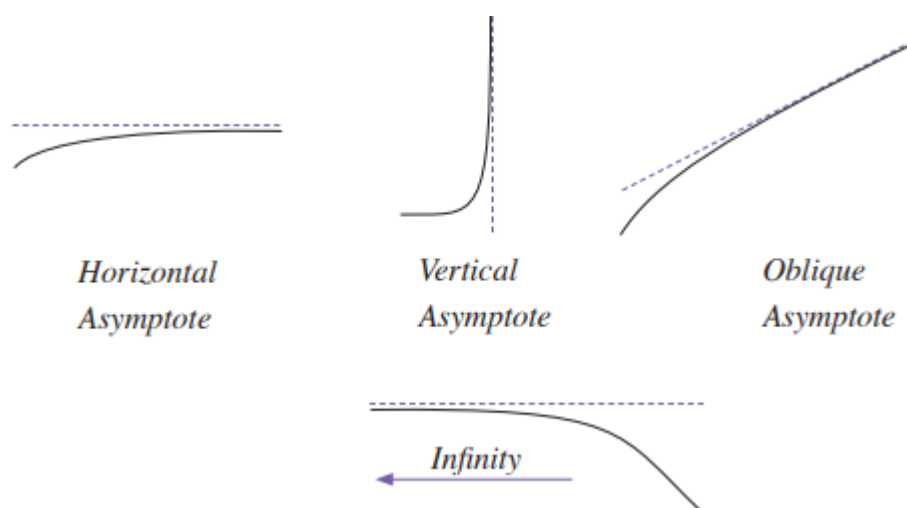
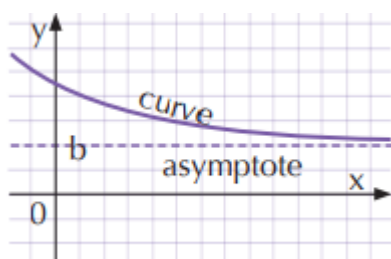


Figure 9: Types of asymptotes

An asymptote can be in a negative direction, the curve can approach from any side (such as from above or below for a horizontal asymptote), or may actually cross over (possibly many times), and even move away and back again. The important point is that: The distance between the curve and the asymptote tends to zero as they head to infinity.

✓ Horizontal asymptotes



It is a horizontal asymptote when: as x goes to infinity (or $-\infty$) the curve approaches some constant value b , $\lim_{x \rightarrow \pm\infty} f(x) = b$; $H.A \equiv y = b$

Examples: Find the equation of the horizontal asymptote to the curve

a. $y = \frac{x+3}{(x+2)(x-1)}$ b. $f(x) = \frac{x^2-1}{x^2+2}$ c. $g(x) = \frac{x^5+2x^2+1}{1-4x^5+3x}$

Solutions: a. $y = \frac{x+3}{(x+2)(x-1)} \Rightarrow y = \frac{x+3}{x^2+x-2} \quad \lim_{x \rightarrow \pm\infty} f(x) = b$;

$H.A \equiv y = b$

$\lim_{x \rightarrow \pm\infty} \frac{x+3}{x^2+x-2} = \lim_{x \rightarrow \pm\infty} \frac{x}{x^2} = \lim_{x \rightarrow \pm\infty} \frac{1}{x} = \frac{1}{\pm\infty} = 0$; $H.A \equiv y = 0$

b. $f(x) = \frac{x^2-1}{x^2+2}$

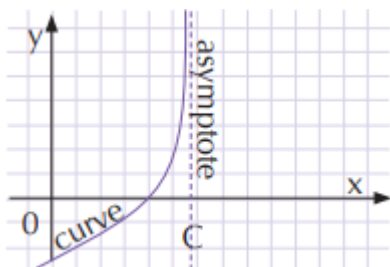
$\lim_{x \rightarrow \pm\infty} \frac{x^2-1}{x^2+2} = \lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2} = 1$; $H.A \equiv y = 1$

c. $g(x) = \frac{x^5+2x^2+1}{1-4x^5+3x} \Rightarrow \lim_{x \rightarrow \pm\infty} \frac{x^5+2x^2+1}{1-4x^5+3x} = \lim_{x \rightarrow \pm\infty} \frac{x^5}{-4x^5} = -\frac{1}{4}$; $H.A \equiv y = -\frac{1}{4}$

Exercises: find horizontal asymptote of the following functions:

a. $g(x) = \frac{x^3 + 2x^2 + 1}{1 - 4x^4 + 3x}$ b. $y = \frac{x - 4}{(2x + 2)(x - 3)}$ c. $f(x) = \frac{x^5 - 3x - 1}{x^2 + 2}$

✓ **Vertical asymptotes**



It is a vertical asymptote when:

as x approaches some constant value c (from the left or right) then the curve goes towards infinity (or $-\infty$). $\lim_{x \rightarrow c} f(x) = \pm\infty$; $VA \equiv x = c$

Examples: Find the equation of the vertical asymptote to the curve

a. $f(x) = \frac{x^3 + x + 1}{x - 2}$ b. $f(x) = \frac{x^3 + x + 1}{x^2 - x - 6}$

Solutions: a. $f(x) = \frac{x^3 + x + 1}{x - 2} \Rightarrow x - 2 = 0 \Rightarrow x = 2 \quad \lim_{x \rightarrow c} f(x) = \pm\infty$;

$VA \equiv x = c$

$\lim_{x \rightarrow 2} \frac{x^3 + x + 1}{x - 2} = \lim_{x \rightarrow 2} \frac{2^3 + 2 + 1}{2 - 2} = \frac{11}{0} = \pm\infty$; $VA \equiv x = 2$

b. $f(x) = \frac{x^3 + x + 1}{x^2 - x - 6} \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x_1 = 3$ and

$x_2 = -2$; here we have two vertical asymptotes

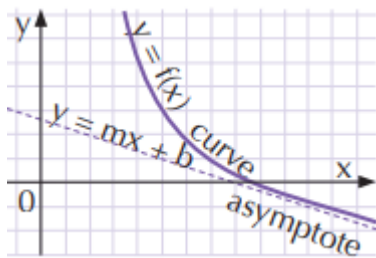
$\lim_{x \rightarrow 3} \frac{x^3 + x + 1}{x^2 - x - 6} = \frac{3^3 + 3 + 1}{3^2 - 3 - 6} = \frac{31}{0} = \pm\infty$; $VA \equiv x = 3$

$$\lim_{x \rightarrow -2} \frac{x^3 + x + 1}{x^2 - x - 6} = \frac{(-2)^3 + (-2) + 1}{(-2)^2 - (-2) - 6} = \frac{-9}{0} = \pm\infty; \text{ VA} \equiv x = -2$$

Exercises: Find the equation of the vertical asymptote to the curve a.

$$y = \frac{x-4}{(2x+2)(x-3)} \quad \text{b. } f(x) = \frac{x^5 - 3x - 1}{x^2 - 2} \quad \text{c. } f(x) = \frac{x^2 - x + 2}{x^2 + x + 1}$$

✓ Oblique asymptotes



It is an oblique asymptote when:

as x goes to infinity (or $-\infty$) then the curve goes towards a line $y = mx + b$

(note: m is not zero as that is a horizontal asymptote); where

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \quad \text{and} \quad b = \lim_{x \rightarrow \infty} f(x) - mx$$

Examples:

Find the equation of the oblique asymptote to the curve

$$\text{a. } f(x) = \frac{2x^2 - 3x + 3}{x - 1} \quad \text{b. } f(x) = \frac{x^2 + 3x - 1}{-2x - 1}$$

Solutions:

$$\text{a. } m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2 - 3x + 3}{x - 1}}{x} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{(x - 1)x} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{x^2 - x} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2} = 2$$

$$b = \lim_{x \rightarrow \infty} f(x) - mx = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{x - 1} - 2x = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3 - 2x(x - 1)}{x - 1} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3 - 2x^2 + 2x}{x - 1}$$

$$b = \lim_{x \rightarrow \infty} \frac{-3x + 2x + 3}{x - 1} = \lim_{x \rightarrow \infty} \frac{-x + 3}{x - 1} = \frac{-x}{x} = -1$$

$$\text{OA} \equiv y = 2x - 1$$

$$\text{b. } f(x) = \frac{x^2 + 3x - 1}{-2x - 1}$$

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{x^2 + 3x - 1}{-2x - 1}}{x} = \lim_{x \rightarrow \infty} \frac{x^2 + 3x - 1}{(-2x - 1)x} = \lim_{x \rightarrow \infty} \frac{x^2 + 3x - 1}{-2x^2 - x} = \lim_{x \rightarrow \infty} \frac{x^2}{-2x^2} = -\frac{1}{2}$$

$$b = \lim_{x \rightarrow \infty} f(x) - mx = \lim_{x \rightarrow \infty} \frac{x^2 + 3x - 1}{-2x - 1} - \left(-\frac{1}{2}x\right) = \lim_{x \rightarrow \infty} \frac{x^2 + 3x - 1}{-2x - 1} + \frac{x}{2} = \lim_{x \rightarrow \infty} \frac{2(x^2 + 3x - 1) + x(-2x - 1)}{(-2x - 1)2}$$

$$b = \lim_{x \rightarrow \infty} \frac{2x^2 + 6x - 2 - 2x^2 - 1x}{-4x - 2} = \lim_{x \rightarrow \infty} \frac{5x - 2}{-4x - 2} = \frac{5x}{-4x} = -\frac{5}{4}$$

$$OA \equiv y = -\frac{1}{2}x - \frac{5}{2}$$

Exercises: Find the equation of the oblique asymptote to the curve

$$1. \text{ A. } f(x) = \frac{x^2 + 2x - 1}{2x + 1} \quad \text{b. } f(x) = \frac{6x^2 - 4x + 2}{2x - 1} \quad \text{c. } f(x) = \frac{x^2 - 1}{x^2 + 2}$$

$$3. \text{ a. } f: x \rightarrow f(x) = \frac{x^2 + 1}{x}. \text{ Given a function } f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow y = \frac{2x^2 + 2}{ax^2 + bx + c}. \text{ find a,}$$

b and c such that the point $(0, -1)$ belongs to the curve and lines with the equation

$x + \frac{2}{3} = 0$ and $x = 1$ are asymptotes to the curve.

b. Lines $x = 2$ and $y = 3$ are asymptotes of function $f(x) = \frac{ax + 5}{bx + 4}$ find a and b.

c. given that the equation $y = -3x + 6$ and $x + 2 = 0$ are asymptotes to the curve

defined in real number by $F(x) = \frac{ax^2 + 3}{x - b}$. Determine the value of real number a and b.

$$1. \text{ A. Find the oblique asymptotes of the curve } y = -1 + \frac{2}{3}\sqrt{x^2 - 4x - 5}$$

B. Find the equations of asymptotes of the following function: $f(x) = \sqrt{x^2 - 1}$

Graphing asymptotes

Examples: 1. Find the asymptotes of $f(x) = \frac{x^2 - 3x}{2x - 2}$ and sketch the graph.

Solution

There is one vertical asymptote V.A $\equiv x = 1$. There is no horizontal asymptote.

The oblique asymptote is found by long division $\frac{x^2 - 3x}{2x - 2} = \frac{1}{2}x - 1 - \frac{2}{2x - 2}$. So O.A $\equiv$

$y = \frac{1}{2}x - 1$ and the sketch graph is:

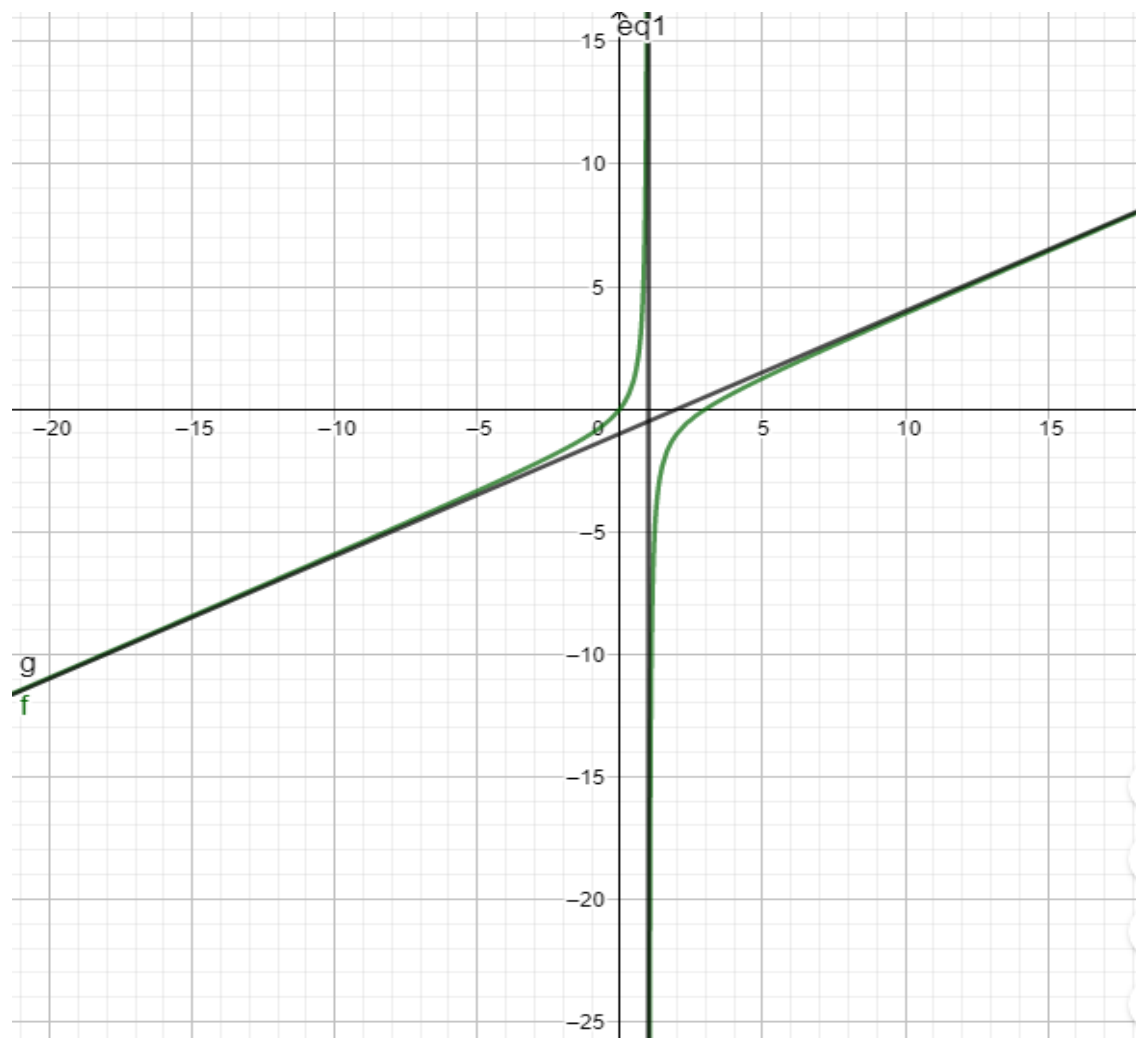


Figure 10: Vertical and Oblique asymptotes to the curve

To graph a rational function, you find the asymptotes and the intercepts, plot a few points, and then sketch in the graph.

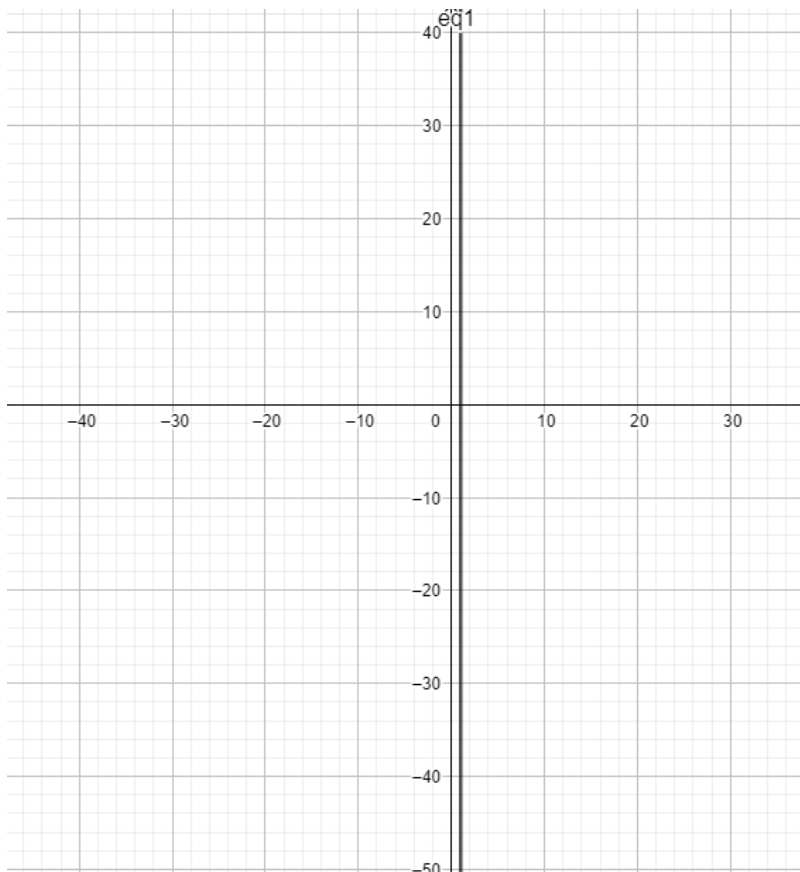
2. Graph the following and find the vertical asymptote, if any, for this rational function.

$$y = \frac{2x + 5}{x - 1}$$

Solution

First find the vertical asymptotes, if any, for this rational function. Since we cannot graph where the function does not exist, and since the function won't exist where there would be a zero in the denominator, we set the denominator equal to zero to find any unwanted points:

$$x - 1 = 0 \Rightarrow x = 1$$

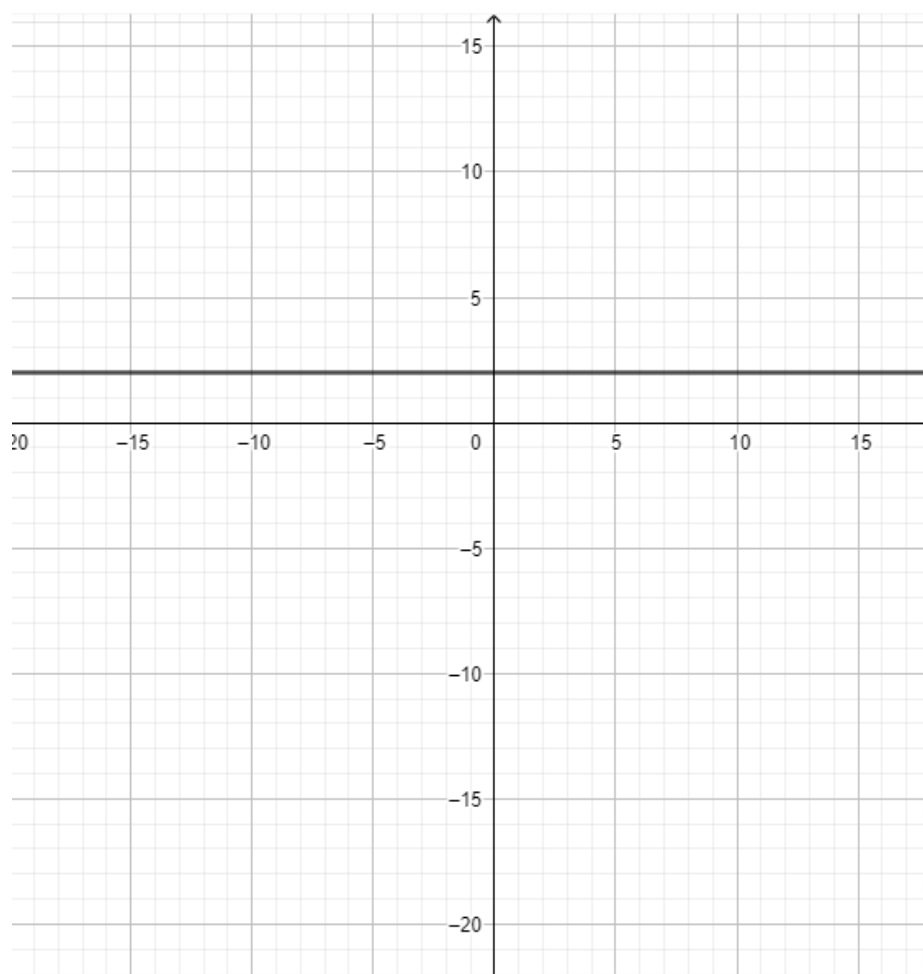


We cannot have $x = 1$, and therefore we have a vertical asymptote there.

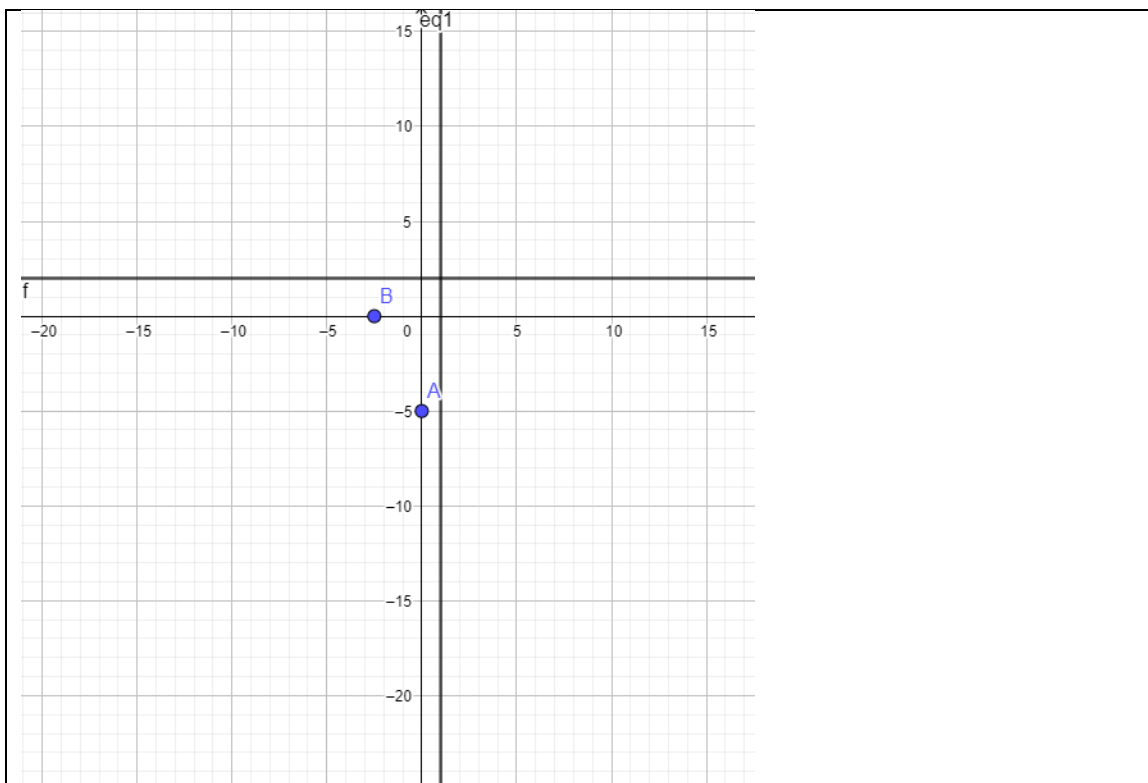
Next, we find the horizontal or slant asymptote. Since the numerator and denominator have the same degree (they are both linear), the asymptote will be

horizontal, not slant, and the horizontal asymptote will be the result of dividing the leading coefficients:

$$y = \frac{2}{1} = 2$$



Then the intercepts are at $(0, -5)$ and $(-2.5, 0)$. We sketch these in:



Now we pick a few more x-values, compute the corresponding y-values, and plot a few more points.

X	-6	-1	2	3	6	8	15
Y	1	-1.5	9	5.5	3.4	3	2.5

Now we plot these points:

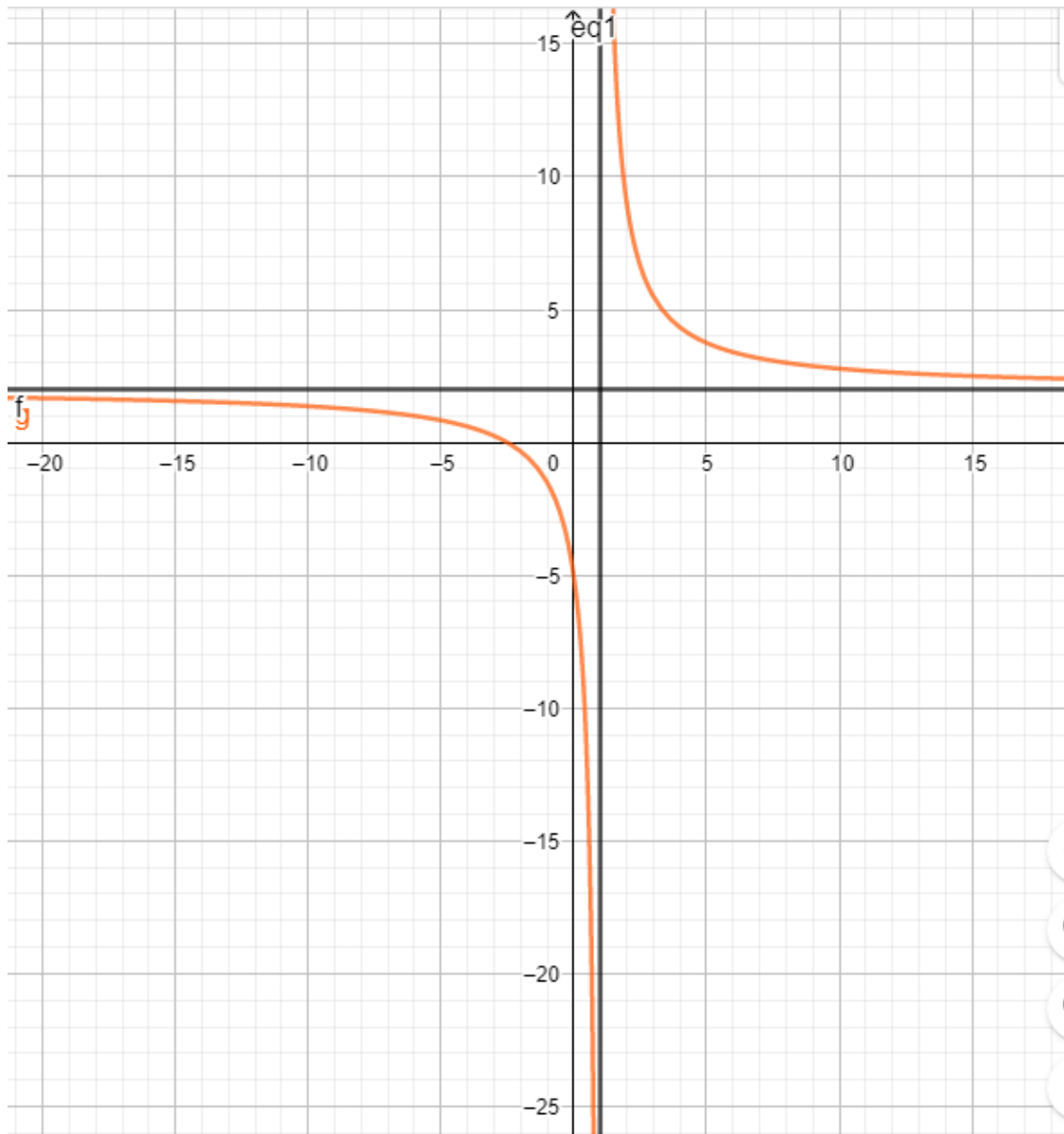


Figure 11: vertical and horizontal asymptotes to the curve

Exercises: Sketch the Graph the following and find their asymptotes

a. $f(x) = \frac{x+2}{x^2+1}$ b. $f(x) = \frac{x^3-8}{x^2+5x+6}$ c. $y = \frac{x^2-x-2}{x-2}$ d. $y = \frac{x^2+2}{x^2-4}$

(Ndorimana, 2016)



Activity 2: Guided Practice



Task 22:

The possible asymptotes of $F(x) = \frac{2x^2 - 3x + 3}{x - 1}$ are: $f(x) = \frac{2x^2 - 3x + 3}{x - 1} \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$$\lim_{x \rightarrow c} f(x) = \pm\infty; \text{ VA} \equiv x = c$$

$$\lim_{x \rightarrow 2} \frac{2x^2 - 3x + 3}{x - 2} = \lim_{x \rightarrow 2} \frac{2(1)^2 - 3(1) + 3}{2 - 2} = \frac{2}{0} = \pm\infty; \text{ VA} \equiv x = 1$$

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2 - 3x + 3}{x - 1}}{x} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{(x - 1)x} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{x^2 - x} = \lim_{x \rightarrow \infty} \frac{2x^2}{x^2} = 2$$

$$b = \lim_{x \rightarrow \infty} f(x) - mx = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3}{x - 1} - 2x = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3 - 2x(x - 1)}{x - 1} = \lim_{x \rightarrow \infty} \frac{2x^2 - 3x + 3 - 2x^2 + 2x}{x - 1}$$

$$b = \lim_{x \rightarrow \infty} \frac{-3x + 2x + 3}{x - 1} = \lim_{x \rightarrow \infty} \frac{-x + 3}{x - 1} = \frac{-x}{x} = -1$$

$$\text{OA} \equiv y = 2x - 1$$

Refer to the above examples corrected find all possible asymptotes of the following functions:

a. $f(x) = \frac{x+8}{x-4}$ b) $f(x) = \frac{x^2-1}{x^2-4}$ c) $y = \frac{x^2+5x+4}{x}$



Activity 3: Application



Task 23:

a. The function $f(x) = \frac{x+4}{x-2}$ models the concentration of a medication in the bloodstream,

where x represents the time in hours. Calculate and sketch the asymptotes of this function to understand how the concentration changes over time and approaches certain limits.

- b. The function $f(x) = \frac{x^2 + 1}{x^2 - 1}$ represents the efficiency of a solar panel, where x represents the intensity of sunlight. Calculate and sketch the asymptotes of this function to analyze how the efficiency changes with varying sunlight intensity.
- c. The function $y = \frac{x^2 - x + 4}{x - 1}$ models the cost per unit of producing items, where x represents the number of units produced. Calculate and sketch the asymptotes of this function to understand how the cost per unit changes as production increases.

Topic 2.4: Derivative of functions.



Activity 1: Problem Solving



Task 24:

A company's profit $P(x)$ (in dollars) from selling x units of a product is modeled by

$$P(x) = -5x^2 + 200x - 1500.$$

- Find the derivative $P'(x)$. What does $P'(x)$ represent in the context of this problem?
- Calculate $P'(10)$ and interpret its value. Does the profit increase or decrease when selling the 10th unit?
- Determine the critical points of $P(x)$ and identify whether they correspond to a maximum or minimum profit.

Key Facts 2.4: Derivative of functions.

- **Derivative of function at a given point**

For a non-linear function with equation $y = f(x)$, slopes of tangents at various points continually change. The gradient of a curve at a point depends on the position of the point on the curve and is defined to be the gradient of the tangent to the curve at that point. In the figure P $(x, f(x))$ is any point on the graph of $y = f(x)$ and Q is a neighboring point $(x + h, f(x + h))$.

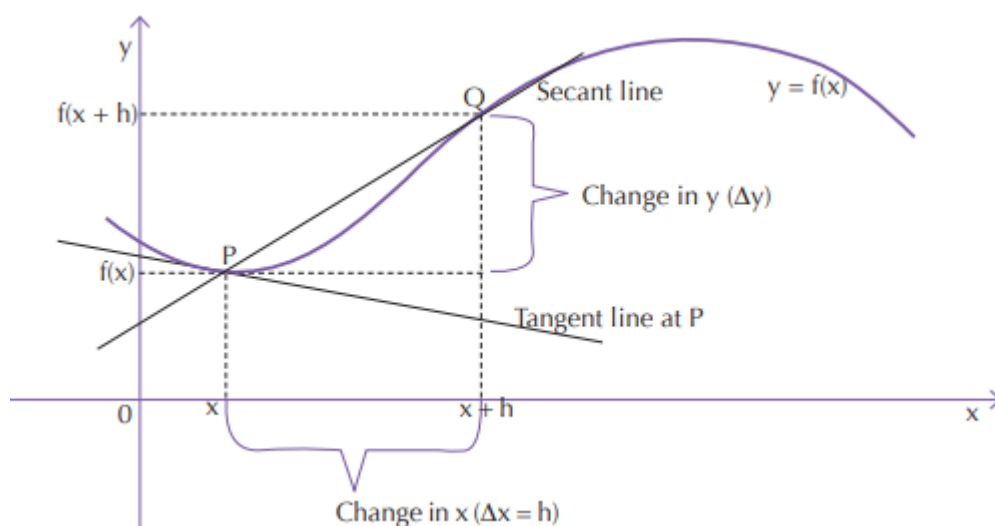


Figure 12: Gradient of a curve

As Q Approaches P along the curve, the gradient of the secant PQ approaches the gradient of the tangent at P. The gradient of the tangent at P is thus defined to be the limit of the gradient of the secant PQ as Q Approaches P along the curve. i.e., as $h \rightarrow 0$. Now the

gradient of PQ is $\frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$ Thus, we define the gradient of the

tangent at P and hence the gradient of the curve at P to be $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

Examples: 1. Find the gradient of the curve $f(x) = x^2 + 4x$ at the point P $(1, 1)$.

Solution: Then the gradient of the secant PQ is

$$\frac{f(x+h) - f(x)}{h} \Rightarrow f(x+h) = (x+h)^2 + 4(x+h) = x^2 + 2hx + h^2 + 4x + 4h^2$$

$$\frac{f(x+h) - f(x)}{h} = \frac{x^2 + 2hx + h^2 + 4x + 4h - (x^2 + 4x)}{h} = \frac{2xh + h^2 + 4h}{h} = \frac{h(2x + h + 4)}{h} = 2x + h + 4$$

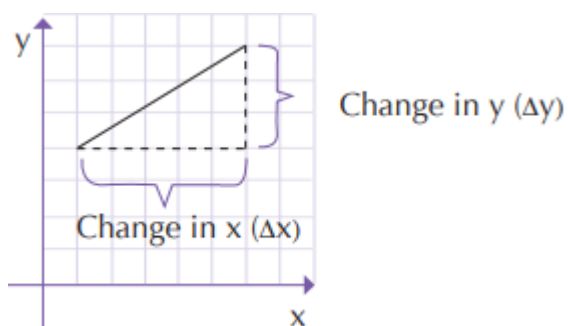
Therefore the gradient of the curve at P is $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} =$

$$\lim_{h \rightarrow 0} 2x + h + 4 = 2x + 0 + 4 = 2x + 4$$

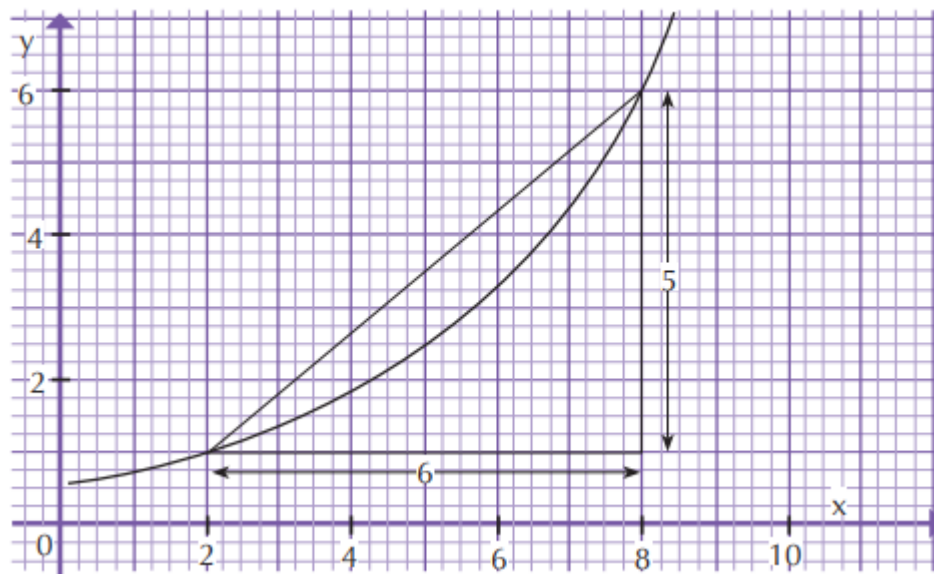
At the point (1, 1), the gradient is $2(1) + 4 = 6$.

Derivative of a function

A derivative is all about slope. Slope = Change in y over Change in x



We can find an average slope between two points.

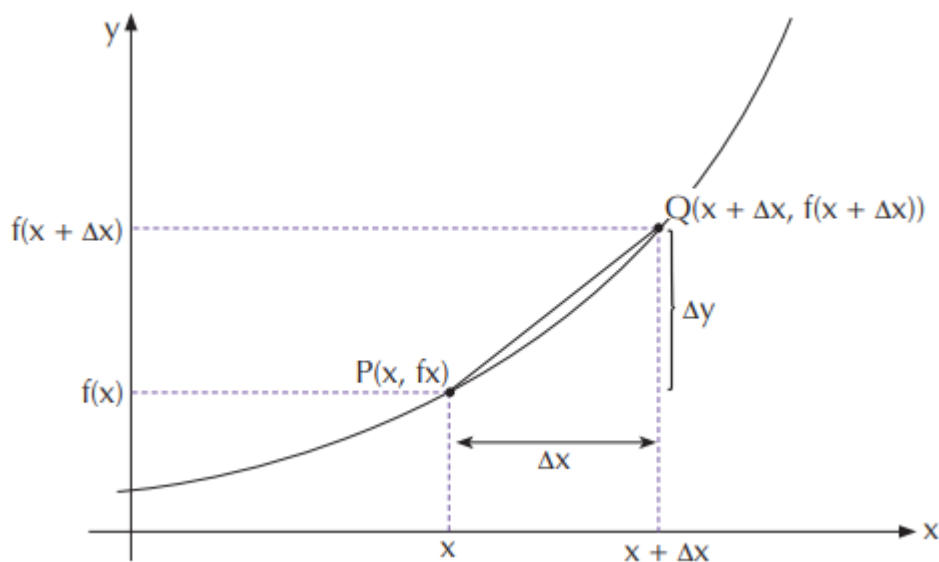


Average slope is $\frac{5}{6}$

How do we get the slope (gradient) at a point?

The derivative of a function $y = f(x)$ at the point $(x, f(x))$ equals the slope of the tangent line to the graph at that point.

Let us illustrate this concept graphically. Let f be a real-valued function and $P(x, f(x))$ be a point on the graph of this function. Let there be another point Q in the neighborhood of P .



It is assumed that this point Q is extremely close to P , thus the coordinates of Q are $(x + \Delta x, f(x + \Delta x))$; where, " Δx " is a value which is very small and provides an appropriate approximation to the slope of tangent line. The straight line PQ has a gradient of

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
 . We now consider what happens to the slope of PQ as Δx gets smaller and smaller. i.e. Q gets nearer to P .

As the value of Δx gets smaller, the two points get closer and the slope of PQ approaches that of the tangent line to the curve at P . As this happens the gradient of PQ will get closer to the slope of the tangent at P . If we take this to the limit, as Δx approaches 0, we will find the slope of the tangent at P and hence the gradient of the curve at P . Gradient at

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Thus, $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ is the slope of the tangent at the point $(x, f(x))$ and is called the derivative of the function $f(x)$.

The derivative of the function with respect to x is the function and is defined as,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Let us see an example. Take the function $f(x) = x^2 + 4x$

We know $f(x) = x^2 + 4x$, and can calculate $f(x + \Delta x)$: Start with: $f(x + \Delta x) = (x + \Delta x)^2 + 4(x + \Delta x)$

Expand $f(x + \Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 + 4x + 4\Delta x$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 4\Delta x + 4x - (x^2 + 4x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 4\Delta x + 4x - x^2 - 4x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 4\Delta x}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x + 4)}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2x + \Delta x + 4 = 2x + 0 + 4 = 2x + 4$$

$$\frac{dy}{dx} = 2x + 4$$

Note:

- The slope (gradient) of the tangent to a curve of $f(x)$ is defined as the slope of the curve $f(x)$, and is the instantaneous rate of change in y with respect to x .
- Finding the slope using the limit method is said to be using first principles.
- A chord (secant) of curve is a straight line segment which joins any two points on the curve.
- A tangent is straight line which touches curve at point

The derivative of a function, also known as slope of a function, or derived function or

simply the derivative, is defined as $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Examples: Find, from the first principles, the derivative functions of a) $f(x) = \frac{2}{3x}$ b)

$$f(x) = \sqrt{2x+2}$$

Solutions: b) $f(x) = \frac{2}{3x}$, $f(x+h) = \frac{2}{3(x+h)} = \frac{2}{3x+3h}$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{3x+3h} - \frac{2}{3x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{2(3x) - 2(3x+3h)}{(3x+3h)(3x)}}{h} = \lim_{h \rightarrow 0} \frac{6x - 6x - 6h}{(3x+3h)(3x)h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{-6h}{(3x+3h)(3x)h} = \lim_{h \rightarrow 0} \frac{-6}{(3x+3h)(3x)} = \frac{-6}{(3x+3(0))(3x)} = \frac{-6}{9x^2} = -\frac{2}{3x^2}$$

c) $f(x) = \sqrt{2x+2}$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \Rightarrow f(x+h) = \sqrt{2(x+h)+2} \Rightarrow f(x+h) = \sqrt{2x+2h+2}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2x+2h+2} - \sqrt{2x+2}}{h} = \frac{0}{0} (I.F)$$

BY rationalize: $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{(\sqrt{2x+2h+2} - \sqrt{2x+2})(\sqrt{2x+2h+2} + \sqrt{2x+2})}{(\sqrt{2x+2h+2} + \sqrt{2x+2})h}$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{((\sqrt{2x+2h+2})^2 - (\sqrt{2x+2})^2)}{(\sqrt{2x+2h+2} + \sqrt{2x+2})h} = \lim_{h \rightarrow 0} \frac{2x+2h+2 - (2x+2)}{(\sqrt{2x+2h+2} + \sqrt{2x+2})h}$$

$$\lim_{h \rightarrow 0} \frac{2x+2h+2 - (2x+2)}{(\sqrt{2x+2h+2} + \sqrt{2x+2})h} = \lim_{h \rightarrow 0} \frac{2h}{(\sqrt{2x+2h+2} + \sqrt{2x+2})h} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2x+2h+2} + \sqrt{2x+2}}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{2x+2(0)+2} + \sqrt{2x+2}} = \frac{2}{2\sqrt{2x+2}} = \frac{1}{\sqrt{2x+2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x+2}} \quad (\text{Ndorimana, 2016})$$

Exercises:

1. If $f(x) = 4x^2 + 3x - 5$, find: (a) $f'(x)$ using the definition of derivative (b) $f'(2)$

(c) $f'(-2)$

2. Find, from the first principles, the derivative functions of a. $f(x) = \sqrt{4x+8}$ b.

$$f(x) = \frac{2x}{x-1}$$

$$f(x) = 3x^3 + 3x^2 - 4x + 5 \quad \text{d.} \quad f(x) = \frac{1}{\sqrt{x}}$$

Notation of differentiation: $f'(x)$ or $\frac{dy}{dx}$

Rules of differentiation

Differentiation is the process of finding the derivative function. If we are given a function $f(x)$ then $f'(x)$ represents the derivative function. However, if we are given y in terms of x

then y' or $\frac{dy}{dx}$ are usually used to represent the derivative function. Note: $\frac{dy}{dx}$ reads 'Dee

y Dee x', or 'the derivative of y with respect to x'. $\frac{dy}{dx}$ is not a fraction. We write dx

instead of " Δx heads towards 0", so "the derivative of f " is commonly written as $\frac{d}{dx}$; so

$$\frac{d}{dx}(x^2) = 2x$$

"The derivative of x^2 equals $2x$."

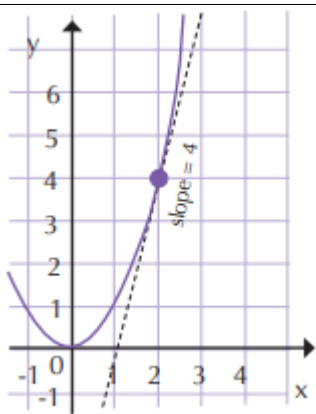


Figure 13: Slope of a curve

What does $\frac{d}{dx}(x^2) = 2x$ mean?

It means that, for the function x^2 , the slope or “rate of change” at any point is $2x$. So when $x = 2$ the slope is $2(2) = 4$, as shown in given Figure: Or when $x = 5$ the slope is $2(5) = 10$, and so on. Note: sometimes $f'(x)$ is also used for “the derivative of”: $f'(x) = 2x$ “The derivative of $f(x)$ equals $2x$ ”.

The process of finding a derivative is called “differentiation”.

Below is the table containing basic rules which can be used to differentiate more complicated functions without using the differentiation from the first principles.

- **Derivative of polynomial, derivative of product of two function and quotient rule of derivative**

all are summarized in the table below:

Name of rules	$F(x)$	$f/(x)$
Differentiating a constant C	C	0
Differentiating of polynomial x^n	x^n	nx^{n-1}
product rule	$u(x).v(x)$	$u'(x)v(x) + u(x)v'(x)$
Sum or difference rule	$u(x) \pm v(x)$	$u/(x) \pm v/(x)$

Quotient rule	$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$
---------------	---------------------	--

Examples: Find the derivative of the following functions:

a. $f(x) = -20$ b. $f(x) = 20x$ c. $g(x) = x^6$ d. $f(x) = 15x^3$
e. $f(x) = x^5 - 5x^4 + 3x^2 - 20x + 25$ f. $f(x) = \frac{2x}{3x-1}$ g. $f(x) = (3x^2 + 1)(x^2 - 4)$

Solutions : a. $\frac{dy}{dx} = 0$ b. $\frac{dy}{dx} = 20$ c. $\frac{dy}{dx} = 6x^{6-1}$ d. $\frac{dy}{dx} = 15(3)x^{3-1}$
 $\frac{dy}{dx} = 6x^5$ $\frac{dy}{dx} = 45x^2$

$f(x) = x^5 - 5x^4 + 3x^2 - 20x + 25$
e. $\frac{dy}{dx} = 5x^{5-1} - 5(4)x^{4-1} + 3(2)x^{2-1} - 20 + 0$ f. $\frac{dy}{dx} = \frac{(2x)'(3x-1) - (3x-1)'(2x)}{(3x-1)^2}$
 $\frac{dy}{dx} = 5x^4 - 20x^3 + 6x - 20$ $\frac{dy}{dx} = \frac{2(3x-1) - 3(2x)}{(3x-1)^2} = \frac{6x-2-6x}{(3x-1)^2}$
 $\frac{dy}{dx} = -\frac{2}{(3x-1)^2}$

$f(x) = (3x^2 + 1)(x^2 - 4)$
g. $\frac{dy}{dx} = (3x^2 + 1)'(x^2 - 4) + (x^2 - 4)'(3x^2 + 1)$ $\Rightarrow \frac{dy}{dx} = 6x(x^2 - 4) + 2x(3x^2 + 1) \Rightarrow$
 $\frac{dy}{dx} = 6x^3 - 24x + 6x^3 + 2x$

$\frac{dy}{dx} = 12x^3 - 22x$

1. Exercises: Find the derivative of the following functions:

a. $f(x) = (x^2 + 6)(x - 2)$

b. $f(x) = (x^2 + 2)^{\frac{1}{2}}$

c. $9x^3 + 7x^2 + 20$

d. $f(x) = \frac{x^2 - 2x + 5}{2x - 1}$

- **Other functions needed to be differentiable**

- ✓ **Derivative of composite functions:**

The chain rule Composite functions are functions like $(3x^2 + 4x)^5$. This function is made up of two simpler functions.

$$y = (3x^2 + 4x)^5 \text{ is } y = u^5 \text{ where } u = 3x^2 + 4x$$

Notice that in the example $(3x^2 + 4x)^5$, if $f(x) = x^5$ and $g(x) = 3x^2 + 4x$ then $f(g(x)) = f(3x^2 + 4x) =$

$(3x^2 + 4x)^5$, All of these functions of the form $g(x) = (f(x))^n$ can be made up in this way where we compose a function of a function. Thus, these functions are called composite functions.

Consider the function $y = (2x + 1)^3$ which is really $y = u^3$ where $u = 2x + 1$.

✚ We see that $\frac{dy}{du} = 3u^2 = 3(2x + 1)^2 = 3(2x + 1)^2$ and $\frac{du}{dx} = 2$

✚ When we expand, we obtain $\frac{dy}{du} = 3u^2 = 3(2x + 1)^2 \times 2 = \frac{dy}{du} \times \frac{du}{dx} = 6(2x + 1)^2$

From the above example we derive the formula of the chain rule:

$$\text{If } y = f(u) \text{ where } u = u(x) \text{ then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ or if } g(x) = (f(x))^n \Rightarrow g'(x) =$$

$$n(f(x))^{n-1} \times f'(x)$$

Examples:

Find derivative of the following functions: a. $f(x) = 4(5x^4 + 7)^{20}$ b.

$$f(x) = (3x^5 - 3x + 5)^{100}$$

Solutions: a. $f(x) = 4(5x^4 + 7)^{20}$

$$f'(x) = 20(4) (5x^4 + 7)^{20-1} (5x^4 + 7)' \Rightarrow \frac{dy}{dx} = 80(5x^4 + 7)^{19} (20x^3)$$

$$\frac{dy}{dx} = 80(20x^3)(5x^4 + 7)^{19} \Rightarrow \frac{dy}{dx} = 1600x^3(5x^4 + 7)^{19}$$

b. $f(x) = (3x^5 - 3x + 5)^{100}$

$$\frac{dy}{dx} = 100(3x^5 - 3x + 5)^{100-1} \frac{d}{dx}(3x^5 - 3x + 5)$$

$$\frac{dy}{dx} = 100(3x^5 - 3x + 5)^{99}(15x^4 - 3)$$

$$\frac{dy}{dx} = 100(15x^4 - 3)(3x^5 - 3x + 5)^{99}$$

$$\frac{dy}{dx} = (1500x^4 - 300)(3x^5 - 3x + 5)^{99}$$

Exercises: work out the derivative of the following functions

a. $f(x) = (3x^6 - 5x + 5)^{1000}$ b. $f(x) = 4(5x + 7)^{10}$ c. $f(x) = 5(5x^4 + 7)^{200}$ d.

$$f(x) = 5(5x^4 + 2x^3 - 3x^2 + 7)^{20}$$

✓ Derivative of irrational functions

✚ Irrational functions are those functions with the form of radicals where

$$g(x) = \sqrt[n]{(f(x))^m}$$

✚ To make the derivative of Irrational functions first we have to use the property from

radical form to exponential form where $g(x) = \sqrt[n]{(f(x))^m} = (f(x))^{\frac{m}{n}}$, this exponential form has the form of composite functions

✚ To make the derivative of irrational functions, we have to proceed the process of making derivative of composite functions

Examples:

Find derivative of the following functions:

a. $g(x) = \sqrt{3x^2 + 1}$ b. $g(x) = \sqrt[4]{(2x^3 - 3)^3}$ c. $g(x) = \sqrt[3]{(4x - 1)^4}$

Solutions: a. $g(x) = \sqrt{3x^2 + 1} \Rightarrow g(x) = (3x^2 + 1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} (3x^2 + 1)^{\frac{1}{2}-1} = \frac{1}{2} (3x^2 + 1)^{-\frac{1}{2}} \frac{d}{dx} (3x^2 + 1) = \frac{1}{2} (3x^2 + 1)^{-\frac{1}{2}} (6x) = \frac{1}{2} \times \frac{6x}{(3x^2 + 1)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2 + 1}}$$

b. $g(x) = \sqrt[4]{(2x^3 - 3)^3} \Rightarrow g(x) = (2x^3 - 3)^{\frac{3}{4}}$

$$\frac{dy}{dx} = \frac{3}{4} (2x^3 - 3)^{\frac{3}{4}-1} \frac{d}{dx} (2x^3 - 3)$$

$$\frac{dy}{dx} = \frac{3}{4} (2x^3 - 3)^{-\frac{1}{4}} (6x^2) = \frac{3}{4} \times 6x^2 (2x^3 - 3)^{-\frac{1}{4}} = \frac{9x^2}{2} \times \frac{1}{(2x^3 - 3)^{\frac{1}{4}}} = \frac{9x^2}{2\sqrt[4]{2x^3 - 3}}$$

c. $g(x) = \sqrt[3]{(4x - 1)^4} = (4x - 1)^{\frac{4}{3}}$

$$\frac{dy}{dx} = \frac{4}{3} (4x - 1)^{\frac{4}{3}-1} \frac{d}{dx} (4x - 1) = \frac{4}{3} (4x - 1)^{\frac{1}{3}} (4)$$

$$\frac{dy}{dx} = \frac{16}{3} (4x - 1)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{16}{3} \sqrt[3]{4x - 1}$$

Exercises : a. $g(x) = \sqrt{\frac{2x-1}{x+4}}$ b. $g(x) = \sqrt[5]{(2x^3 - 3)^2}$ c. $g(x) = \sqrt[4]{\left(\frac{2x}{x+2}\right)^3}$ d.

$$g(x) = \sqrt[4]{(3x^3 - 3)^3}$$

✓ Derivative of simple trigonometric functions

To find the derivative of simple trigonometric functions, first we have to remember the simple trigonometric formulas as we studied in level three, where:

a. $\tan x = \frac{\sin x}{\cos x}$ b. $\cot x = \frac{\cos x}{\sin x}$ c. $\sec x = \frac{1}{\cos x}$ d. $\csc(x) = \frac{1}{\sin x}$ e.

$\sin^2(x) + \cos^2(x) = 1$ f. $\sec^2(x) = 1 + \tan^2(x)$ g. $\csc^2(x) = 1 + \cot^2(x)$.

✓ Derivative of trigonometric functions:

Let $A = \sin x$; $B = \cos x$, $C = -\sin x$ and $D = -\cos x$

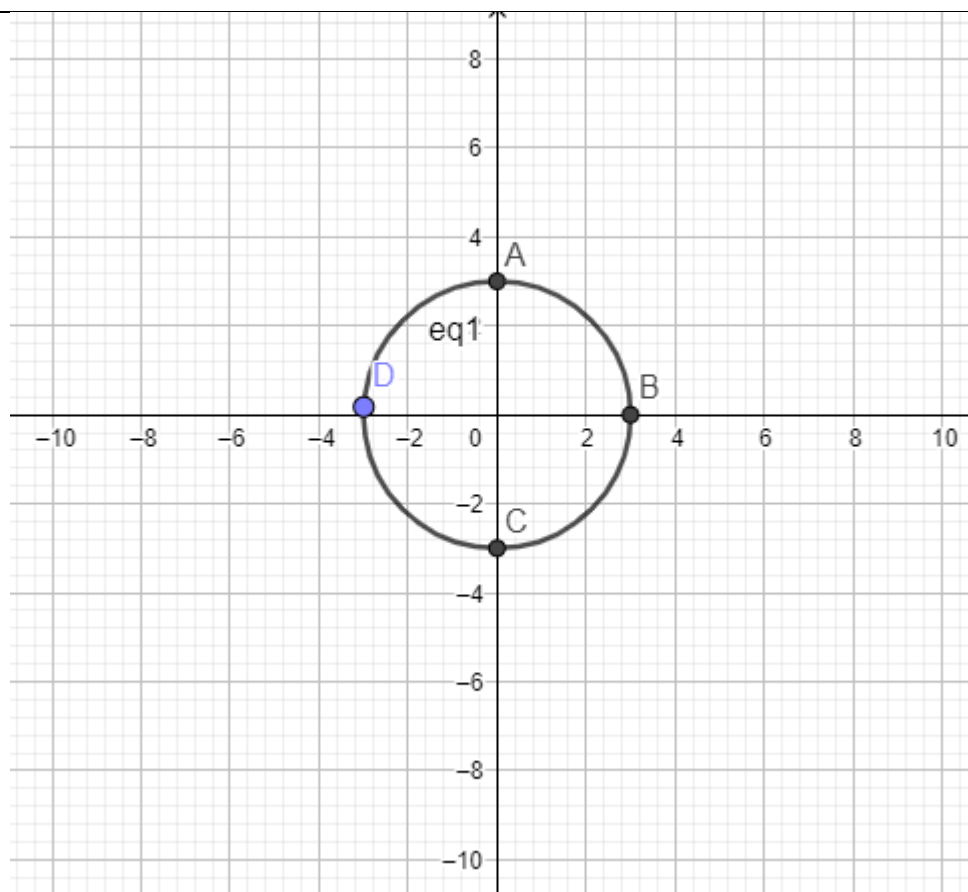


Figure 14: Trigonometric circle

We make derivative of trigonometric functions by moving clockwise:

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(-\sin x) &= -\cos x & \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(-\cos x) &= \sin x & \frac{d}{dx}(\cos x) &= -\sin x \end{aligned}$$

If we differentiate the composite functions, where u is another function, then

$$\frac{d}{dx}(\sin u) = u' \cos u \quad \text{and} \quad \frac{d}{dx}(\cos u) = -u' \sin u$$

Examples:

find derivative of the following functions:

a. $f(x) = \sin(4x^3 + 10)$ b. $g(x) = 3\cos(5x^4 + 3x - 3)$

Solutions: a. $\frac{dy}{dx} = (4x^3 + 10)' \cos(4x^3 + 10) \Rightarrow \frac{dy}{dx} = 12x^2 \cos(4x^3 + 10)$

b.

$$g(x) = 3 \cos(5x^4 + 3x - 3) \Rightarrow \frac{dy}{dx} = 3(5x^4 + 3x - 3)'(-\sin(5x^4 + 3x - 3))$$

$$\frac{dy}{dx} = (-3)(20x^3 + 3)\sin(5x^4 + 3x - 3)$$

$$\frac{dy}{dx} = (-60x^3 - 9)\sin(5x^4 + 3x - 3)$$

Exercises:

find derivative of the following functions:

a. $f(x) = \sin(5x^3 - 10)$ b. $g(x) = 5\cos(4x^4 - 4x - 2)$

Derivative of other trigonometric ratios:

The function $f(x) = \tan x = \frac{\sin x}{\cos x}$ is differentiable on $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z}$ and the

function $f(x) = \cot x$ is differentiable on $\mathbb{R} \setminus \{k\pi\}, k \in \mathbb{Z}$.

✓ **Given** $f(x) = \tan x = \frac{\sin x}{\cos x}$ **find its derivative**

$$\frac{dy}{dx} = \frac{(\sin x)'(\cos x) - (\cos x)'(\sin x)}{(\cos x)^2} = \frac{\cos(x)\cos(x) - (-\sin x)(\sin x)}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2(x) \quad \frac{dy}{dx} = \sec^2(x) \text{ or } \frac{dy}{dx} = 1 + \tan^2 x$$

✓ **For** $\cot x = \frac{\cos x}{\sin x}$

$$\frac{dy}{dx} = \frac{(\sin x)(\cos x)' - (\cos x)(\sin x)'}{(\sin x)^2} = \frac{-\cos(x)\cos(x) - \sin x(\sin x)}{\sin^2(x)} = \frac{-(\cos^2 x + \sin^2 x)}{\sin^2(x)}$$

$$\frac{dy}{dx} = \frac{-1}{\sin^2 x} = -\csc^2(x) \quad \frac{dy}{dx} = -\csc^2(x) \quad \text{or} \quad \frac{dy}{dx} = -(1 + \cot^2 x)$$

✓ **For** $\sec x = \frac{1}{\cos x}$

$$\frac{dy}{dx} = \frac{(1)'(\cos x) - (\cos x)'(1)}{(\cos x)^2} = \frac{(0)\cos(x) - (-\sin x)(1)}{\cos^2(x)} = \frac{\sin x}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x \quad \frac{dy}{dx} = \tan x \sec x$$

✓ For $f(x) = \csc(x) = \frac{1}{\sin x}$

$$\frac{dy}{dx} = \frac{(\sin x)(1)' - (1)(\sin x)'}{(\sin x)^2} = \frac{0 - \cos x}{\sin^2(x)} = \frac{-\cos x}{\sin^2(x)} = -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = -\cot x \csc x$$

$$\frac{dy}{dx} = -\cot x \csc x$$

If we differentiate the composite functions, where u is another function, then

$$\frac{d}{dx}(\tan u) = u' \sec^2 u \quad ; \quad \frac{d}{dx}(\cot u) = -u' \csc^2 u \quad ; \quad \frac{d}{dx}(\sec x) = u' \tan u \sec u \quad \text{and}$$

$$\frac{d}{dx}(\csc u) = -u' \csc u \cot u$$

Examples:

find derivative of the following functions:

a. $f(x) = \tan(4x^3 + 10)$ b. $g(x) = 3\cot(5x^4 + 3x - 3)$ c. $g(x) = 3\sec(5x^4 + 3x - 3)$

d. $g(x) = 3\csc(5x^4 + 3x - 3)$

Solutions: A. $f(x) = \tan(4x^3 + 10)$

$$\frac{dy}{dx} = (4x^3 + 10)' \sec^2(4x^3 + 10)$$

$$\frac{dy}{dx} = 12x^2 \sec^2(4x^3 + 10)$$

b. $g(x) = 3\cot(5x^4 + 3x - 3)$

$$\frac{dy}{dx} = 3(5x^4 + 3x - 3)' (-\csc^2(5x^4 + 3x - 3))$$

$$\frac{dy}{dx} = -3(20x^3 + 3)\csc^2(5x^4 + 3x - 3)$$

$$\frac{dy}{dx} = (-60x^3 - 9)\csc^2(5x^4 + 3x - 3)$$

c. $g(x) = 3\sec(5x^4 + 3x - 3)$

$$\frac{dy}{dx} = 3(5x^4 + 3x - 3)' \tan(5x^4 + 3x - 3)\sec(5x^4 + 3x - 3)$$

$$\frac{dy}{dx} = 3(20x^3 + 3)\tan(5x^4 + 3x - 3)\sec(5x^4 + 3x - 3)$$

$$\frac{dy}{dx} = (60x^3 + 9)\tan(5x^4 + 3x - 3)\sec(5x^4 + 3x - 3)$$

d. $g(x) = 3\csc(5x^4 + 3x - 3)$

$$\frac{dy}{dx} = 3(5x^4 + 3x - 3)' (-\cot(5x^4 + 3x - 3)\csc(5x^4 + 3x - 3))$$

$$\frac{dy}{dx} = -3(20x^3 + 3)\cot(5x^4 + 3x - 3)\csc(5x^4 + 3x - 3)$$

$$\frac{dy}{dx} = (-60x^3 - 9)\cot(5x^4 + 3x - 3)\csc(5x^4 + 3x - 3)$$

Derivative of the form: If we differentiate the composite functions, where u is another

function, then $\frac{d}{dx}(\sin^n u) = n(\sin u)^{n-1}(\sin u)'$; $\frac{d}{dx}(\cos u) = n(\cos u)^{n-1}(\cos u)'$;

$\frac{d}{dx}(\tan^n u) = n(\tan u)^{n-1}(\tan u)'$ and $\frac{d}{dx}(\cot^n u) = n(\cot u)^{n-1}(\cot u)'$

For examples:

find derivative of the following functions

a. $g(x) = \sin^5(2x+1)$ b. $g(x) = 3\tan^5(2x^2 + x)$

We put the given function in the form of composite functions where

$$g(x) = (\sin(2x+1))^5$$

Let $f(x) = \sin(2x+1)$, our function becomes

$$g(x) = (f(x))^n \Rightarrow g'(x) = n(f(x))^{n-1} f'(x)$$

$$g'(x) = 5(\sin(2x+1))^{5-1} (\sin(2x+1))'$$

$$\frac{dy}{dx} = 5(\sin(2x+1))^4 (2x+1)' \cos(2x+1)$$

$$\frac{dy}{dx} = 5(\sin(2x+1))^4 2 \cos(2x+1)$$

$$\frac{dy}{dx} = 10 \sin^4(2x+1) \cos(2x+1)$$

b. $g(x) = 3 \tan^5(2x^2 + x)$

$$g'(x) = 5 \times 3 (\tan(2x^2 + x))^{5-1} (\tan(2x^2 + x))'$$

$$g'(x) = 15 (\tan(2x^2 + x))^4 (2x^2 + x)' \sec^2(2x^2 + x)$$

$$\frac{dy}{dx} = 15 (\tan^4(2x^2 + x)) (4x+1) \sec^2(2x^2 + x) = 15(4x+1) \tan^4(2x^2 + x) \sec^2(2x^2 + x)$$

$$\frac{dy}{dx} = (60x+15) \tan^4(2x^2 + x) \sec^2(2x^2 + x)$$

(Emmanuel N. & Pacifique I, 2017)

Exercises :

find derivative of the following functions

a. $g(x) = \cos^7(2x+1)$ b. $g(x) = 3 \sec^6(2x^2 + x)$ c. $g(x) = \cos(5x^6 + 3x^2 - 3x)$

d. $g(x) = -3 \sin(5x^{10} - 3x - 3)$ e. $g(x) = 3 \cot^8(2x^2 + x)$ f. $g(x) = 3 \csc^5(2x^2 + x)$

g. $f(x) = x^2 \tan 2x$ h. $f(x) = \frac{\sin 3x}{2x}$ i. $f(x) = \sqrt{2x+1} \cos x$

- **Higher order derivatives or Successive derivatives**

If f is a function which is differentiable on its domain, then f' is a derivative function. If, in addition, f' is differentiable on its domain, then the derivative of f' exists and is denoted by

f'' ; it is the function given by $f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$ and is called the second

derivative of $f(x)$. If, in addition, f'' is differentiable on its domain, then the derivative of f'' is denoted by f''' and is called the third derivative of $f(x)$.

In general, the n th derivative of $f(x)$, where n is a positive integer, denoted by $f^n(x)$, is

defined to be the derivative of the $(n-1)^{\text{th}}$ derivative of $f(x)$. For $n > 1$, $f^n(x)$ is called a higher-order derivative of $f(x)$. If we differentiate the function y n times in successive differentiation, the resulting function is called the n th derivative of y and is denoted as

$\frac{d^n y}{dx^n}$ or $y(n)$ or $f(n)(x)$. The process of finding the successive derivatives of a function is

known as successive differentiation. Thus, if $y = f(x)$, the successive differential of $f(x)$ are

$$\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \dots, \frac{d^{(n)} y}{dx^{(n)}}.$$

Examples:

Find the fifth derivative of the following function and find $f^{(5)}(-2)$:

$$f(x) = x^6 - 2x^5 + 3x^4 - 4x^3 + 2x^2 + 50x - 60$$

$$f'(x) = 6x^5 - 10x^4 + 12x^3 - 12x^2 + 4x + 50$$

$$f''(x) = 30x^4 - 40x^3 + 36x^2 - 24x + 4$$

$$f'''(x) = 120x^3 - 120x^2 + 72x - 24$$

$$f^{(4)}(x) = 360x^2 - 240x + 72$$

$$\frac{d^5 y}{dx^5} = 720x - 240 \quad f''''(-2) = 720(-2) - 240 \quad f''''(-2) = -1440 - 240 = -1680$$

Exercises:

Given $f(x) = 4x^3 - 2x^2 + 3x - 20$; calculate

i. $f'(x)$ ii. $f''(x)$ iii. $f'(0)$ iv. $f'(2)$ v. $f''(2)$

(Emmanuel N. & Pacifique I, 2017)



Activity 2: Guided Practice



Task 25:

Example: Derivative of $f(x) = x^5 - 5x^4 + 3x^2 - 20x + 25$ is

$$f(x) = x^5 - 5x^4 + 3x^2 - 20x + 25$$

$$\frac{dy}{dx} = 5x^{5-1} - 5(4)x^{4-1} + 3(2)x^{2-1} - 20 + 0$$

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 6x - 20$$

Refer to the corrected example find the derivative of the following functions:

1. $f(x) = x^{15} - 5x^7 + 3x^5 - 20x + 255$

2. $f(x) = x^9 - 5x^8 + 3x^6 - 200x + 205$



Activity 3: Application



Task 26:

- a. The function $g(x) = \sqrt[4]{(5x^3 - 4)^7}$ models the growth rate of a plant, where x represents the number of days since planting. Calculate the differentiation of this function to determine the rate at which the plant's growth changes over time.

- b. The function $g(x) = \sqrt[4]{(x^3 - 3)^5}$ represents the speed of a car, where x represents the time in seconds. Calculate the differentiation of this function to find the acceleration of the car at any given time.
- c. The function $f(x) = 4(5x^4 + 20x)^{30}$ models the population of a species, where x represents the number of years. Calculate the differentiation of this function to determine the rate of population growth over time.
- d. The function $g(x) = \cos(5x^6 + 3x^2 - 3x)$ represents the oscillation of a pendulum, where x represents the time in seconds. Calculate the differentiation of this function to find the velocity of the pendulum at any given time.
- e. The function $g(x) = 3\sec^6(2x^2 + x)$ models the intensity of light passing through a medium, where x represents the depth of the medium in meters. Calculate the differentiation of this function to determine how the light intensity changes with depth.
- f. The function $f(x) = \frac{-2x+5}{2x-1}$ represents the efficiency of a machine, where x represents the number of hours of operation. Calculate the differentiation of this function to find the rate at which the machine's efficiency changes over time.

Topic 2.5: Application of derivative



Activity 1: Problem Solving



Task 27:

A particle is moving along the x -axis such that its position, $x(t)$ meters to the right of the origin at time t seconds, is given by $x(t) = t^3 - 9t^2 + 24t - 18$

- a. Describe the particle motion during the first five seconds
- b. Calculate the distance travelled in that time

Key Facts 2.5: Application of derivative

- **Determination of equation of tangent line at a given point**

Tangent line to a curve of function Consider a curve $y = f(x)$. If P is the point with x-coordinate a, then the slope of the tangent at this point is $f'(a)$. The equation of the tangent

is by equating slopes and is $\frac{y - f(a)}{x - a} = f'(a) \Rightarrow T \equiv y - f(a) = f'(a)(x - a)$

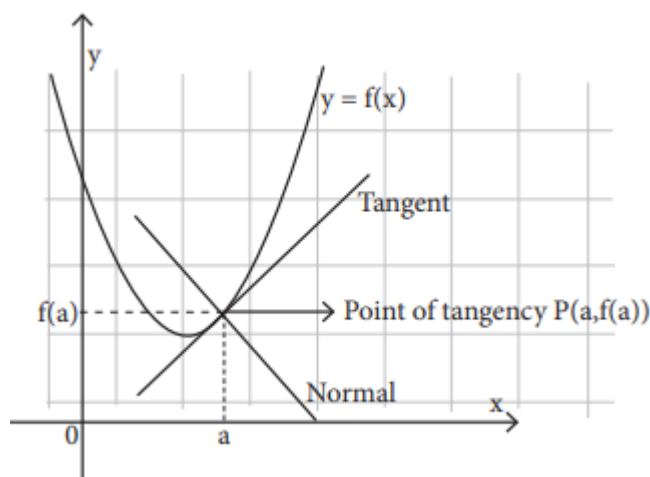


Figure 15: Tangent and normal lines

- **Determination of equation of normal line at a given point**

A normal to a curve is a line which is perpendicular to the tangent at the point of contact.

Therefore, if the slope of the tangent at $x = a$ is $f'(a)$, then the slope of a normal at $x = a$ is

$-\frac{1}{f'(a)}$. This comes from the fact that the product of gradients of two perpendicular lines is -1 .

$$N \equiv y - f(a) = -\frac{1}{f'(a)}(x - a)$$

Note: If a tangent touches $y = f(x)$ at (a, b) then it has equation

$\frac{y - f(a)}{x - a} = f'(a) \Rightarrow T \equiv y - f(a) = f'(a)(x - a)$. Vertical and horizontal lines have equations of the form $x = k$ and $y = c$ respectively, where c and k are constants.

Examples:

1. Find the equation of the tangent to $f(x) = x^2 + 2$ at the point where $x = 1$.

2. Find the tangent and normal equation of the following functions

a. $f(x) = 2x^2 - 2x + 1$ at $x_0 = 1$

b. $f(x) = \frac{x-1}{2x-1}$ at $p\left(2, \frac{1}{3}\right)$

Solutions: 1. $f(x) = x^2 + 2$ at the point where $x = 1$

$$f'(x) = 2x \Rightarrow f'(1) = 2 \times 1 \Rightarrow \text{gradient} = 2$$

2. a. $T \equiv y - f(a) = f'(a)(x - a)$

$$y_0 = f(1) = 2(1)^2 - 2(1) + 1 \Rightarrow y_0 = 1$$

$$f'(x) = 4x - 2 \Rightarrow f'(1) = 4(1) - 2 \Rightarrow f'(x) = 2$$

$$T \equiv y - 1 = 2(x - 1) \Rightarrow T \equiv y - 1 = 2x - 2 \Rightarrow T \equiv y = 2x - 2 + 1$$

$$T \equiv y = 2x - 1$$

$$N \equiv y - f(a) = -\frac{1}{f'(a)}(x - a)$$

$$N \equiv y - 1 = -\frac{1}{2}(x - 1) \Rightarrow N \equiv y - 1 = -\frac{1}{2}x + \frac{1}{2} \Rightarrow N \equiv y = -\frac{1}{2}x + \frac{1}{2} + 1$$

$$N \equiv y = -\frac{1}{2}x + \frac{3}{2}$$

b. $f(x) = \frac{x-1}{2x-1}$ at $p\left(2, \frac{1}{3}\right)$

$$T \equiv y - f(a) = f'(a)(x - a)$$

$$y_0 = \frac{1}{3}; x_0 = 2$$

$$\frac{dy}{dx} = \frac{(x-1)'(2x-1) - (2x-1)'(x-1)}{(2x-1)^2} = \frac{1(2x-1) - 2(x-1)}{(2x-1)^2}$$

$$f'(x) = \frac{2x-1-2x+2}{(2x-1)^2} = \frac{1}{(2x-1)^2} \Rightarrow f'(2) = \frac{1}{(2(2)-1)^2} = \frac{1}{(4-1)^2} = \frac{1}{9}$$

$$T \equiv y - \frac{1}{3} = \frac{1}{9}(x-2) \Rightarrow T \equiv y - \frac{1}{3} = \frac{1}{9}x - \frac{2}{9} \Rightarrow T \equiv y = \frac{1}{9}x - \frac{2}{9} + \frac{1}{3}$$

$$T \equiv y = \frac{1}{9}x - \frac{2}{9} + \frac{1}{3} \Rightarrow T \equiv y = \frac{1}{9}x + \frac{1}{9}$$

$$N \equiv y - f(a) = -\frac{1}{f'(a)}(x-a)$$

$$y_0 = \frac{1}{3}; x_0 = 2 \text{ and } f'(2) = \frac{1}{9}$$

$$N \equiv y - \frac{1}{3} = -\frac{1}{\frac{1}{9}}(x-2) \Rightarrow N \equiv y - \frac{1}{3} = -9(x-2) \Rightarrow N \equiv y - \frac{1}{3} = -9x + 18$$

$$N \equiv y = -9x + 18 + \frac{1}{3} \Rightarrow N \equiv y = -9x + \frac{55}{3}$$

Exercises:

1. Find the equation of any horizontal tangent to $y = x^3 - 12x + 2$.
2. For the curve of the function $y = 2x^3 + 3x^2 - 12x + 1$, find the equations of all possible horizontal tangents to the curve. (Ndorimana, 2016)

- **Other application of derivative**

- ✓ **Kinematic meaning of derivative**

Motion of a body on a straight line Consider a body moving along the x-axis such that its displacement, x meters on the right of the origin 0 after a time t seconds ($t \geq 0$), is given by $x = f(x)$. The average velocity of the body in the time interval $[t, t+h]$ is given by

$$\vec{v} = \frac{f(t+h) - f(t)}{h} \quad (h \neq 0) \text{ or } v = \text{total displacement over total time taken.}$$

In order to find the instantaneous velocity of the body at time t seconds, we find the average velocity in the time interval $[t, t+h]$ and let h take smaller and smaller values.

In fact the instantaneous velocity of the body at time t seconds is defined to be

$$\vec{v} = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

This is the first derivative of the function $x = f(t)$ and so $v(t) = \frac{dx}{dt} = f'(t)$. Then

differentiating with respect to time the 'dot' notation is often used.

Thus, if $x = f(t)$, then $v(t) = \frac{dx}{dt} = f'(x)$ We define the velocity to be the rate of change of displacement. Acceleration is defined as the rate of change of velocity. Thus, the acceleration of the body moving along a straight line with displacement $x(t)$ at time t and

is given by:

$$a(t) = \frac{d}{dt}(v(t)) = v'(t) = \frac{d^2}{dt^2}(x(t)) = x''(t)$$

Velocity is a vector quantity and so the direction is critical. If the body is moving towards the right (the positive direction of the x -axis), its velocity is positive and if it is moving towards the left, its velocity is negative. Therefore, the body changes motion when velocity changes sign. A sign diagram of the velocity provides a deal with information regarding the motion of the body.

Geometric interpretation of derivatives

If the function $y = f(x)$ is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it

is the rate of change of y with respect to x . Since $f''(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called convexity (bending) of a curve.

If $x = t$ is time and if $y = s(t)$ is displacement function of moving object, then $s'(t) = \frac{ds}{dt}$ is

the velocity function.

The derivative of velocity i.e. the second derivative of the displacement function is $s''(t)$

or $\frac{d^2y}{dx^2}$; it is the rate of change of the velocity function, which is, the acceleration

function.

Examples:

a. use the position function a. $s(t) = 5t^3 - t^2 + 10t, t = 1s$

b. $s(t) = -2 \cos t - \sin(t), t = \frac{\pi}{2}$. find velocity and acceleration

Correction: a. $s(t) = 5t^3 - t^2 + 10t, t = 1s$

$$v(t) = s'(t), v(t) = 15t^2 - 2t + 10 \Rightarrow v(1) = 15(1)^2 - 2(1) + 10 \Rightarrow v(1) = 25 - 2 = 23m/s$$

$$A(t) = v'(t) \quad a(t) = \frac{d}{dt}(15t^2 - 2t + 10) \Rightarrow a(t) = 30t - 2 \Rightarrow a(1) = 30(1) - 2 \Rightarrow a(1) = 28m/s^2$$

b. $s(t) = -2 \cos t - \sin(t), t = \frac{\pi}{2}$

$$v(t) = s'(t); v(t) = 2 \sin t - \cos t \Rightarrow v\left(\frac{\pi}{2}\right) = 2 \sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \Rightarrow v = 2m/s$$

$$A(t) = v'(t) \quad a(t) = \frac{d}{dt}(2 \sin t - \cos t) \Rightarrow a(t) = 2 \cos t + \sin t \Rightarrow a\left(\frac{\pi}{2}\right) = 2 \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)$$

$$a\left(\frac{\pi}{2}\right) = 1m/s^2$$

Exercises:

- The distance of 's' meters at time 't' seconds travelled by a particle is given by $s = t^3 - 9t^2 + 24t - 18$. Find the velocity when acceleration is zero and also find the acceleration when the velocity is zero.
- A body moves along the x-axis so that at time t seconds $x(t) = t^3 + 3t^2 - 9t$. Find:
 - where and when the body comes to rest
 - the maximum speed of the body in the first 1 second of motion
 - the maximum velocity of the body in the first 1 second of motion
 - the total distance travelled by the body in the first 2 seconds of motion.

✓ **Limit by using Hospital's rule**

If numerical functions $f(x)$ and $g(x)$ are such that $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ (I.F),

then to remove the indetermination, we proceed as follows, through Hospital's rule:

- Differentiate separately, the numerator and the denominator, to get $f'(x)$ and $g'(x)$;

$$2. \text{ Calculate } \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}$$

$$3. \text{ Then } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x_0)}{g'(x_0)}$$

Note that: - the process can be repeated if necessary;

Hospital's rule is used only if we have indetermination $\frac{0}{0}, \frac{\infty}{\infty} (I.F)$.

Examples: Indeterminate form with the form $\frac{0}{0}$

$$1. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0}, \text{ this is the indeterminate form (I.F)}$$

By using hospital rule, we move out this I.F

$$\lim_{x \rightarrow 2} \frac{(x^2 - 4)'}{(x - 2)'} = \lim_{x \rightarrow 2} \frac{2x}{1} = 2(2) = 4$$

$$2. \lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} = \frac{2 - 2}{\sqrt{2} - \sqrt{2}} = \frac{0}{0} (IF)$$

We eliminate the indeterminate form by hospital rule:

$$\lim_{x \rightarrow 2} \frac{(x - 2)'}{(\sqrt{x} - \sqrt{2})'} = \lim_{x \rightarrow 2} \frac{1}{\frac{x'}{2\sqrt{x}} - 0} = \lim_{x \rightarrow 2} \frac{1}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow 2} 2\sqrt{x} = 2\sqrt{2}$$

$$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \frac{0}{0} (I.F) \Rightarrow \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$3. \lim_{x \rightarrow 1} \frac{(x^{\frac{1}{3}} - 1)'}{(x - 1)'} = \frac{\frac{1}{3}x^{\frac{1}{3}-1} - 0}{1} = \lim_{x \rightarrow 1} \frac{1}{3x^{\frac{2}{3}}} = \lim_{x \rightarrow 1} \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{1}} = \frac{1}{3}$$

Exercises: by using hospital rule find limit of the following: a) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ b) $\lim_{x \rightarrow 2} \frac{x - 3}{\sqrt{x} - \sqrt{3}}$ c)

$$\lim_{x \rightarrow 5} \frac{\sqrt{x} - \sqrt{5}}{2x - 10} \quad \text{d) } \lim_{x \rightarrow -1} \frac{x^3 - 2x - 1}{3x^2 - 2x - 5} \quad \text{e) } \lim_{x \rightarrow 2} \frac{x^4 - 4x^3 + 4x^2}{x^3 - 3x^2 + 4} \quad \text{f) } \lim_{x \rightarrow 0} \frac{\sqrt[5]{(1+x)^3} - 1}{x} \quad \text{g)}$$

$$\lim_{x \rightarrow 4} \frac{\sqrt{x-3} - 1}{x - 4} \quad \text{h) } \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt[3]{x-2}} \quad \text{i) } \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \quad \text{j) } \lim_{x \rightarrow 2} \frac{x^4 - 16}{x^2 - 4} \quad (\text{Ndorimana, 2016})$$

- **Stationary point/extrema points/turning points or Maximum and minimum points of a function**

This is a point on the graph $y = f(x)$ at which f is differentiable and $f'(x) = 0$. The term is also used for the number c such that $f'(c) = 0$. The corresponding value $f(c)$ is a stationary value. A stationary point c can be classified as one of the following, depending on the behavior of f in the neighborhood of c :

- A local maximum, if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c ,
- A local minimum, if $f'(x) < 0$ to the left of c and $f'(x) > 0$ to the right of c ,
- Neither local maximum nor minimum
- If we have two extrema points the maximum point is given by the small value of x and minimum is given by large value of x .

Note: Maximum and minimum values are termed as extreme values.

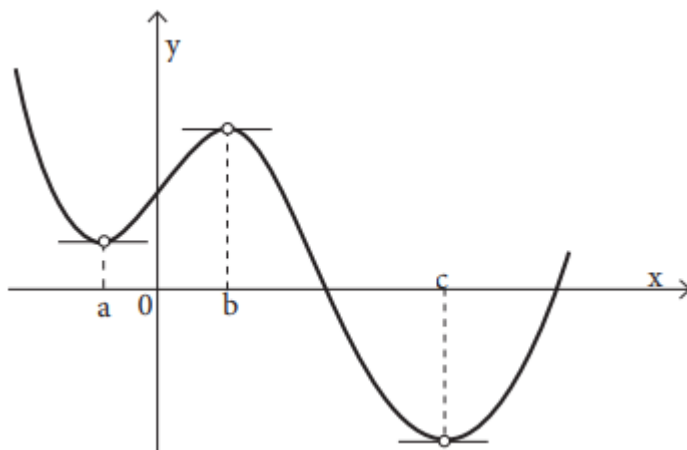


Figure 16: Extrema points

The points a , b and c are stationary points.

✓ **Steps of finding stationary points**

1. Find derivative of the given function
2. Equalize the given function with zero and solve for x
3. Replace the value of x in the given equation in order to get the value of y .

Examples: Find the stationary point of the function defined by $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$.

Solution: step 1. $\frac{dy}{dx} = \frac{3x^2}{3} + \frac{2x}{2} - 2 \Rightarrow \frac{dy}{dx} = x^2 + x - 2$

Step 2. $\frac{dy}{dx} = 0 \Rightarrow x^2 + x - 2 = 0$

$$\begin{array}{lcl} (x+2)(x-1) = 0 & & \\ x^2 - x + 2x - 2 & x+2 = 0 & \Rightarrow x_1 = -2 \\ x(x-1) + 2(x-1) \Rightarrow x-1 = 0 & & x_2 = 1 \end{array}$$

Step 3. $y_1 = f(x_1) = f(-2)$
 $y_1 = \frac{(-2)^3}{3} + \frac{(-2)^2}{2} - 2(-2) \Rightarrow y_1 = \frac{-8}{3} + \frac{4}{2} + 4 = 3.33$

$$y_2 = f(x_2) = f(1)$$

$$y_2 = \frac{1^3}{3} + \frac{1^2}{2} - 2(1) = -1.17$$

The stationary points are $(-2, 3.33)$ and $(1, -1.17)$ where $(-2, 3.33)$ is maximum and $(1, -1.17)$ is minimum.

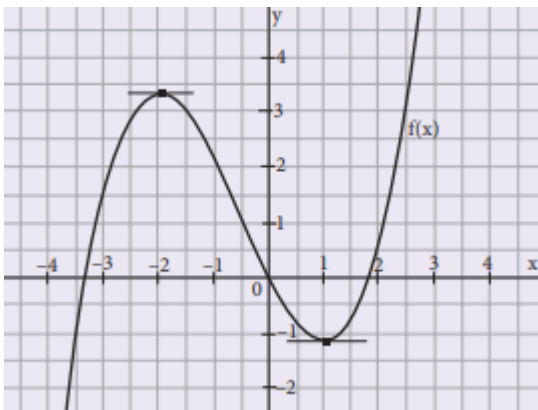


Figure 17: Minimum and maximum points

Exercises: Find the stationary points and state the nature of turning points of the function defined by

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$

- **Increasing and decreasing functions**

A real function f is increasing in or on an interval I if $f(x_1) \leq f(x_2)$ whenever x_1 and x_2 are in I with

$x_1 < x_2$. Also, f is strictly increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. A real function f is decreasing in or on an interval I if $f(x_1) \geq f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is strictly decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Below is increasing function

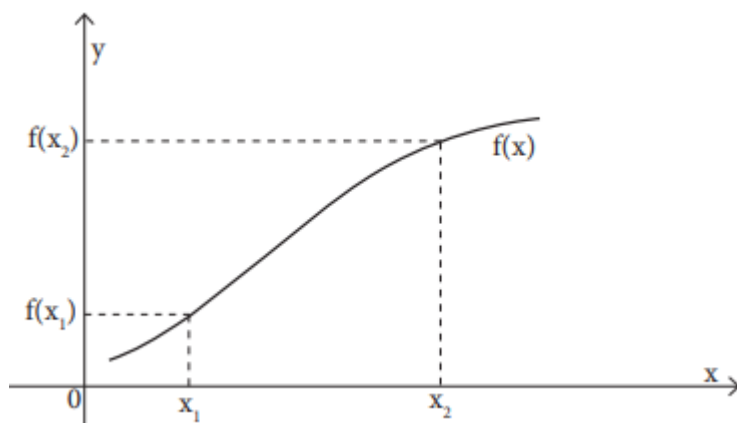


Figure 18: Increasing function

Below is decreasing function

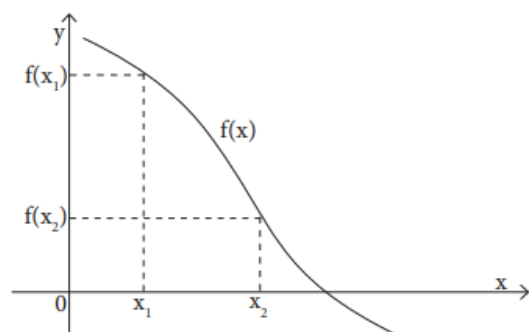


Figure 19: Decreasing function

✓ Meaning of the sign of the derivative

If we recall that the derivative of a function yields the slope of the tangent to the curve of the function. It appears that a function is increasing at a point where the derivative is positive and decreasing where the derivative is negative.

Examples: 1. Given the function $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$. Determine the interval where the graph of the function is increasing and where it is decreasing.

Solutions:

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$$

$$\frac{dy}{dx} = \frac{3x^2}{3} + \frac{2x}{2} - 2 \Rightarrow \frac{dy}{dx} = x^2 + x - 2$$

$$x^2 - x + 2x - 2$$

$$x(x-1) + 2(x-1) \Rightarrow (x+2)(x-1) = x^2 + x - 2$$

✓ **Table of signs**

X	$-\infty$	-2	1	$+\infty$
Factors				
X+2	-----	0	+++++	+++++
x-1	-----	-----	0	+++++
f/(x)=(x+2)(x-1)	+++++	0	-----	0+++++
F(x)	Increasing	3.33 decreasing	-1.17	increasing

Thus, f(x) is strictly increasing on $]-\infty, -2[\cup]1, +\infty[$

f(x) is strictly decreasing on $] -2, 1[$

2. Given the function $f(x) = 2x^3 + 9x^2 + 12x + 20$. Determine the interval where the graph of the function is increasing and where it is decreasing.

Solutions:

$$f(x) = 2x^3 + 9x^2 + 12x + 20$$

$$\frac{dy}{dx} = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2)$$

$$\frac{dy}{dx} = 0 \Rightarrow 6(x^2 + 3x + 2) = 0 \Rightarrow x^2 + 3x + 2 = 0$$

$$(x + 2)(x + 1) = 0$$

We have the following summary table of sign of $f'(x)$

x	$-\infty$	-2	-1	$+\infty$
Function				
$x + 2$	$-$	0	$+$	$+$
$x + 1$	$-$		0	$+$
$f'(x)$	$+$	0	0	$+$
$f(x)$				

Thus, $f(x)$ is strictly increasing on $]-\infty, -2[\cup]-1, +\infty[$

$f(x)$ is strictly decreasing on $] -2, -1[$

Exercises: Determine the interval where the graph of the following functions is increasing and where it is decreasing.

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$ (Ndorimana, 2016)

- **Concavity and inflection point on a graph**

- ✓ **Point of inflection**

A point of inflection is a point on a graph $y = f(x)$ at which the concavity changes. If f' is continuous at a , then for $y = f(x)$ to have a point of inflection at a . it is necessary that $f''(a) = 0$, and so this is the usual method of finding possible points of inflection.

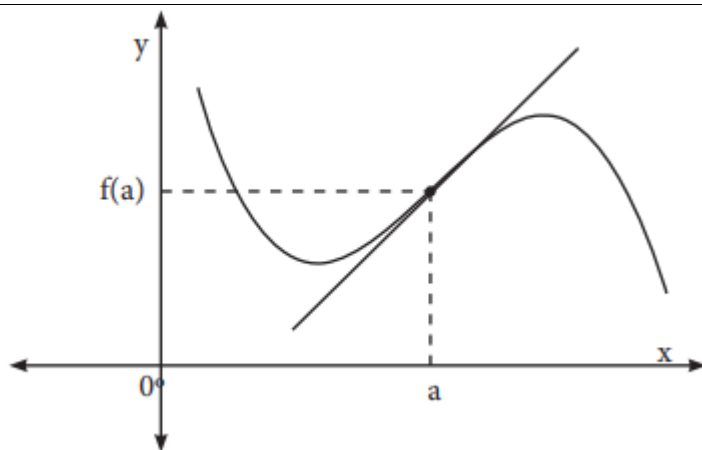


Figure 20: Inflection point

However, the condition $f''(a) = 0$ is not sufficient to ensure that there is a point of inflection at a ; it must be shown that $f''(x)$ is positive to one side of a and negative to other side. Thus, if $f(x) = x^4$ then

$f''(0) = 0$; but $y = x^4$ does not have a point of inflection at 0 since $f''(x)$ is positive to both sides of 0.

✓ **Steps of finding inflection point/concavity point**

1. Find the second derivative of the given function
2. Equalize second derivative with zero in order to get the value of x
3. Replace the value of x in the given function in order to get the value of y .

Example:

Find the inflection point of the function defined by $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$

Solution:

$$f'(x) = x^2 + x - 2$$

$$f''(x) = 2x + 1$$

$$f''(x) = 0 \Leftrightarrow 2x + 1 = 0$$

$$x = -\frac{1}{2}$$

by replacing the value of x in a given function you get $f\left(-\frac{1}{2}\right) = 1.08$

The inflection point to the curve is at $x = -\frac{1}{2}$ and is $(-0.5, 1.08)$.

Below is the graph illustrating the inflection point:

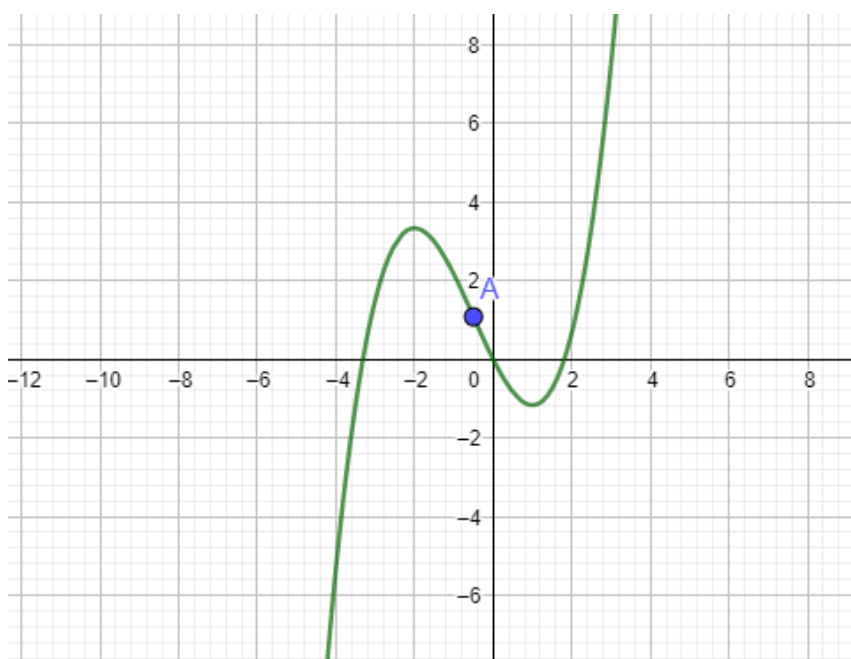


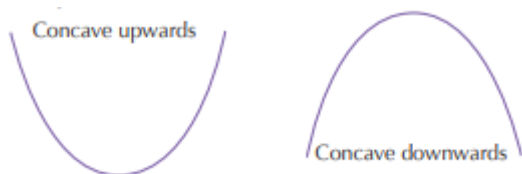
Figure 21: Concavity point

Exercises: Find the inflection point of the following functions

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$

- **Concavity**

At a point of graph $y = f(x)$, it may be possible to specify the concavity by describing the curve as either concave up or concave down at that point, as follows:



1. A curve is said to be concave downwards (or concave) in an interval $]a, b[$ if $f''(x) < 0$ for all $x \in]a, b[$.
2. A curve is said to be concave upwards (or convex) in an interval $]a, b[$ if $f''(x) > 0$ for all $x \in]a, b[$.

Example:

Determine the interval where the graph of $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$ has concave upward and concave downward.

Solution: $f'(x) = x^2 + x - 2$

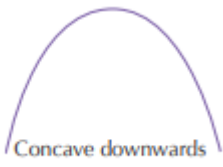
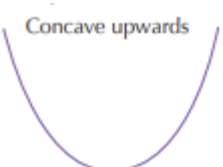
$f''(x) = 2x + 1$

$f''(x) = 0 \Leftrightarrow 2x + 1 = 0$

$x = -\frac{1}{2}$

by replacing the value of x in a given function you get $f\left(-\frac{1}{2}\right) = 1.08$

✓ **Table of sign:**

X Factors	$-\infty$	-0.5	$+\infty$
2x+1	-----	0	+++++
F(x)		1.08	

Therefore, concave downward = $]-\infty, -0.5[$

Concave upward = $] -0.5, +\infty[$

Exercises: Determine the interval where the graph of the following functions has concave upward and concave downward.

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$ (Ndorimana, 2016)



Activity 2: Guided Practice



Task 28:

1. find the extrema points
2. State the values of x for which f are increasing
3. Find the x -coordinate of each inflection point of $f(x)$.
4. State the values of x for which the curve of $f(x)$ is concave upwards and downwards.
5. Find the x -coordinate of each point of inflection



Activity 3: Application



Task 29:

1. The function $f(x) = 4x - 3x^2$ models the trajectory of a ball thrown into the air, where x represents the horizontal distance in meters and $f(x)$ represents the height in meters. Find the equation of the tangent and normal to the trajectory at the point $(2, -4)$ to analyze the ball's path at that specific point.
2. The function $y = x^2 + 4x + 1$ represents the profit of a company, where x represents the number of products sold in thousands and y represents the profit in thousands of dollars. Find the equation of the tangent and normal to the profit function at $x = 3$ to understand the rate of change of profit when 3,000 products are sold.
3. The function $y = \frac{t^3}{1+t}$ models the concentration of a drug in the bloodstream, where t represents the time in hours and y represents the concentration in milligrams per liter. Find the equation of the tangent and normal to the concentration function at $t = 1$ to analyze how the drug concentration changes after 1 hour.

Topic 2.6: Sketching curve of algebraic function



Activity 1: Problem Solving



Task 30:

Consider the cubic function $f(x) = x^3 - 3x^2 - 9x + 27$.

- Calculate its variation table
- Find its axes of symmetry
- Calculate the additional points
- Construct the graph of given function.

Key Facts 2.6: Sketching curve of algebraic function

- Establishing required parameters

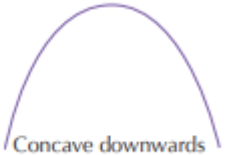
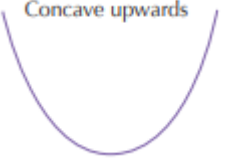
- ✓ Variation table

Variation table is a combination of increasing and decreasing table with concavity table.

For example: find the Variation table of $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$.

Variation Table

X	$-\infty$	-2	-0.5	1	$+\infty$
Factors					
$X+2$	-----	-	0	+++++	+++++
$x-1$	-----	-----	-	0	+++++
$\frac{dy}{dx} = (x+2)(x-1)$	+++++	+++++	0	-----	0+++++
$\frac{d^2y}{dx^2} = 2x+1$	-----	-----	0	+++++	+++++

$F(x)$	 Concave downwards  Concave upwards 1.08				
	Increasing	3.3	decreasing	-1.17	increasing

Exercises: find the variation table of the following functions

downward.

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$

- Additional points**

The points of the given functions are extrema points, inflection point and you have to add the new points in order to draw the curve.

For examples: The additional point of the curve $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$ are

X	-2	1	-0.5	1.96	-3.96
Y	3.33	-1.17	1.08	0.52	-4.94

- Sketching graph of polynomial function**

To sketch the graph of polynomial function we have to join extrema points, inflection points and the additional points on the same graph.

Examples: Join extrema points, inflection points and additional points of

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$$

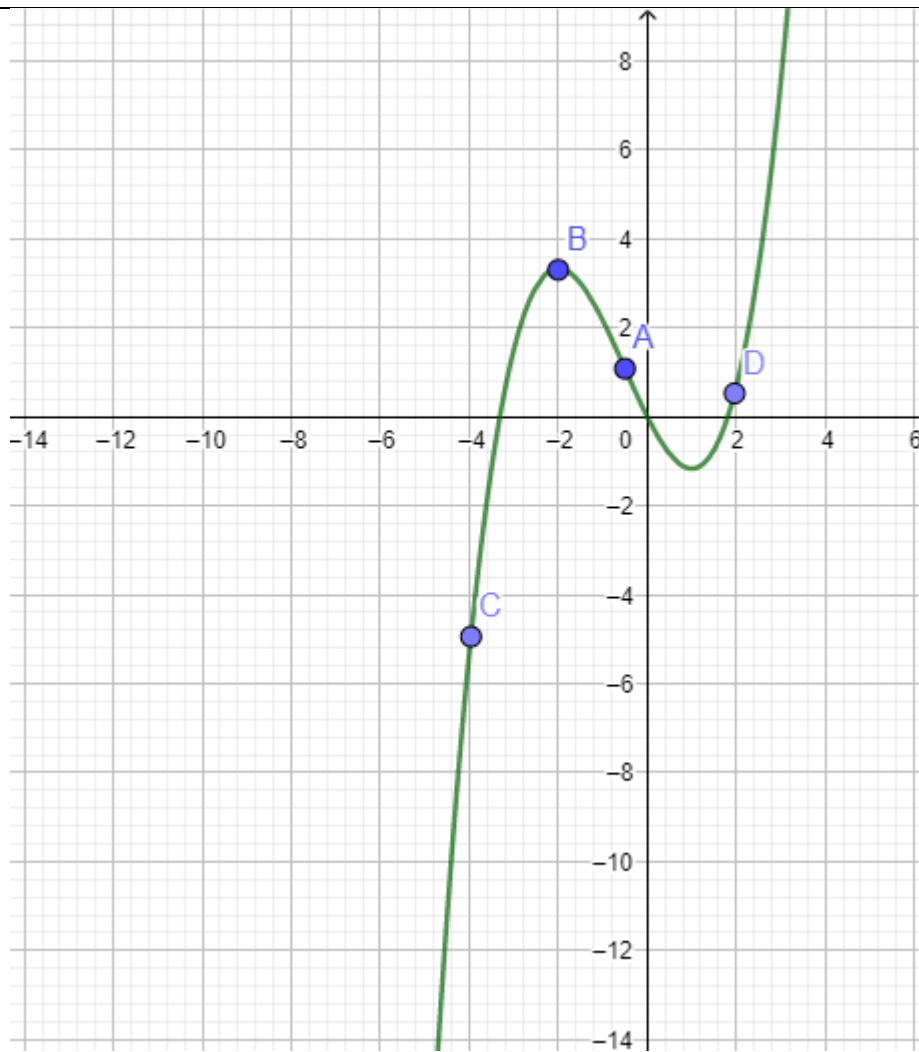


Figure 22: Graph of polynomial with third degree.

Exercises

1: construct the extrema points, inflection points and other additional points on the graph of the following functions:

a. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$. b. $f(x) = -\frac{x^3}{3} - \frac{x^2}{2} + 6x - 2$

2. Given the function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

- State the values of x for which f is increasing
- Find the x -coordinate of each extreme point of $f(x)$.
- State the values of x for which the curve of $f(x)$ is concave upwards.

- d. Find the x-coordinate of each point of inflection.
- e. Sketch the general shape of the graph of $f(x)$ indicating the extrema points and points of inflection.
- f. The motion of the particle performing damped vibration is given by

$$f(t) = \frac{t^3}{3} - \frac{t^2}{2} - 2t + 2, \text{ y is being the displacement from its mean position at time (t).}$$

Show that y is maximum at $t = -1$ s.

3. Given $f(x) = \frac{x^4}{12} + \frac{x^3}{6} - x^2 + 2x - 1$, find

- a. Find the inflection points
- b. Showing where the function has concave up and concave down.

- **Sketching graph of rational function**

To sketch the graph of rational functions you have to follow the following steps:

1. To find the domain of definition of the given functions.
2. To find the limit at the boundaries from domain of functions
3. To find the asymptotes to the curve from the finding limits.
4. To find the stationary points.
5. To show where the function is increasing or is decreasing.
6. To find the inflection points.
7. To show where the concave is upward or downward.
8. To calculate the variation table.
9. To calculate the additional points
10. To construct the graph of rational functions by adding asymptotes.

Example: Given the function f of real variable x defined by $f(x) = \frac{x^2 - 1}{x^2 - 4}$

1. What is the domain of definition of $f(x)$?
2. State the asymptotes to the curve.
3. state the nature of turning point
4. show where the function is increasing or decreasing

5. find the inflection point
 6. Show where the concave is upward or downward.
 7. find the variation table
 8. Find the coordinates of the point at which the curve cuts the coordinates axes; and the additional points.
- ✓ Sketch the graph of the curve in Cartesian plan

solution:

a. $df = \mathbb{R} \setminus \{-2, 2\}$

b. $\lim_{x \rightarrow \infty} f(x) = 1, \lim_{x \rightarrow \pm 2} f(x) = \infty,$
 $HA \equiv Y = 1$
 $VA \equiv x = \pm 2$

c. $f'(x) = \frac{-6x}{(x^2 - 4)^2} \quad f'(x) = 0, \frac{-6x}{(x^2 - 4)^2} = 0 \quad -6x = 0, x = 0$

$f(0) = \frac{1}{4}$, turning point is $(0, \frac{1}{4})$ is maximum point

d. increasing and decreasing table

X	$-\infty$	-2	0	2	$+\infty$
Factors					
$f'(x) = \frac{-6x}{(x^2 - 4)^2}$	++++++		+++ +++++0 -----		----- --
F(x)	Increases		Increases $\frac{1}{4}$ Decreasing		Decreasing

Function increases = $] -\infty, -2[\cup] -2, 0[$ and decreases = $] 0, 2[\cup] 2, +\infty[$

e. $f''(x) =$

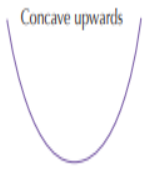
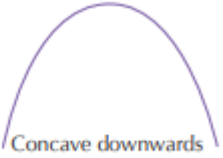
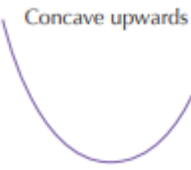
$$\left(\frac{-6x}{(x^2-4)^2} \right)' = \frac{(-6x)'(x^2-4)^2 - ((x^2-4)^2)'(-6x)}{(x^2-4)^4} = \frac{-6(x^2-4)^2 - 2(x^2-4)(2x)(-6x)}{(x^2-4)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2-4)[(-6)(x^2-4) - 2(2x)(-6x)]}{(x^2-4)^4} = \frac{-6x^2 + 24 + 24x^2}{(x^2-4)^3} = \frac{18x^2 + 24}{(x^2-4)^3}$$

$$f''(x) = 0 \Rightarrow \frac{18x^2 + 24}{(x^2-4)^3} = 0 \Rightarrow 18x^2 + 24 = 0 \Rightarrow 18x^2 = -24 \Rightarrow x^2 = \frac{-24}{18} \Rightarrow x = \sqrt{-\frac{24}{18}}$$

which means that there is no real solution, therefore our function has not inflection points

f. Concave upward or downward

X	$-\infty$	-2	2	$+\infty$
Factors				
$\frac{d^2y}{dx^2} = \frac{18x^2 + 24}{(x^2-4)^3}$	+++++	-----	-----	+++++
F(x)	Concave upwards 	Concave downwards 	Concave downwards	Concave upwards 

Function is concave upward = $]-\infty, -2[\cup]2, +\infty[$ and is concave downward = $]-2, 2[$

g. Variation table

X	$-\infty$	-2	0	2	$+\infty$
Factors					
$f'(x) = \frac{-6x}{(x^2-4)^2}$	+++++	+++	0	---	---

$\frac{d^2y}{dx^2} = \frac{18x^2 + 24}{(x^2 - 4)^3}$	+++++	-----	+++++
F(x)	Increases	Increase $\frac{1}{4}$ Decrease	Decreasing
	Concave upwards	Concave downwards	Concave upwards

h. axes of symmetry

we say y-intercept if x=0

$$f(x) = \frac{x^2 - 1}{x^2 - 4} \quad f(0) = \frac{0^2 - 1}{0^2 - 4} = \frac{1}{4} \Rightarrow p\left(0, \frac{1}{4}\right)$$

We say x-intercept if y=0

$$0 = \frac{x^2 - 1}{x^2 - 4} \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \sqrt{1} \Rightarrow x = \pm 1$$

We have $p(1,0)$ and $p(-1,0)$

Additional points

X	-1	1	0	-5.32	-2.22	-2.67	2.19	2.64	4.58	-1.86	1.83
Y	0	0	$\frac{1}{4}$	1.27	4.27	1.96	4.72	2.01	1.18	-4.46	-3.67

Here is the graph of $f(x) = \frac{x^2 - 1}{x^2 - 4}$

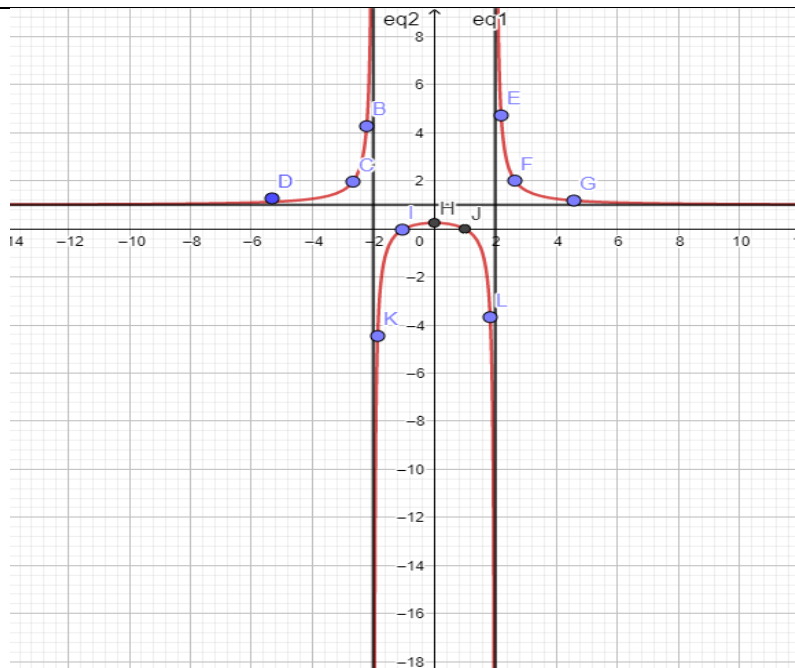


Figure 23: Graph of rational function

Exercises:

1. The function f of real variable is defined by: $f(x) = \frac{2x^2 + 4x + 5}{x^2 + 1}$.

- What is its definition set of domain of definition?
- Calculate the limits of the function f bounded by the definition set and deduce the equation of the asymptotes to the curve represented by the function f .
- Determine the sign of the function with the direction of variation and construct the table of variations.
- Find the coordinates of the points at which the curve cuts the coordinates axes.
- Find the coordinates of the point at which the line $y=2$ cuts the curve.
- Plot the coordinates of the curve in a Cartesian plane.

2. Let $f : x \rightarrow f(x) = \frac{x+1}{x-5}$.

- Find the domain of definition.
- Is $f(x)$ odd or even function?
- Find asymptotes to the curve.
- Find the stationary points and variation table.
- Sketch the graph on Cartesian plane. (Ndorimana, 2016)



Activity 2: Guided Practice

To sketch the graph of rational functions you have to follow the following steps:

1. To find the domain of definition of the given functions.
2. To find the limit at the boundaries from domain of functions
3. To find the asymptotes to the curve from the finding limits.
4. To find the stationary points.
5. To show where the function is increasing or is decreasing.
6. To find the inflection points.
7. To show where the concave is upward or downward.
8. To calculate the variation table.
9. To calculate the additional points
10. To construct the graph of rational functions by adding asymptotes.



Task 31:

By using the steps above study and sketch $f(x) = \frac{x^2 - 2x - 8}{x - 1}$



Activity 3: Application



Task 32:

1. The function $f : x \rightarrow f(x) = \frac{x+1}{x-5}$ models the rate of infection spread in a population, where x represents the number of days since the outbreak and $f(x)$ represents the infection rate. Study and sketch this function to understand how the infection rate changes over time and identify any critical points or asymptotes.

2. The function $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 6$ represents the profit of a manufacturing company, where x represents the number of units produced in thousands and $f(x)$ represents the profit in thousands of dollars. Study and sketch this function to analyze the profit trends and identify the production levels that maximize or minimize profit.
3. The function $f(x) = \frac{2x^2 + 4x + 5}{x^2 + 1}$ models the absolute error in a scientific measurement, where x represents the time in seconds and $f(x)$ represents the error magnitude. Study and sketch this function to understand how the measurement error varies over time and identify any periods of stability or instability.



Formative Assessment

- Find the derivative of the following functions
 - $f(x) = (x^2 + 6)(x - 2)$
 - $(x^2 + 2)^{\frac{1}{2}}$
 - $f(x) = (3x^3 + 4x + 1)\sin x$
 - $9x^3 + 7x^2 + 20$
 - $f(x) = \frac{x^2 + 2x + 5}{\sqrt{4x + 1}}$
- Find the domain and range of the following functions:
 - $f(x) = \frac{2x + 3}{x - 1}$
 - $f(x) = \frac{2x + 1}{3x - 4}$
 - $f(x) = 2x^2 - 8$
 - $g(x) = 2x + 4$
 - $f(x) = \frac{5}{x}$
 - $g(x) = x^2 + 4x - 5$
- Determine the following limit:
 - $\lim_{x \rightarrow 5} \frac{\sqrt{x} - \sqrt{5}}{2x - 10}$
 - $\lim_{x \rightarrow -1} \frac{x^3 - 2x - 1}{3x^2 - 2x - 5}$
 - $\lim_{x \rightarrow 2} \frac{x^4 - 4x^3 + 4x^2}{x^3 - 3x^2 + 4}$
 - $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$
 - $\lim_{x \rightarrow 0} \frac{\sqrt[5]{(1+x)^3} - 1}{x}$
 - $\lim_{x \rightarrow 4} \frac{\sqrt{x-3} - 1}{x-4}$
 - $\lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt[3]{x-2}}$
 - $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x^2 - 4}$
 - $\lim_{x \rightarrow 2} \frac{x+1}{x+4}$
- Calculate and sketch the asymptotes of the following functions
 - $f(x) = \frac{x+4}{x-2}$
 - $f(x) = \frac{x^2+1}{x^2-1}$
 - $y = \frac{x^2-x+4}{x-1}$

5.

- a. Given a function $f: \mathbb{R} \rightarrow \mathbb{R}: x \rightarrow y = \frac{2x^2 + 2}{ax^2 + bx + c}$. find a, b and c such that the point $(0, -1)$ belongs to the curve and lines with the equation $x + \frac{2}{3} = 0$ and $x = 1$ are asymptotes to the curve.
- b. Lines $x = 2$ and $y = 3$ are asymptotes of function $f(x) = \frac{ax + 5}{bx + 4}$ find a and b.
- c. given that the equation $y = -3x + 6$ and $x + 2 = 0$ are asymptotes to the curve defined in real number by $F(x) = \frac{ax^2 + 3}{x - b}$. Determine the value of real number a and b.

6.

- a. Find the oblique asymptotes of the curve $y = -1 + \frac{2}{3}\sqrt{x^2 - 4x - 5}$
 - b. Find the equations of asymptotes of the following function: $f(x) = \sqrt{x^2 - 1}$
7. The motion of the particle performing damped vibration is given by $f(t) = \frac{t^3}{3} - \frac{t^2}{2} - 2t + 2$, y is being the displacement from its mean position at time (t) . Show that y is maximum at $t = -1$.

8.

- a. use the position function a. $s(t) = 5t^3 - t^2 + 10t, t = 1s$
- b. $s(t) = 2 \cos t + \sin(t), t = \frac{\pi}{2}$. Find velocity and acceleration

9. The distance of 's' meters at time 't' seconds travelled by a particle is given by $s = t^3 - 9t^2 + 24t - 18$. Find the velocity when acceleration is zero and also find the acceleration when the velocity is zero.

10. If the tangent to $y = 2x^3 + mx^2 - 3$ at the point where $x = 2$ has slope 4, find the value of m

11. Find the equation of the tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

12. Given the function $f(x) = \frac{x^2 - 2x - 8}{x - 1}$

- a. Find the domain of definition
- b. Study the parity of $f(x)$
- c. Find asymptotes to the curve
- d. Compute the first derivative and study its signs.
- e. Compute the second derivative and study its signs.
- f. Sketch the graph



Points to Remember

- Domain and range of function.
- Parity of function (even or odd)
- Limit of function
- Asymptotes to the curve
- Derivative by using definition
- Derivative by properties
- Application of differentiation such as tangent equation, normal equation, kinematic meaning of derivative
- Extrema points
- Increasing and decreasing function
- Inflection points
- Concavities (concave upwards or concave downwards)
- Variation table
- Intercepts
- Additional points
- Graph of function.



Self-Reflection

1. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify a function as a rule and recognize rules that are not functions.					
Determine the domain and range of a function					
Construct composition of functions					
Find whether a function is even, odd, or neither					
Find whether a function is even, odd, or neither					
Demonstrate an understanding of operations on polynomials, rational and irrational functions, and find the composite of two functions.					
Evaluate the limit of a function and extend this concept to determine the asymptotes of the given function.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Evaluate derivatives of functions using the definition of derivative					
Define and evaluate from first principles the gradient at a point					
Distinguish between techniques of differentiation to use in an appropriate context					
Apply the formula for the slope of the tangent line using the derivative at the given point					
Recall the relationship between the tangent and normal lines (normal line is perpendicular to the tangent).					
Interpret the derivative as the rate of change of position (velocity) in kinematics					
Explain the application of L'Hopital's Rule to resolve indeterminate forms in limits					
Define maximum and minimum points using					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
the first and second derivative tests.					
Recall how the sign of the first derivative indicates increasing or decreasing behaviour					
Explain the definition of an inflection point as where concavity changes.					
Identify concave up and concave down intervals based on the sign of the second derivative					
Contrast the purpose of a variation table to summarize critical points and intervals of increase/decrease					
Identify additional key points (e.g., intercepts, turning points) that help in sketching the graph					
Recall the behaviour of polynomial functions based on degree and leading coefficient.					
Perform operations on functions.					
Apply different properties of functions					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
to model and solve related problems in various practical contexts					
Analyse, model and solve problems involving linear or quadratic functions and interpret the results					
Calculate limits of certain elementary functions					
Develop introductory calculus reasoning					
Apply informal methods to explore the concept of a limit including one sided limits.					
Use the concepts of limits to determine the asymptotes to the rational and polynomial functions.					
Use first principles to determine the gradient of the tangent line to a curve at a point					
Apply the concepts and techniques of differentiation to model, analyse and					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
solve rates or optimization problems in different situations					
Identify various differentiation techniques, such as product differentiation, market differentiation, and pricing differentiation					
Compute the slope and equation of the tangent line at a specific point on the curve.					
Determine the equation of the normal line by using the negative reciprocal of the tangent slope.					
Use the derivative to calculate velocity and acceleration from a position-time function					
Apply L'Hopital's Rule to evaluate limits involving $0/0$ or ∞/∞ forms					
Identify critical points and use the first or second derivative to determine maxima or minima.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Identify critical points and use the first or second derivative to determine maxima or minima.					
Determine the inflection point by finding where the second derivative changes sign.					
Classify intervals as concave up or concave down by analysing the second derivative.					
Construct a variation table to organize information about a function's behaviour					
Plot additional points to refine the graph and ensure accuracy in function representation.					
Sketch the graph of a polynomial by identifying key features like roots, turning points, and end behaviour.					
Sketch the graph of a rational function by determining vertical and horizontal					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
asymptotes and analysing intercepts					
Be consistent and persistent when performing operations on functions.					
Show concern on patience, mutual respect and tolerance when applying properties of functions.					
Show concern on patience, mutual respect and tolerance when solving problems about polynomial, rational and irrational functions					
Show concern on the importance, the use and determination of limit of functions					
Appreciate the use of parity of functions.					
Appreciate the use of gradient as a measure of rate of change (economics)					
Be consistent and persistent when using the concepts of limits					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
to determine the asymptotes to the rational and polynomial functions					
Think logically when determining the gradient of the tangent line to a curve at a point					
Be consistent and persistent when applying techniques of differentiation to model, analyse and solve rates or optimization problems in different situations					
Show interest in derivatives to help in the understanding optimization problem					
Think logically when calculating the tangent line equation.					

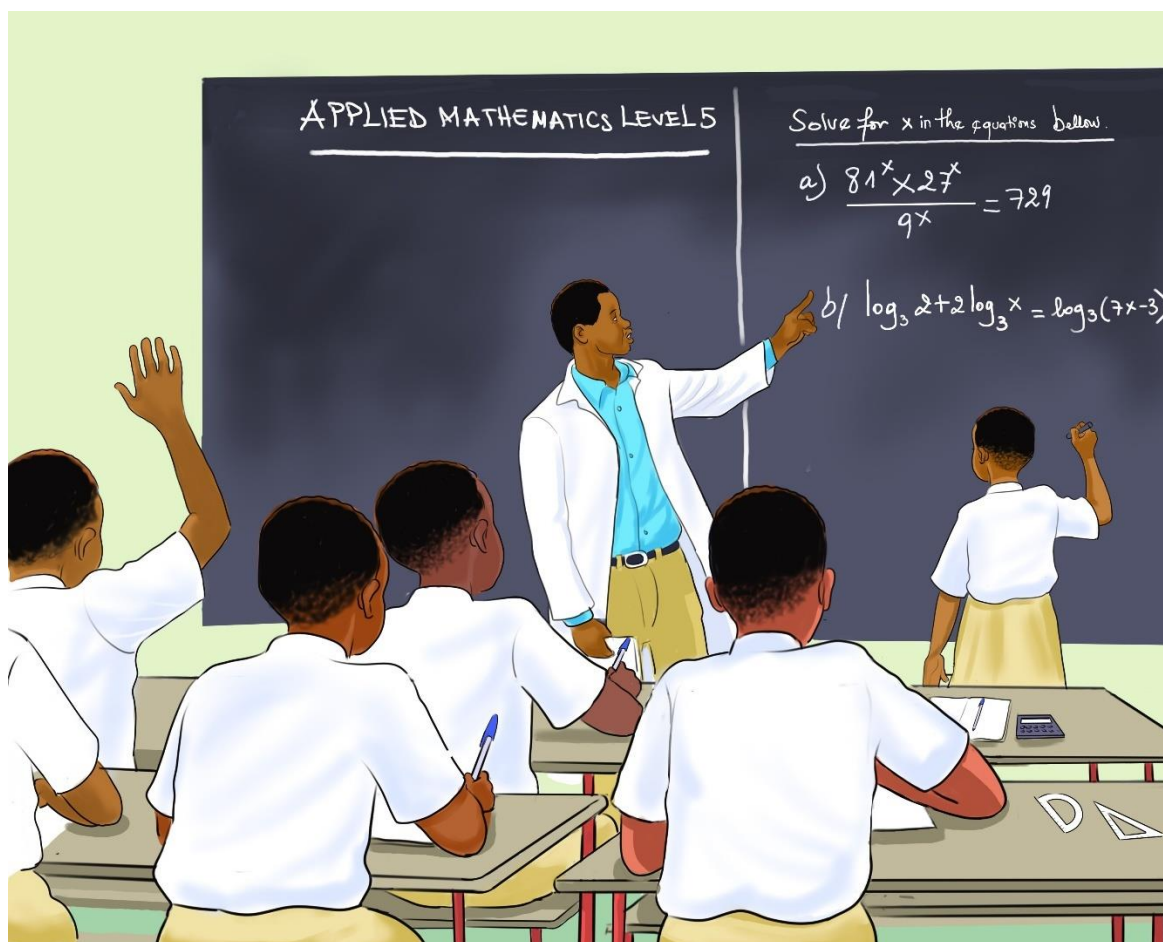
My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Think logically on the consideration of perpendicular slopes when working with normal lines.					
Foster curiosity about how calculus connects to real-life motion and change.					
Be consistent and persistent when faced with challenging limit problems and the use of appropriate rules					
Think logically when determining extreme values of a function.					
Show interest in systematic analysis to accurately determine function behaviour over intervals.					
Think logically when determining the inflection point by finding where the					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
second derivative changes sign					
Instil precision and clarity when analysing the concavity of functions.					
Think logically when using variation tables.					
Be consistent and persistent when selecting and plotting critical points					
Think logically and step-by-step analysis when graphing polynomial functions.					
Think logically when analysing and sketching rational functions.					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	1.	1.
2.	2.	2.
3.	3.	3.

UNIT 3: EXPONENTIAL AND LOGARITHMIC EQUATIONS



Unit summary

This unit provides you with the knowledge, skills and attitudes required to Apply exponential and logarithmic expressions required to Apply Basic Mathematical Analysis, Statistics and Probability. It covers the Properties for exponential expressions, Properties for logarithmic expression, Exponential equations, Logarithmic equations and Logarithms and exponential applications.

Self-Assessment: Unit 3

- Look at the unit illustration in the Manuals and together discuss:
 - What is an exponential function?
 - What is logarithmic function?
 - What is relationship between logarithmic expression and exponential expression?
 - How to solve logarithmic equations and exponential equations?
 - How to apply logarithm and exponential in real life situation?
- Fill out the below self-assessment. Think about yourself: do you think you can do this? How well? Read the statements across the top. Assess your level of knowledge, skills and attitudes under this unit.
 - There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills and attitudes acquired during the learning process
 - Think about yourself: do you think you have the knowledge, skills or attitudes to do the task? How well?
 - Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.
- At the end of this unit, you will assess yourself again.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Define the exponential expressions.					
Enumerate the Laws of exponents (e.g., product rule, quotient rule, power rule).					
Explain The concept of zero and negative exponents.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe The relationship between exponential and radical expressions					
Define logarithms as the inverse of exponents.					
List Logarithmic properties (product, quotient, power rules).					
Differentiate Common and natural logarithms (base 10 and base e).					
Describe The relationship between exponential and logarithmic functions.					
Explain The general form of exponential equations					
Define the concept of exponential growth and decay.					
Identify the Methods for solving exponential equations (e.g., equating bases, logarithmic conversion).					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Explain the general form of logarithmic equations.					
Enumerate the methods for solving logarithmic equations (e.g., converting to exponential form).					
Enumerate the restrictions on the domain of logarithmic equations					
Explain the real-world contexts involving exponential growth and decay.					
Describe the applications of logarithms in solving real-world problems (e.g., sound intensity, earthquake magnitude).					
Explain the significance of the natural base e and its applications					
Simplify exponential expressions using exponent rules.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Solve problems involving the multiplication and division of exponential terms.					
Translate between exponential and radical forms					
Evaluate expressions with fractional or negative exponents					
Simplify logarithmic expressions using properties.					
Convert between exponential and logarithmic forms					
Expand and condense logarithmic expressions					
Evaluate logarithms using known values or approximations					
Solve exponential equations algebraically by isolating the variable.					
Apply logarithms to solve equations with non-matching bases					
Use exponential equations to model					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
and solve real-world problems (e.g., population growth)					
Solve logarithmic equations algebraically, ensuring valid solutions.					
Apply logarithmic properties to simplify and solve equations					
Interpret solutions in applied contexts, such as finance or science.					
Model real-world scenarios using exponential and logarithmic equations.					
Interpret parameters in exponential and logarithmic models					
Solve application problems involving growth and decay					
Think logically when simplifying exponential expressions using exponent rules.					
Value the utility of exponent rules in simplifying expressions					

My experience Knowledge, skills and attitudes	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Maintain accuracy and attention to detail when performing calculations					
Develop curiosity about the connections between logarithmic and exponential functions.					
Show interest about the role of logarithms in real-world applications.					
Be consistent and persistent when working with logarithmic transformations					
Think logically when solving equations requiring multiple steps.					
Appreciate the importance of exponential models in understanding natural phenomena.					
Think logically when interpreting solutions in context.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Think logically when resolving domain issues or extraneous solutions.					
Be consistent and persistent when recognizing the versatility of logarithms in mathematical modeling.					
Think logically when Fostering a methodical approach to solving multi-step equations.					
Appreciate the relevance of exponential and logarithmic concepts in everyday life.					
Show interest in tackling real-world problems using mathematical tools					
Think logically when maintaining an open-minded approach to learning from applied problem-solving scenarios					
Be consistent and persistent when					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
interpreting parameters in exponential and logarithmic models.					
Think logically when solving application problems involving growth and decay.					



Key Competencies:

Knowledge	Skills	Attitudes
1. Define the exponential expressions.	1. Simplify exponential expressions using exponent rules.	1. Think logically when simplifying exponential expressions using exponent rules.
2. Enumerate the Laws of exponents (e.g., product rule, quotient rule, power rule).	2. Solve problems involving the multiplication and division of exponential terms.	2. Value the utility of exponent rules in simplifying expressions
3. Explain The concept of zero and negative exponents.	3. Translate between exponential and radical forms	3. Maintain accuracy and attention to detail when performing calculations
4. Describe The relationship between exponential and radical expressions	4. Evaluate expressions with fractional or negative exponents	4. Develop curiosity about the connections between logarithmic and exponential functions.
5. Define logarithms as the inverse of exponents.	5. Simplify logarithmic expressions using properties.	5. Show interest about the role of logarithms in real-world applications.

Knowledge	Skills	Attitudes
6. List Logarithmic properties (product, quotient, power rules).	6. Convert between exponential and logarithmic forms	6. Be consistent and persistent when working with logarithmic transformations
7. Differentiate Common and natural logarithms (base 10 and base e).	7. Expand and condense logarithmic expressions	7. Think logically when solving equations requiring multiple steps.
8. Describe The relationship between exponential and logarithmic functions.	8. Evaluate logarithms using known values or approximations	8. Appreciate the importance of exponential models in understanding natural phenomena.
9. Explain The general form of exponential equations	9. Solve exponential equations algebraically by isolating the variable.	9. Think logically when interpreting solutions in context.
10. Define the concept of exponential growth and decay.	10. Apply logarithms to solve equations with non-matching bases	10. Think logically when resolving domain issues or extraneous solutions.
11. Identify the Methods for solving exponential equations (e.g., equating bases, logarithmic conversion).	11. Use exponential equations to model and solve real-world problems (e.g., population growth)	11. Be consistent and persistent when recognizing the versatility of logarithms in mathematical modeling.
12. Explain the general form of logarithmic equations.	12. Solve logarithmic equations algebraically, ensuring valid solutions.	12. Think logically when fostering a methodical approach to solving multi-step equations.
13. Enumerate the methods for solving logarithmic equations (e.g., converting to exponential form).	13. Apply logarithmic properties to simplify and solve equations	13. Appreciate the relevance of exponential and logarithmic concepts in everyday life.
14. Enumerate the restrictions on the	14. Interpret solutions in applied contexts, such as finance or science.	14. Show interest in tackling real-world problems using mathematical tools

Knowledge	Skills	Attitudes
domain of logarithmic equations		
15. Explain the real-world contexts involving exponential growth and decay.	15. Model real-world scenarios using exponential and logarithmic equations.	15. Think logically when maintaining an open-minded approach to learning from applied problem-solving scenarios
16. Describe the applications of logarithms in solving real-world problems (e.g., sound intensity, earthquake magnitude).	16. Interpret parameters in exponential and logarithmic models	16. Be consistent and persistent when interpreting parameters in exponential and logarithmic models.
17. Explain the significance of the natural base e and its applications	17. Solve application problems involving growth and decay	17. Think logically when solving application problems involving growth and decay.



Discovery activity:



Task 33

Considering that you go in bank to open a savings account that will generate an interest on the deposited money, the bank promises you that the interest will be calculated on the previous accumulated interest. If you need to check your final value after 5 years, explain how you can proceed? Does exponential or logarithmic functions/equations helpful to this situation? Justify your point of view.

Topic 3.1: Properties for exponential expressions



Activity 1: Problem Solving



Task 34:

Consider the function $h(x) = 3^x$

a. Complete the following table

X	-10	-1	0	1	10
$h(x) = 3^x$					

b. Discuss whether $\forall x \in \mathbb{R}, h(x) \in \mathbb{R}$ and deduce the domain of $h(x)$

c. Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$.

Key Facts 3.1: Properties for exponential expressions

- Definition of exponential expressions**

X With $a \neq 0$ and $a \neq 1$, an exponential function is written as function in the form

$f(x) = a^x$, where a is the base and x is an exponent. An exponential function is also defined as:

$\exp_a : \mathbb{R} \rightarrow \mathbb{R}_0^+ : x \mapsto y = \exp_a^x$. For simplicity we write $\exp_a^x = a^x$.

If the base a of exponential function $f(x) = a^x$ is equal to e where $e = 2.71\dots$, we have natural exponential function denoted by $f(x) = e^x$ or $f(x) = \exp(x)$. The exponential function $y = e^x$ has graph which is like the graph of $f(x) = a^x$ where $a > 1$.

- Evaluation of exponential expression**

On the same graph of sketch the graph of $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$

X	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

$g(x)$	9	3	0	$\frac{1}{3}$	$\frac{1}{9}$
--------	---	---	---	---------------	---------------

Solution:

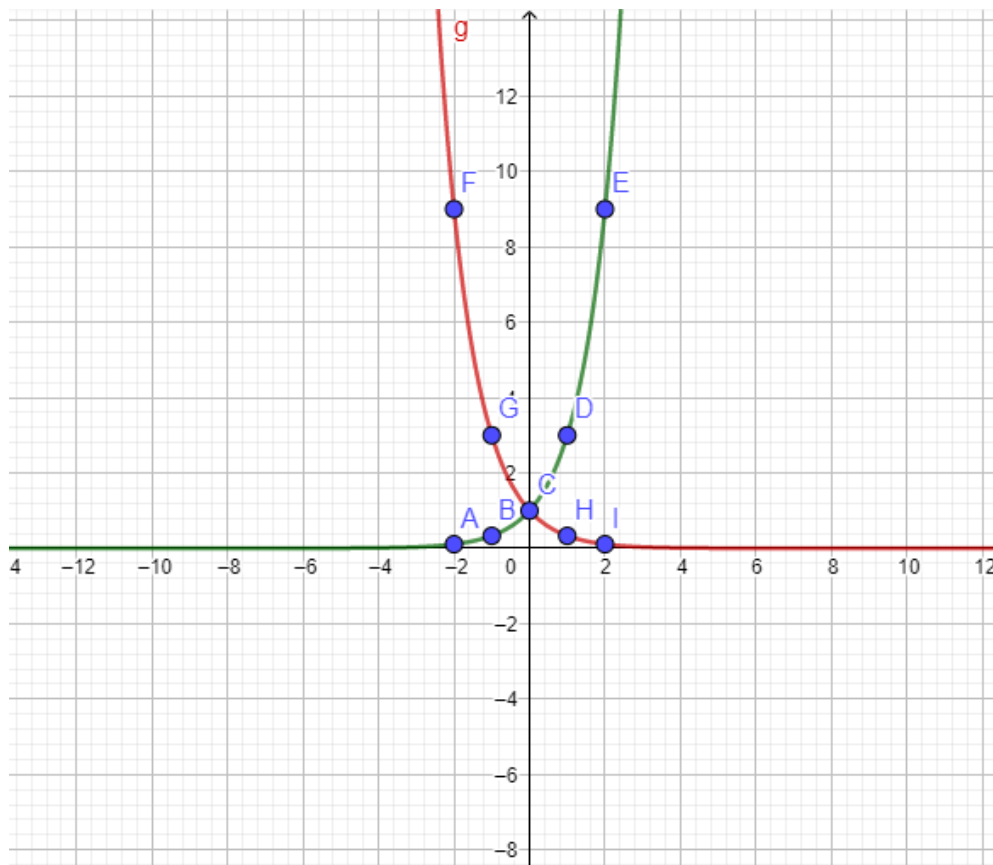


Figure 24: Exponential functions

From this, graph $f(x)$ is increasing while $g(x)$ decrease and they meet at point $(0,1)$. It is very important to note that exponential function $f(x) = a^x$ is:

- Ever zero
- Always positive
- Only taking value 1 when $x=0$

Exercise: sketch the following functions on the same Cartesian plane with its : $y = (2)^x$

$$y = \left(\frac{1}{2}\right)^x, \text{ for } -3 \leq x \leq 3$$

- **Properties for exponential expressions**

For $a > 0$, and $a \neq 1$, $m, n \in \mathbb{R}$

✓ $a^m \times a^n = a^{m+n}$

for examples: work out the following: a. $2x \times x^3$ b. $3^2 \times 3^4$

Solutions: a. $2x \times x^3 = 2x^{1+3} = 2x^4$ b. $3^2 \times 3^4 = 3^{2+4} = 3^6$

✓ $a^m : a^n = a^{m-n}$

For examples: calculate a. $x^5 \div x^3$ b. $4^2 \div 4^{10}$

Solutions: a. $x^5 \div x^3 = x^{5-3} = x^2$ b. $4^{2-10} = 4^{-8}$

✓ $(a^m)^n = a^{m \cdot n}$

Examples: $(3^2)^4 = 3^{2 \times 4} = 3^8$

✓ $a^{-n} = \frac{1}{a^n}$

Examples: a. $x^{-2} = \frac{1}{x^2}$ b. $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

✓ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Examples: a. $x^{\frac{3}{4}} = \sqrt[4]{x^3}$ b. $16^{\frac{1}{2}} = \sqrt[2]{16} = 4$

✓ $a^{\log_a b} = b$

Examples: a. $5^{\log_5 x} = x$ b. $10^{\log 5} = 5$ c. $e^{\ln y} = y$

(Icyimpaye, 2017)



Activity 2: Guided Practice



Task 35:

Using properties of exponential where $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, $a^m \times a^n = a^{m+n}$ and $a^m : a^n = a^{m-n}$ work

out: $\frac{\sqrt[3]{27} \times 9^2}{3^3}$



Activity 3: Application



Task 36:

1. Simplifying Expressions:

In a computer science algorithm, the efficiency of a process is modeled by the expression $(2^3)^4$, where the exponents represent the number of steps in the process. Simplify this expression using the properties of exponents to determine the total number of steps.

2. Combining Exponents:

In a biology experiment, the growth rate of bacteria is modeled by the equation $x^m \times x^n = x^{12}$, where m and n represent the growth rates in different phases. Given that $m = 7$, find the value of n to understand the growth rate in the second phase.

3. Negative Exponents:

In a physics experiment, the intensity of light is given by the expression $\frac{1}{2^{-3}}$, where the exponent represents the attenuation factor. Rewrite this expression without fractions and simplify it to determine the light intensity.

4. Distributive Property with Exponents:

In an engineering project, the force exerted by a machine is modeled by the expression $(3x^2y^3)^4$, where x and y represent different parameters of the machine. Simplify this expression using the rules of exponents to analyze the force distribution.

5. Division of Exponents:

In a chemistry experiment, the concentration of a solution is modeled by the expression $\frac{a^8}{a^5}$, where a represents the amount of a substance. Simplify this expression and explain the property used to understand the concentration change.

Topic 3.2: Properties for logarithmic expression



Activity 1: Problem Solving



Task 37:

For the following values of x : 50, 100, $\frac{1}{2}$, 0.7, 0.8, -30, -20, -5, 0.9, 10, 20, and 40

1. Draw and complete the table of values for $\log_{10}^x$. What do you notice about the value of $\log_{10}^x$ for $x < 0$?
2. Discuss the values of $\log_{10}^x$ for $0 < x < 1$

Key Facts 3.2: Properties for logarithmic expression

- **Definition of logarithmic expression**

For a positive constant a , $a \neq 1$, we call logarithmic function, the function:

$\mathbb{R} \rightarrow \mathbb{R} : x \mapsto y = \log_a^x$. In the expression $y = \log_{10}^x$, y is referred to as the logarithm, a is the base, and x is the argument. $\log_{10}^x$ means the power to which a must be raised to produce x . If the base is 10, it is not necessary to write the base, and we say decimal logarithm or common logarithm or Brigg's logarithm. So, the notation will become $y = \log x$. $\log x$ means the power to which 10 must be raised to produce x . If the base is e , then $\log_e^x$ is usually written $\ln x$ which is called natural logarithm. $\ln x$ means the power to which e must be raised to produce x . For every a with $a > 0$ with $a \neq 1$, the graph of logarithmic function $\log_b^x$ is the inverse of exponential function $\exp_b^x$, it follows that the graph of $\log_b^x$ and that of $\exp_b^x$ are symmetric about the line $x = y$. The following graph shows $y = \log_2^x$ logarithmic function which is the inverse of exponential function $y = 2^x$, here the base is 2.

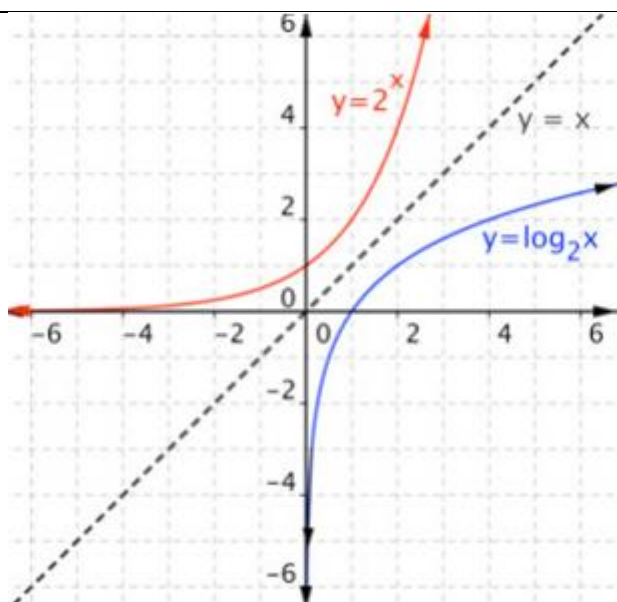


Figure 25: Exponential function and its inverse

Exercises:

a. Sketch the following function in Cartesian plane with its inverse: $y = \left(\frac{1}{2}\right)^x$,

$$-3 \leq x \leq 3$$

b. Sketch the following function in Cartesian plane with its inverse: $y = (3)^x$,

$$-2 \leq x \leq 2$$

• Change of base law

If u ($u > 0$) and if a and b are positive real numbers different from 1,

$$\log_b^u = \frac{\log_{10}^u}{\log_{10}^b} = \frac{\log u}{\log b} \text{ or } \log_b^u = \frac{\log_e^u}{\log_e^b} = \frac{\ln u}{\ln b}. \text{ This means that if you have a logarithm}$$

in any other base, you can convert it in **the decimal logarithm** or in **natural**

logarithm in the following way where $a = 10$: $\log_b^u = \frac{\log_{10}^u}{\log_{10}^b} = \frac{\log u}{\log b}$ or $a = e$:

$$\log_b^u = \frac{\log_e^u}{\log_e^b} = \frac{\ln u}{\ln b}.$$

$$\text{This is for example: } \log_2^5 = \frac{\log_{10}^5}{\log_{10}^2} = \frac{\log 5}{\log 2} \approx 2.322 \text{ or } \log_2^5 = \frac{\log_e^5}{\log_e^2} = \frac{\ln 5}{\ln 2} \approx 2.322$$

Exercises : Calculate : a. $\log_3^{10}$ b. $\log_5^7$ c. $\log_3^{11}$

Note: Let b be a positive number. The logarithm of any positive number x to be the base b , written as $\log_b^x$ represents the exponent to which b must be raised to obtain x . That is, $y = \log_b^x$ and $b^y = x$.

Examples: a. $\log_2^8 = 3$ since $2^3 = 8$ b. $\log_2^{64} = 6$ since $64 = 2^6$

- **Evaluation of logarithmic expression**

$\forall x, y \in]0, +\infty[, a \in]0, +\infty[\setminus \{1\}$:

✓ $\log_a^{xy} = \log_a^x + \log_a^y$

Example: $\log_2^{5x} = \log_2^5 + \log_2^x$

✓ $\log_a^{\frac{x}{y}} = \log_a^x - \log_a^y$

Example: $\log_3^{\frac{7}{5}} = \log_3^7 - \log_3^5$

✓ $\log_a^{\frac{1}{y}} = -\log_a^y$

Example: $\log_2^{\frac{1}{5}} = -\log_2^5$

✓ $\log_a^{x^r} = r \log_a^x$

Examples: $\log_3^{3^2} = 2 \log_3^3 = 2$

✓ $\log_a^b = \frac{\log_c^b}{\log_c^a}$ (change of base law).

Example: $\log_2^5 = \frac{\log 5}{\log 2} \approx 2.322$

✓ $\log_a^b = c \Leftrightarrow b = a^c$ (to change from logarithm to exponential and vice versa)

Example: $\log_7^{49} = 2 \Leftrightarrow$ In exponential form we have $49 = 7^2$

✓ $\log_b^a = \frac{1}{\log_a^b}$

Example: $\log_{16}^4 = \frac{1}{\log_4^{16}} = \frac{1}{\log_4^{4^2}} = \frac{1}{2 \log_4^4} = \frac{1}{2}$

✓ $\log_a^a = 1$

✓ $\log_a^1 = 0$ (REB, 2022)



Activity 2: Guided Practice



Task 38:

By using product rule, quotient rule, power rule and change of base rule perform the following questions:

- Simplify $\log_2^8 + \log_2^9$ using logarithmic properties.
- Simplify $\log_5^{125} - \log_5^{25}$.
- Rewrite $3\log_3^2$ as a single logarithmic expression.
- Use the change of base formula to evaluate $\log_5^{100}$ in terms of base 10 logarithms.



Activity 3: Application



Task 39:

In a financial model, the growth of an investment is represented by the expression $\log_a^{\frac{x^3}{y^2z}}$, where x,y and z represent the investment amount and a is a constant representing the initial investment. Express this logarithmic expression in terms of $\log_a^x, \log_a^y$ and $\log_a^z$ to analyze the growth rate and performance of the investment over time.

Topic 3.3: Exponential equations



Activity 1: Problem Solving



Task 40:

For each of the following exponential equation solve for x or t.

a. $5 + e^{0.2t} = 10$

b. $e^{2x} = 3e^x$

c. $e^{2x} = e^x + 12$

d. $e^t = 12 - 32e^{-t}$

Key Facts 3.3: Exponential equations

- **Domain of validity**

With the definition $y = f(x) = a^x$ and the restrictions that $a > 0$ and that $a \neq 1$,

the domain of an exponential function is the set of all real numbers or $]-\infty, +\infty[$. The range is the set of all positive real numbers or $]0, +\infty[$ because there is no exponent that can turn $y = f(x) = a^x$ into a zero or into a negative result.

Note: Generally, if $u(x)$ is a defined function of x, the range and domain of the function $f(x) = a^{u(x)}$ will depend on $u(x)$.

Example 1

Determine the domain and the range of the function $f(x) = 3^{\sqrt{2x}}$.

Solution :

domain : $2x \geq 0 \Leftrightarrow x \geq 0 \quad \text{dom} f = [0, +\infty[$

$$f(x) = 3^{\sqrt{2x}} \Rightarrow y = 3^{\sqrt{2x}} \Rightarrow \log y = \log 3^{\sqrt{2x}} \Rightarrow \log y = \sqrt{2x} \log 3 \Rightarrow \frac{\log y}{\log 3} = \sqrt{2x}$$

range: $\left(\frac{\log y}{\log 3}\right)^2 = (\sqrt{2x})^2 \Rightarrow \frac{\log^2 y}{\log^2 3} = 2x \Rightarrow x = \frac{\log^2 y}{2 \log^2 3}$

condition: $y > 0$ $range =]0, +\infty[$

Example 2

Find the domain and the range of $f(x) = 3^{\frac{x+1}{x-2}}$

Domain: condition $x - 2 \neq 0 \Rightarrow x \neq 2$ $domf = \mathbb{R} \setminus \{2\}$

For range we find x in terms of y

$$\begin{aligned} f(x) &= 3^{\frac{x+1}{x-2}} \Rightarrow y = 3^{\frac{x+1}{x-2}} \Rightarrow \log y = \log 3^{\frac{x+1}{x-2}} \Rightarrow \log y = \frac{x+1}{x-2} \log 3 \\ x \log y - 2 \log y &= x \log 3 + \log 3 \Rightarrow x \log y - x \log 3 = \log 3 + 2 \log y \\ x(\log y - \log 3) &= \log 3 + 2 \log y \Rightarrow x = \frac{\log 3 + 2 \log y}{\log y - \log 3} \end{aligned}$$

Here we have two conditions: $y > 0$ which is $]0, +\infty[$ **and**

$$\log y - \log 3 \neq 0 \Rightarrow \log y \neq \log 3 \Rightarrow y \neq 3$$

Then we find the intersection between two conditions:

$$Range =]0, 3[\cup]3, +\infty[$$

Exercises: Discuss and determine the domain and range of the following functions

a. $f(x) = 4^{\sqrt{x^2-4}}$ b. $g(x) = e^{\frac{x+2}{x-3}}$ c. $f(x) = e^{\sqrt{x}}$ d. $f(x) = e^{\sqrt{x^2-4}}$ e. $f(x) = 3^{x+4}$

- **Solving exponential equation Using like bases Using logarithms Activities:**

For which value(s), each function $f(x)$ below can be defined. Explain.

(1) a. $f(x) = e^{x+2}$ b. $f(x) = e^{x+2}$.

(2) Detect the value of x , if $2^{1-x} = 6$

Equations that involve powers (exponents) as terms of their expressions are referred to as exponential equations. Such equations can sometimes be solved by appropriately applying the properties of exponents or introducing logarithms within expression.

- **Solve exponential equations using one- to- one property**

To solve exponential function using properties follow the three steps:

- Rewrite both sides of the equation as an exponential expression with the same base.
- Since the bases are equal, then the exponents must be equal. Set the exponents equal to each other.
- Solve the equations and check answers

Example 1:

Solve the following exponential equations

a. $3^{x+2} = 27^x$ b. $4^{x-1} = 16$

Solutions

a. Domain of validity $x \in \mathbb{R}$,

$$3^{x+2} = 27^x \Rightarrow 3^{x+2} = 3^{3x} \Rightarrow x+2 = 3x \Rightarrow x-3x = -2 \Rightarrow -2x = -2 \Rightarrow x = 1, \text{ then, } x=1 \text{ Therefore, solution } S = \{1\}$$

b. Domain of validity $x \in \mathbb{R}$,

$$4^{x-1} = 16 \Rightarrow 4^{x-1} = 4^2 \Rightarrow x-1 = 2 \Rightarrow x = 2+1 \Rightarrow x = 3 \text{ Therefore, solution } S = \{3\}$$

Example 2:

Solve the following exponential equations a. $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$ B. $2^{2y} + 3(2^y) = 4$

Solution

a. $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$ Domain of validity $x \in \mathbb{R}$ then

$$e^{-x^2} = e^{2x} \cdot e^{-3} \Rightarrow e^{-x^2} = e^{2x-3} \Rightarrow -x^2 = 2x-3 \Rightarrow -x^2 - 2x + 3 = 0, \text{ The solution of this equation gives } x=-3 \text{ and } x=1, \text{ The solution set is } \{-3, 1\}$$

b. $2^{2y} + 3(2^y) = 4$

$$(2^y)^2 + 3(2^y) = 4$$

Let $t = 2^y \Rightarrow t^2 + 3t - 4 = 0$ by solving $t=-4$ and $t=1$

Replacing the value of x in the equation $2^y = t$ For $t = -4$: $2^y = -4$ doesn't exist For $x = 1$: $2^y = 1 \Rightarrow 2^y = 2^0 \Rightarrow y = 0, S = \{0\}$

- **Solve an exponential equation by taking logarithms for both sides**

To solve an exponential equation using logarithms the four steps followed:

- Isolate the exponential expression
- Take logarithms for both sides
- Rewrite exponential side as a linear expression
- Solve the obtained equation and check the answer

Example 1: solve an exponential equation $2e^{x+1} - 4 = 12$

Solution: Let first isolate the exponential expression,

$$2e^{x+1} - 4 = 12 \Rightarrow e^{x+1} = 8 \Rightarrow \ln e^{x+1} = \ln 8 \Rightarrow (x+1) \ln e = \ln 8$$

$$x+1 = \ln 8 \Rightarrow x = \ln 8 - 1 \Rightarrow x \approx 1.079 \Rightarrow S = \{1.079\}$$

Example 2 Solve each of the following equations

a. $e^x = 5$ b. $10 + e^{0.1t} = 14$

Solution: a. $e^x = 5$ Domain of validity $x \in \mathbb{R}$

$$e^x = 5 \Rightarrow \ln e^x = \ln 5 \Rightarrow x \ln e = \ln 5 \Rightarrow x = \ln 5 \Rightarrow S = \{\ln 5\}$$

b. $10 + e^{0.1t} = 14$ Domain of validity $t \in \mathbb{R}$

$$10 + e^{0.1t} = 14 \Rightarrow e^{0.1t} = 14 - 10 \Rightarrow e^{0.1t} = 4 \Rightarrow \ln e^{0.1t} = \ln 4 \Rightarrow 0.1t \ln e = \ln 4$$

$$0.1t = \ln 4 \Rightarrow t = \frac{1}{0.1} \ln 4 \Rightarrow t = 10 \ln 4 \Rightarrow S = \{10 \ln 4\}$$

Example 3. Solve each equation a. $5 + 3^{t-4} = 7$ b. $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

Solution: a.

$$5 + 3^{t-4} = 7 \Rightarrow 3^{t-4} = 7 - 5 \Rightarrow 3^{t-4} = 2 \Rightarrow \ln 3^{t-4} = \ln 2 \Rightarrow (t-4) \ln 3 = \ln 2$$

$$t - 4 = \frac{\ln 2}{\ln 3} \Rightarrow t = 4 + \frac{\ln 2}{\ln 3}$$

Therefore, $t = 4 + \frac{\ln 2}{\ln 3}$

B. $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

Let $t = 2^{2x}$ then $3t^2 - 7t + 4 = 0$ $D = 1 \Rightarrow t = \frac{4}{3}$ or $t = 1$

Replacing the value of k in the equation

$$t = \frac{4}{3} \Rightarrow 2^{2x} = \frac{4}{3} \Rightarrow \ln 2^{2x} = \ln \left(\frac{4}{3} \right) \Rightarrow 2x \ln 2 = \ln 4 - \ln 3$$

and

$$x = \frac{\ln 4 - \ln 3}{2 \ln 2}$$

$$t = 1 \Rightarrow 2^{2x} = 1 \Rightarrow \ln 2^{2x} = \ln 1 \Rightarrow 2x \ln 2 = 0 \Rightarrow x = \frac{0}{2 \ln 2} \Rightarrow x = 0$$

$$S = \left\{ 0, \frac{\ln 4 - \ln 3}{2 \ln 2} \right\}$$

Exercises:

1. Solve each equation for x or t.

a. Solve: **a.** $2e^{-x+1} - 5 = 9$ **b.** $\frac{50}{1+12e^{0.02x}} = 10.5$

b. Solve in the following equations in the set of real numbers: $\frac{e^x + e^{-x}}{2} = 1$

c. Find the value of marked letter in each equation:

$9^t + 3^t = 12$ **b.** $2^x + 2^{x-1} = \frac{3}{2}$ **c.** $5^m \sqrt[m]{8^{m-1}} = 500$ (REB, 2022)



Activity 2: Guided Practice



Task 41:

To perform exponential equation through this way: $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

Let $t = 2^{2x}$ then $3t^2 - 7t + 4 = 0$ $D = 1 \Rightarrow t = \frac{4}{3}$ or $t = 1$

Replacing the value of k in the equation

$$t = \frac{4}{3} \Rightarrow 2^{2x} = \frac{4}{3} \Rightarrow \ln 2^{2x} = \ln \left(\frac{4}{3} \right) \Rightarrow 2x \ln 2 = \ln 4 - \ln 3$$

$$x = \frac{\ln 4 - \ln 3}{2 \ln 2}$$

Referring to the steps above Solve the following equations:

a. $4^{x+1} + 31.2^{x-1} = 2$

b. $2^x + \frac{1}{2^x - 7} = 9$

c. $81^x + 81^{1-x} = 30$



Activity 3: Application



Task 42:

1. In a population growth model, the number of individuals in a species is given by the equation $2^{4x} - 6.2^{3x} + 6.2^x - 1 = 0$, where x represents the time in years. Solve this equation to determine the critical points in time when the population reaches certain thresholds.

2. In a chemical reaction, the concentration of two reactants is modeled by the equation $\frac{2^x}{4} - \frac{3^x}{9} = 0$, where x represents the time in minutes. Solve this equation to find the time at which the concentrations of the two reactants are equal.

Topic 3.4: Logarithmic equations



Activity 1: Problem Solving



Task 43

1. Use calculator to complete the following tables:

a.

X	$\ln x$
-0.8	
-0.6	
-0.4	
-0.2	
0	

b.

X	$\ln x$
0.2	
0.4	
0.6	
0.8	
1	

c.

X	$\ln x$
1.5	
2	
2.5	
3	
3.5	

2. Using the tables, give your observation for

- Negative x values and zero.
- x Values between 0 and 1.
- x Values greater than 1.

3. Plot a graph of condition for domain of definition $f(x) > 0$ for $x > 0$.

Key Facts 3.4: Logarithmic equations

- Domain of validity**

Domain and range of natural logarithmic function

The Natural logarithm of x is denoted $\ln x$ or $\log_e x$.

$\ln x$ is defined on positive real numbers, $]0, +\infty[$ and its range is all real numbers.

✓ For $y = \ln(h(x))$, condition for domain of definition $h(x) > 0$

✓ $\log(f(x))$, condition for domain of definition $f(x) > 0$

✓ $\log_a^{f(x)}$, condition for domain of definition $f(x) > 0$

For examples: 1. Find the domain and range of the following functions:

a. $y = \ln x$ b. $y = \ln\left(\frac{x+1}{x-2}\right)$ c. $y = \log_5^{x+1}$ d. $y = \ln(x^2 + 1)$

Solutions: a. $y = \ln x$ condition for domain of definition is $x > 0$,
 $\text{dom}f =]0, +\infty[$

Range: $y = \ln x \Rightarrow \log_e^x = y \Rightarrow x = e^y \Rightarrow \text{range} = \mathfrak{R}$

b. $y = \ln\left(\frac{x+1}{x-2}\right)$, condition for domain of definition is $\frac{x+1}{x-2} > 0$

Table of signs:

F(x) x	$-\infty$	-1	2	$+\infty$
$x+1$	-	-	-	-
$x-2$	-	-	-	-
$\frac{x+1}{x-2}$	+++++	0	-----	+++++

$\text{Dom}f =]-\infty, -1[\cup]2, +\infty[$

Range:

$y = \ln\left(\frac{x+1}{x-2}\right) \Rightarrow \log_e^{\frac{x+1}{x-2}} = y \Rightarrow \frac{x+1}{x-2} = e^y \Rightarrow x+1 = e^y(x-2) \Rightarrow x+1 = xe^y - 2y$

$x - xe^y = -1 - 2y \Rightarrow x(1 - e^y) = -1 - 2y \Rightarrow x = \frac{-1 - 2y}{1 - e^y}$

Condition for range is $1 - e^y \neq 0 \Rightarrow e^y \neq 1 \Rightarrow e^y \neq e^0 \Rightarrow y \neq 0 \Rightarrow \text{range} = \mathbb{R} \setminus \{0\}$
(Icyimpaye, 2017)

- **Logarithmic equations involving one unknown**

Logarithmic equation in $\mathbb{R}$ is the equation containing the unknown within the logarithmic expression. To solve logarithmic equations, the following steps are followed:

- ✓ Set existence conditions for solution(s) of equation.
- ✓ Express logarithms in the same base
- ✓ Use logarithmic properties to obtain: $\log_a^{u(x)} = \log_a^{v(x)} \Rightarrow u(x) = v(x)$; **where** $u(x)$ **and** $v(x)$ are the functions in x .
- ✓ Make sure that the value(s) of unknown verifies the conditions above. The properties of logarithms can be used to solve logarithmic equations.

Example 1: Solve the following equation $\log_x^{49} = 2$

Solution: condition of validity $x: x > 0, x \neq 1$ From the equation $\log_x^{49} = 2$, we change logarithmic equation to exponential equation to get $x^2 = 49$. x for $x = \pm 7$, for $x > 7$ therefore, solution set is $\{7\}$

Example 2: Solve each equation

a. $\log_3^{(x+1)} = \log_3^2$ b. $\log_{x-2}^3 = 1$ c. $\log_2^{x+14} + \log_2^{(x+2)} = 6$

Solution: a. $\log_3^{(x+1)} = \log_3^2$

Condition: $x+1 > 0 \Leftrightarrow x > -1$, then $\log_3^{(x+1)} = \log_3^2$ simplify to obtain

$$x+1=2 \Rightarrow x=1 \Rightarrow S = \{1\}$$

b. $\log_{x-2}^3 = 1$ condition: $x-2 > 0, x > 2$ and $x-2 \neq 1 \Rightarrow x \neq 3$, this means $x \in]2, 3[\cup]3, +\infty[$

$$\text{Then } \log_{x-2}^3 = 1 \Rightarrow \log_{x-2}^3 = 1 \log_{x-2}^{x-2} \Rightarrow 3 = x-2 \Rightarrow x=5 \Rightarrow S = \{5\}$$

OR $\log_a^b = c \Leftrightarrow b = a^c \Rightarrow \log_{x-2}^3 = 1 \Leftrightarrow 3 = (x-2)^1 \Rightarrow 3+2 = x \Rightarrow x = 5 \Rightarrow$
 $S = \{5\}$

c. $\log_2^{(x+14)} + \log_2^{(x+2)} = 6$

Condition: $x+14 > 0 \Rightarrow x > -14$, and $x+2 > 0 \Leftrightarrow x > -2$ then

$$\log_2^{(x+14)} + \log_2^{(x+2)} = 6 \Rightarrow$$

$$\log_2^{(x+14)} + \log_2^{(x+2)} = 6 \log_2^2 \Rightarrow \log_2^{(x+14)} + \log_2^{(x+2)} = \log_2^{2^6} \Rightarrow$$

$$(x+14)(x+2) = 64 \Rightarrow$$

$$\begin{aligned} x^2 + 14x + 28 - 64 &= 0 & \Rightarrow (x+18)(x-2) &= 0 & \Rightarrow x-2 &= 0 \\ x^2 + 14x - 36 &= 0 & \Rightarrow x+18 &= 0 & \Rightarrow x &= -18 \end{aligned}$$

(REB, 2023)

- **Systems of equations involving logarithms**

In solving systems of equations using logarithms, like one-variable logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can be used in the creation of a one-variable equation that is simple to solve.

Example: solve the following system of equations $\begin{cases} \log x + \log y = 1 \\ \log(10x) - \log y = 2 \end{cases} \Rightarrow$

$$\begin{cases} \log x + \log y = 1 \\ \log(10x) - \log y = 2 \end{cases}$$

$$\log x + \log 10x = 3$$

Use logarithmic property of addition to solve for x the new equation

$$\begin{aligned} \Rightarrow \log x(10x) &= 3 & \Rightarrow \log_{10}^{10x^2} &= 3 \Rightarrow 10x^2 = 10^3 \\ \log 10x^2 &= 3 & \Rightarrow x^2 &= \frac{1000}{10} \Rightarrow x = \sqrt{100} \Rightarrow x = \pm 10, \text{ since } -10 \text{ is} \end{aligned}$$

undefined, we reject it and keep $x = 10$.

Substituting the value of $x = 10$ in any of the equations in the system, we get $\log 10 + \log y = 1$, $\log y = 0 \Rightarrow \log_{10}^y = 0 \Rightarrow y = 10^0 \Rightarrow$. This gives the value of $y = 1$. The solution is $S = \{(10,1)\}$.

Example 2: solve the following system of equations
$$\begin{cases} \log x + \log y = 1 \\ y = 2x + 1 \end{cases}$$

Solution Let use substitution method, take equation (1), $\log x + \log y = 1 \Rightarrow \log xy = 1$, $xy = 10 \Rightarrow x = \frac{10}{y}$ and apply property to get $xy = 10 \Rightarrow x = \frac{10}{y}$. Replace the value of x in $y = 2\left(\frac{10}{y}\right) + 1 \Rightarrow y^2 - y - 2 = 0$

By solving this quadratic equation $y = \{-5, 4\}$. Since at -4 is not defined then, we take $y = 5$. By replacing $y = 5$ in equation (2) we get $x = 2$. The solution for the given system of equation is given by $S = \{2, 5\}$.

- **Natural logarithmic equations**

The Natural logarithmic equation in $\mathfrak{R}$ is the equation containing the unknown within the natural logarithmic expression. To solve Natural logarithmic equations, the following steps are followed:

- ✓ Set existence conditions for solution(s) of equation.
- ✓ Use logarithmic properties to obtain: $\ln u(x) = \ln v(x) \Rightarrow u(x) = v(x)$; where $u(x)$ and $v(x)$ are the functions in x .
- ✓ Make sure that the value(s) of unknown verifies the conditions above.

Note:

- ✓ $y = \ln x = \log_e^x \Leftrightarrow e^x = y$ (as inverse)

The equation $\ln x = 1$ has a unique solution; the irrational number 2.71828182845904523536... and this number is represented by letter e . Thus, if $\ln x = 1 \Rightarrow \log_e^x = 1 \Rightarrow x = e$

The properties of natural logarithms can be used to solve natural logarithmic equations.

Example 1: Solve each equation a. $\ln x - \ln 5 = 0$ b. $3 + 2\ln x = 7$ c.

$$\ln 2x + \ln(x + 2) = \ln 6$$

Solution: a. $\ln x - \ln 5 = 0$

condition of validity: $x > 0$ Then $\ln x - \ln 5 = 0 \Rightarrow \ln x = \ln 5 \Rightarrow x = 5$

Therefore, the solution $S = \{5\}$

a. $3 + 2\ln x = 7$

condition of validity: $x > 0$ Then

$$3 + 2\ln x = 7 \Rightarrow 2\ln x = 7 - 3 \Rightarrow \ln x = \frac{4}{2} \Rightarrow \ln x = 2$$

$$x = e^2$$

Therefore, the solution $S = \{e^2\}$.

b. $\ln 2x + \ln(x + 2) = \ln 6$

conditions of validity: $2x > 0$ and $x + 2 > 0 \Leftrightarrow x > 0$ and $x > -2$

Domain of validity: $x \in]-2, +\infty[\cap]0, +\infty[\Rightarrow x \in]0, +\infty[$

Solving:

$$\ln 2x + \ln(x + 2) = \ln 6 \Rightarrow \ln 2x(x + 2) = \ln 6 \Rightarrow 2x^2 + 4x = 6 \Rightarrow 2x^2 + 4x - 6 = 0$$

By solving $x=1$ and $x=-3$

As $x \in]0, +\infty[$, then $S = \{1\}$

- **Systems of equations involving natural logarithms**

In solving systems of equations using natural logarithms, like one-variable natural logarithmic equations require the same set of techniques like logarithmic identities and exponents, which help to rephrase the natural logarithms in ways that make it easier to solve for the variables. Algebraic procedures like substitution and elimination can use in the creation of a one-variable equation that is simple to solve.

Example: solve the following system of equations $\begin{cases} \ln(xy) = 7 \\ \ln\left(\frac{x}{y}\right) = 1 \end{cases} \Rightarrow \begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases}$

solve the system of equations started eliminating variable y

$$\begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases} \Rightarrow \begin{cases} \ln x + \ln y = 7 \\ \ln x - \ln y = 1 \end{cases} \Rightarrow \ln x = 4 \Rightarrow x = e^4$$

$$2\ln x = 8$$

replacing the value of $\ln x$ in equation (1),

$$\ln x + \ln y = 7 \Rightarrow 4 + \ln y = 7 \Rightarrow \ln y = 7 - 4 \Rightarrow \ln y = 3 \Rightarrow y = e^3$$

Solution of the system is $S = \{e^4, e^3\}$.

Exercises:

1. Solve each of the following equations

a. $\ln x = 0$ b. $\ln(x^2 - 1) = \ln(4x - 1) - 2\ln 2$ c. $2\ln 4x = 7$ d. $\ln x + \ln 4 = 0$

(REB, 2022)



Activity 2: Guided Practice



Task 44:

1. Use the properties for logarithm to determine the value of x in the following expressions:

a. $\log x = 2$

b. $\log(100x) = 2 + \log 4$

c. $\log_2^x = -3$

d. $\log_2^x = 5 - \log_2^{x+4}$

2. Find the domain and range of the following functions:

a. $y = \ln(x^2 + 1)$ b. $f(x) = \log_5^{x+1}$

3. For each of the following function solve for x:

a. $\log(35 - x^3) = 3\log(5 - x)$

b. $\log(1 - x) = -1$

c. $\log(3x - 2) + \log(3x - 1) = \log(4x - 3)^2$

4. In $\mathbb{R}^2$, solve the following equations: $\begin{cases} x + y = 9 \\ \log x + \log y = \log 14 \end{cases}$ (Icyimpaye, 2017)



Activity 3: Application



Task 45:

1. Let three functions be defined by:

In a biological study, the growth of a population is modeled by the following logarithmic functions, where x represents the time in days. Determine for which value(s) of x each function is defined to understand the time periods during which the population growth can be accurately modeled.

a. $f(x) = \ln(x + 2)$

b. $f(x) = \ln x$

c. $\ln(x^2 - 5x + 6)$

2. Use the properties of logarithms to determine the value of x in the following expressions:

In a financial analysis, the time required for an investment to reach a certain value is modeled by logarithmic equations. Use the properties of logarithms to find the value of x (time in years) in the following scenarios:

a. $\ln x = 10$

b. $\ln x = 3$

3. Solve the following system: In an environmental study, the relationship between two species' populations is modeled by a system of logarithmic equations, where x and y represent the populations of the two species.

Solve the system to find the equilibrium populations: $\begin{cases} 2\ln x + 3\ln y = -2 \\ 3\ln x + 5\ln y = -4 \end{cases}$

Topic 3.5: Logarithms and exponential applications



Activity 1: Problem Solving



Task 46:

For how many Years must 1 000 FRW be invested at 10% in order to accumulate 1 600Frw?

Key Facts 3.5: Logarithms and Exponential applications

- **Compound interest**

A compound interest plan pays interest on interest already earned. The value of an investment depends not only on the interest rate, but how frequently the interest is compounded. If, for example, a \$100 investment is made with 5% interest compounded annually, after one year, the investment will be worth \$105. The next year, the interest added to the value of the investment will be 5% of the \$105.

Compound interest causes the amount of interest earned to increase with every compounding period.

If P is the principal, n is the number of years, r is the interest rate per period, k is the number of periods per year, and A the total amount at the end of periods, then

$$A(t) = p \left(1 + \frac{r}{k} \right)^{kt}.$$

Examples: 1. A 1,000 FRW deposit is made at a bank that pays 12% compounded annually. How much will be on the account at the end of 10 years?

Solution: $A(t) = p \left(1 + \frac{r}{k} \right)^{kt}$ $p=1000$, $r=12\%$, $k=1$, and $t=10$

$A = 1000(1 + 0.12)^{10} = 1000(1.12)^{10} = 1000(3.1038) = 3105.8$ Therefore, 3,105.8 FRW will be on the account at the end of 10 years.

Now if the compound interest is paid monthly for the same number of principle, we can compute the sum as follows:

Solution: In this example, the compounded is monthly, so the number of periods is k

$$A(t) = p \left(1 + \frac{r}{k} \right)^{kt}$$

=12, P= 1000, r=12%, t=10

$$A = 1000 \left(1 + \frac{0.12}{12} \right)^{12 \times 10} = 1000(3.3004) = 3300.4$$

Therefore, 3,300.4 FRW will be on the account at the end of 10 years. (REB, 2022)

- **Other application related to compound interest**

- ✓ **Discrete and continuous compound interest rates**

If you put money in a savings account then the bank will pay you interest (a percentage of your account balance) at the end of each time of period, typically one day, one week, one month, quarterly, semiannually, or annually. The method whereby the interest is calculated and added to the principal at specific periods is called **discrete compounding**. For example, if the time of period is one month this process is called **Monthly compounding**. The term compounding refers to the fact that interest is added to your account each month. If it is one day is called **daily compounding**. The exponential model that describes this situation is called **discrete compounding interest formula**; it is given by: $A(t) = p \left(1 + \frac{r}{k} \right)^{kt}$, where P is the principal, n is the number of years, r is the interest rate per period, k is the number of periods per year, t is time and A the total amount at the end of periods.

- **Continuous compound interest**

If we start with discrete compound interest formula and let the number of times compounded per year approaches ∞ , then we end up with what is known as continuous compounding then the balance at time t years is given by $A = P_0 e^{rt}$ where P_0 is the principal amount, r is annual interest rate and time t years.

Example:

1. If the principal money is \$10 000 the annual interest rate is 5% and the interest is compounded continuously. What will be the balance after 40 years?

Solution: $P_0 = 10000, r = 5\% = 0.05, t = 40,$

$$A = P_0 e^{rt} \Rightarrow P(40) = 10000e^{0.05 \times 40} = 73890.56$$

The balance after 40 years is \$73890.56.

2. Find the accumulated amount after 3 years if \$1 000 is invested at 8% per year compounded

- a. daily, (b) continuously

solution: a) Using the compound interest formula with $P = 1\ 000, r = 0.08, k = 365,$ and $t = 3,$ we find

$$A(t) = p \left(1 + \frac{r}{k} \right)^{kt} \Rightarrow A = 1000 \left(1 + \frac{0.08}{365} \right)^{365 \times 3} \Rightarrow A \approx 1271.22$$

- b. Using the continuous compound interest formula with $P = 1\ 000, r = 0.08,$ and $t = 3,$ we find $A = P_0 e^{rt} = 1000e^{0.08 \times 3} \approx 1271.25,$ Note that the two solutions are very close to each other. (lcyimpaye, 2017)

- **population growth**

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for n periods will be $P_n = P_0(1+r)^n$.

Examples: 1. The town of Gray rock had a population of 10,000 in 1960 and 12,000 in 1970.

- a. Assuming an exponential growth model, estimate the population in 1980.
b. What is the doubling time for the town's population?

Solution:

a. For exponential growth model $P_n = P_0(1+r)^n$. Let in 1960, have $P_0 = 10000$.

Thus, in 1970, we have $P_{10} = 12000$ while in 1980, we have P_{20} .

$$P_n = P_0(1+r)^n \Rightarrow P_{10} = P_0(1+r)^n \Rightarrow 12000 = 10000(1+r)^{10} \Rightarrow \frac{12000}{10000} = (1+r)^{10} \Rightarrow 1.2 = (1+r)^{10}$$

$$\sqrt[10]{1.2} = 1+r \Rightarrow r = 0.018399376$$

$$P_n = P_0(1+r)^n$$

$$P_{20} = P_0(1+r)^n \Rightarrow P_{20} = 10000(1+0.018399376)^{20} = 14,400$$

The population in 1980 is 14,000.

- c. The doubling time for the town's population means the time for which

$$P_n = 2P_0$$

$$2P_0 = P_0(1+r)^n \Rightarrow \frac{2P_0}{P_0} = (1+r)^n \Rightarrow 2 = (1+r)^n \text{ by multiplying ln in each side}$$

we get

$$\ln 2 = n \ln(1+r) \Rightarrow n = \frac{\ln 2}{\ln(1+r)} \Rightarrow n = 38 \text{ years}$$

Hence, the doubling time for the town's population is 38 years.

Exercises: 1. The population, P , of an island t years after January 1st 2016 is given by this formula $P = 4200(1.04)^t$

- What was the population of the island on January 1st 2016?
- What is the constant rate?
- Work out the population of the island on January 1st 2021.

- **population decay**

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve depreciation value problems. Hence solve the following problem:

During an experiment, a scientist notices that the number of bacteria halves every second. If there were 2.3×10^3 bacteria at the start of the experiment, how many bacteria were left after 5 seconds. Give your answer in standard form correct to two significant figures.

Depreciation (or decay) is negative growth. If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is

$$v_t = v_0(1-r)^t.$$

Example: 1. If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 1,000 cells?

Solution: Using the equation, $v_t = v_0(1-r)^t$; $v_t = 1000$, $v_0 = 5000000$, $r=0.45$.

$$v_t = v_0(1-r)^t \Rightarrow 1000 = 5000000(1-0.45)^t \Rightarrow \frac{1000}{5000000} = (0.55)^t \Rightarrow 0.0002 = (0.55)^t$$

$$\ln(0.0002) = \ln(0.55)^t \Rightarrow \frac{\ln(0.0002)}{\ln(0.55)} = t \Rightarrow t = 14.2$$

It will take about 14.2 minutes for the cell population to drop below a 1,000 count.

(Icyimpaye, 2017)



Activity 2: Guided Practice



Task 47:

- a. By using the compound interest perform the following question:

If you deposited 5,000 FRW into an account paying 6% annual interest compounded monthly, how long will you wait until there is 8,000 FRW on the account?

- b. If you deposited 8,000 FRW into an account paying 7% annual interest compounded quarterly, how long will you wait until there is 12,400 FRW on the account?
- c. At 3% annual interest compounded monthly, how long will it take to double the amount of money deposited in question 1? (REB, 2022)
- d. In a certain experiment, the number of bacteria reduces by a quarter each second. If the number of bacteria initially was X, write a formula that can be used to calculate the number of bacteria, V, remaining after t seconds.
- e. The population of a particular town on July 1, 2011 was 20,000. If the population decreases at an average annual rate of 1.4%, how long will it take for the population to reach 15,300?

(Icyimpaye, 2017)



Activity 3: Application



Task 48:

Find out how exponential and logarithmic functions are used to solve population growth problems. Hence solve the following problem: Betty is investigating the growth in the population of a certain type of bacteria in her flask. At the start of day 1, there are 1,000 bacteria in flask. The population of bacteria grows exponentially at the rate of 50% per day. Find the population of bacteria in her flask at the start of day 5.



Formative Assessment

1.

a. Express: $\log_a \frac{x^3}{y^2z}$ in terms of $\log_a^x, \log_a^y$ and $\log_a^z$

b. solve: a. $2^{3x} = 3^{2x-1}$ b. $e^{\ln x^2} - 9 = 0$ c. $e^x - 12 = \frac{-5}{e^{-x}}$

2. solve the following equation:

a. $2e^{2x} - e^x - 6 = 0$

b.
$$\begin{cases} 2 \ln x + 3 \ln y = -2 \\ 3 \ln x + 5 \ln y = -4 \end{cases}$$

3. Solve : a. $4 \log_a^{\sqrt{x}} - \log_a^{3x} = \log_a^{x^{-2}}$ b. $2 \log_2^x + \log_x^2 = 3$ c. $\left(\frac{5}{2}\right)^x = 0.16$ d. $4^x - 10 \cdot 2^x + 16 = 0$

4. Sketch the following function in Cartesian plane with its inverse: $y = \left(\frac{1}{2}\right)^x, -3 \leq x \leq 3$

5. In $\mathbb{R}^2$, solve the following equation:

a.
$$\begin{cases} x^2 + y^2 = 221 \\ \log_5^x + \log_5^y = \log_5^{110} \end{cases}$$

b. Solve in real number $\log x + \log(x^2 + 2x - 1) - \log 2 = 0$

6. If you deposit 6,500 FRW into an account paying 8% annual interest compounded monthly, how much money will be on the account after 7 years?

7. How much money would you need to deposit today at 9% annual interest compounded monthly to have 12,000 FRW on the account after 6 years?
8. The population of a city increased by 5.2% for the year 2014. At the beginning of 2015 the population of the city was 1,560,000. Betty assumes that the population will continue to increase at a constant rate of 5.2% each year. Use Betty's assumption to:
 - a. Estimate the population of the city at the beginning of 2017. Give your answer corrected to 3 significant figures.
 - b. Work out the year in which the population of the city will reach 2,000,000.



Points to Remember

- Definition of exponential expression
- Domain and range of exponential functions
- Properties of exponential functions.
- Definition of logarithmic functions
- Domain and range of logarithmic functions
- Properties of logarithmic functions
- Change of base law.
- Change from logarithm into exponential and vice versa.
- Exponential equations.
- Logarithmic equations.
- Compound interest.
- Discrete and continuous compounding
- Population growth.
- Population decay



Self-Reflection

1. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Define the exponential expressions.					
Enumerate the Laws of exponents (e.g., product rule, quotient rule, power rule).					
Explain The concept of zero and negative exponents.					
Describe The relationship between exponential and radical expressions					
Define logarithms as the inverse of exponents.					
List Logarithmic properties (product, quotient, power rules).					
Differentiate Common and natural logarithms (base 10 and base e).					
Describe The relationship between exponential and logarithmic functions.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Explain The general form of exponential equations					
Define the concept of exponential growth and decay.					
Identify the Methods for solving exponential equations (e.g., equating bases, logarithmic conversion).					
Explain the general form of logarithmic equations.					
Enumerate the methods for solving logarithmic equations (e.g., converting to exponential form).					
Enumerate the restrictions on the domain of logarithmic equations					
Explain the real-world contexts involving exponential growth and decay.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe the applications of logarithms in solving real-world problems (e.g., sound intensity, earthquake magnitude).					
Explain the significance of the natural base e and its applications					
Simplify exponential expressions using exponent rules.					
Solve problems involving the multiplication and division of exponential terms.					
Translate between exponential and radical forms					
Evaluate expressions with fractional or negative exponents					
Simplify logarithmic expressions using properties.					
Convert between exponential and logarithmic forms					
Expand and condense logarithmic expressions					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Evaluate logarithms using known values or approximations					
Solve exponential equations algebraically by isolating the variable.					
Apply logarithms to solve equations with non-matching bases					
Use exponential equations to model and solve real-world problems (e.g., population growth)					
Solve logarithmic equations algebraically, ensuring valid solutions.					
Apply logarithmic properties to simplify and solve equations					
Interpret solutions in applied contexts, such as finance or science.					
Model real-world scenarios using exponential and logarithmic equations.					
Interpret parameters in exponential and logarithmic models					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Solve application problems involving growth and decay					
Think logically when simplifying exponential expressions using exponent rules.					
Value the utility of exponent rules in simplifying expressions					
Maintain accuracy and attention to detail when performing calculations					
Develop curiosity about the connections between logarithmic and exponential functions.					
Show interest about the role of logarithms in real-world applications.					
Be consistent and persistent when working with logarithmic transformations					
Think logically when solving equations requiring multiple steps.					

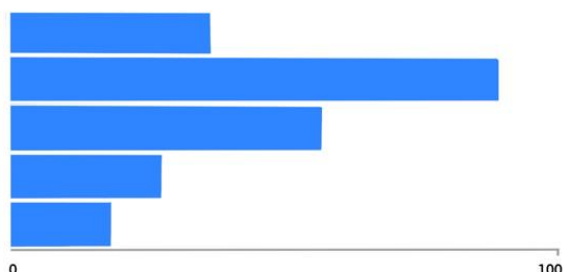
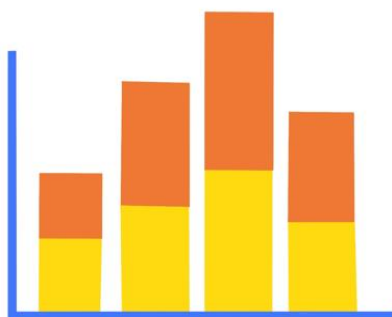
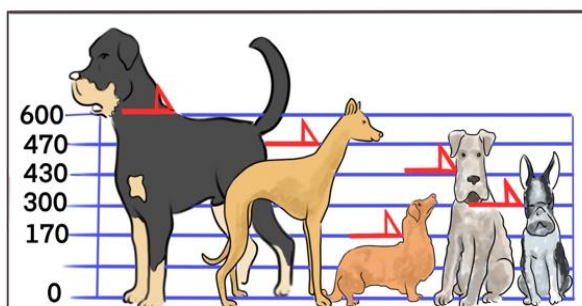
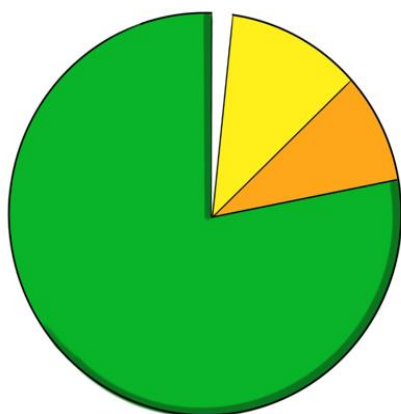
My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Appreciate the importance of exponential models in understanding natural phenomena.					
Think logically when interpreting solutions in context.					
Think logically when resolving domain issues or extraneous solutions.					
Be consistent and persistent when recognizing the versatility of logarithms in mathematical modeling.					
Think logically when Fostering a methodical approach to solving multi-step equations.					
Appreciate the relevance of exponential and logarithmic concepts in everyday life.					
Show interest in tackling real-world problems using mathematical tools					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Think logically when maintaining an open-minded approach to learning from applied problem-solving scenarios					
Be consistent and persistent when interpreting parameters in exponential and logarithmic models.					
Think logically when solving application problems involving growth and decay.					

2. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	1.	1.
2.	2.	2.
3.	3.	3.

UNIT 4: NUMERICAL AND GRAPHICAL METHODS TO DISPLAY DATA



Unit summary

This unit provides you with the knowledge, skills and attitudes required to Apply numerical and graphical methods to display data required to Apply Basic Mathematical Analysis, Statistics and Probability. It covers the Ungrouped quantitative data, Measures of central tendency and Measures of dispersion.

Self-Assessment: Unit 1

1. Referring to the unit illustration above and a picture of dogs above, answer the following questions:
 - a. What is the length of each dog?
 - b. What are the mean heights of the dogs?
 - c. What is the square of each height subtract the mean height?
 - d. Work out the average of those squared differences. What do you notice about the average?

2. Fill out the below self-assessment. Think about yourself: do you think you can do this? How well? Read the statements across the top. Assess your level of knowledge, skills and attitudes under this unit.
 - a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills and attitudes acquired during the learning process
 - b. Think about yourself: do you think you have the knowledge, skills or attitudes to do the task? How well?
 - c. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

4. At the end of this unit, you will assess yourself again.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe the components of pie charts (e.g., sectors, proportions) and bar charts (e.g., axes, bars, scales).					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Explain the principles of designing effective pie charts and bar charts (e.g., proportionate scaling, labeling, and appropriate use cases).					
Explain the concepts of mean, median, and mode, including their formulas and when each is most appropriate.					
Identify the strengths and limitations of mean, median, and mode in different data contexts					
Define the variance, standard deviation and the coefficient of variation					
Analyze and critically interpret data and infer conclusion					
Explain the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Analyze and compare information represented in pie charts and bar charts to extract meaningful insights.					
Construct clear and visually accurate pie charts and bar charts using manual or digital tools.					
Calculate and interpret mean, median, and mode from various data sets.					
Compare and justify the use of different measures of central tendency in analyzing specific datasets					
Determine the measures of dispersion of a given statistical series.					
Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Express the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					
Think logically when analyzing and comparing information represented in pie charts and bar charts to extract meaningful insights.					
Commit to ensuring accuracy and clarity when creating data visualizations to support ethical and transparent reporting.					
Think logically when choosing the appropriate measure of central tendency to ensure accurate data Representation					
Commit to using the most suitable measure to provide fair and meaningful insights while avoiding misleading conclusions					
Appreciate the importance of					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
measures of dispersion in the interpretation of data					
Show concern on how to use the standard deviation as a measure of variability of data.					
Think logically when expressing the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					



Key Competencies:

Knowledge	Skills	Attitudes
1. Describe the components of pie charts (e.g., sectors, proportions) and bar charts (e.g., axes, bars, scales).	1. Analyze and compare information represented in pie charts and bar charts to extract meaningful insights.	1. Think logically when analyzing and comparing information represented in pie charts and bar charts to extract meaningful insights.
2. Explain the principles of designing effective pie charts and bar charts (e.g., proportionate scaling, labeling, and appropriate use cases).	2. Construct clear and visually accurate pie charts and bar charts using manual or digital tools.	2. Commit to ensuring accuracy and clarity when creating data visualizations to support ethical and transparent reporting.

Knowledge	Skills	Attitudes
3. Explain the concepts of mean, median, and mode, including their formulas and when each is most appropriate.	3. Calculate and interpret mean, median, and mode from various data sets.	3. Think logically when choosing the appropriate measure of central tendency to ensure accurate data Representation
4. Identify the strengths and limitations of mean, median, and mode in different data contexts	4. Compare and justify the use of different measures of central tendency in analyzing specific datasets	4. Commit to using the most suitable measure to provide fair and meaningful insights while avoiding misleading conclusions
5. Define the variance, standard deviation and the coefficient of variation	5. Determine the measures of dispersion of a given statistical series.	5. Appreciate the importance of measures of dispersion in the interpretation of data
6. Analyze and critically interpret data and infer conclusion	6. Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data	6. Show concern on how to use the standard deviation as a measure of variability of data.
7. Explain the coefficient of variation as a measure of spread of a set of data as a proportion of its mean	7. Express the coefficient of variation as a measure of spread of a set of data as a proportion of its mean	7. Think logically when expressing the coefficient of variation as a measure of spread of a set of data as a proportion of its mean



Discovery activity:



Task 49

Consider a portfolio that has achieved the following returns: Q1 =+10%, Q2 =-3%, Q3 =+8%, Q4 =+12%, Q5 =-7%, Q6 =+12% and Q7 =+3% over seven quarters.

- What is the average return on investment?
- Which return of the portfolio is in the middle?
- Which return of the portfolio that has been achieved frequently?

Topic 4.1: Ungrouped quantitative data



Activity 1: Problem Solving



Task 50:

The chart below presents data on why teenagers drink. Use the information shown in the chart to answer the following questions:



- For what reason do the highest numbers of teenager's drink?
- What percentage of teenager's drink because they are bored or upset?

Key Facts 4.1.: Ungrouped quantitative data

- Pie diagrams.**

A pie chart, sometimes called a circle chart, is a way of summarizing a set of data in circular graph. This type of chart is a circle that is divided into sections or wedges according to the percentage of frequencies in each category of the distribution. Each part is represented in degrees.

To present data using pie chart, the following steps are respected:

Step 1: Write all the data into a table and add up all the values to get a total.

Step 2: To find the values in the form of a percentage divide each value by the total and multiply by 100. That means that each frequency must also be converted to a percentage by using the formula

$$\% = \frac{f}{n} \times 100\%$$

Step 3: To find how many degrees for each pie sector we need, we take a full circle of 360° and use the formula: Angle of sector = $\frac{\text{frequency of } S \times 360}{\text{total frequency}}$

Since there are 360° in a circle, the frequency for each class must be converted into a proportional part of the circle. This conversion is done by using the formula:

$$\text{Degrees} = \frac{f}{n} \times 360^\circ$$

Where f is frequency for each class and n is the sum of the frequencies. Hence, the following conversions are obtained. The degrees should sum to 360.

Step 4: Once all the degrees for creating a pie chart are calculated, draw a circle (pie chart) using the calculated measurements with the help of a protractor, and label each section with the name and percentages or degrees.

Example 1. In the summer, a survey was conducted among 400 people about their favorite beverages: 2% like cold-drinks, 6% like Iced-tea, 12% like Cold-coffee, 24% like Coffee and 56% like Tea.

- How many people like tea?
- How many more people like coffee than cold coffee?
- What is the total central angle for iced tea and cold-drinks?
- Draw a pie chart to represent the provided information.

Solution: Total number of people = 400

$$\text{a. Number of people like tea} = 400 \times \frac{56}{100} = 224$$

$$\text{b. Number of people like coffee} = 400 \times \frac{24}{100} = 96$$

$$\text{Number of people like cold-coffee} = 400 \times \frac{12}{100} = 48$$

$$\text{Number of people like coffee more than cold-coffee} = 96 - 48 = 48$$

$$\text{Number of people like iced-tea} = 400 \times \frac{6}{100} = 24$$

$$\text{Number of people like cold-drinks} = 400 \times \frac{2}{100} = 8$$

$$\text{c. Central angle for iced-tea} = \frac{24}{400} \times 360^\circ = 21.6^\circ$$

$$\text{Central angle for cold-drinks} = \frac{8}{400} \times 360^\circ = 7.2^\circ$$

$$\text{Total central angle} = 21.6^\circ + 7.2^\circ = 28.8^\circ$$

d.

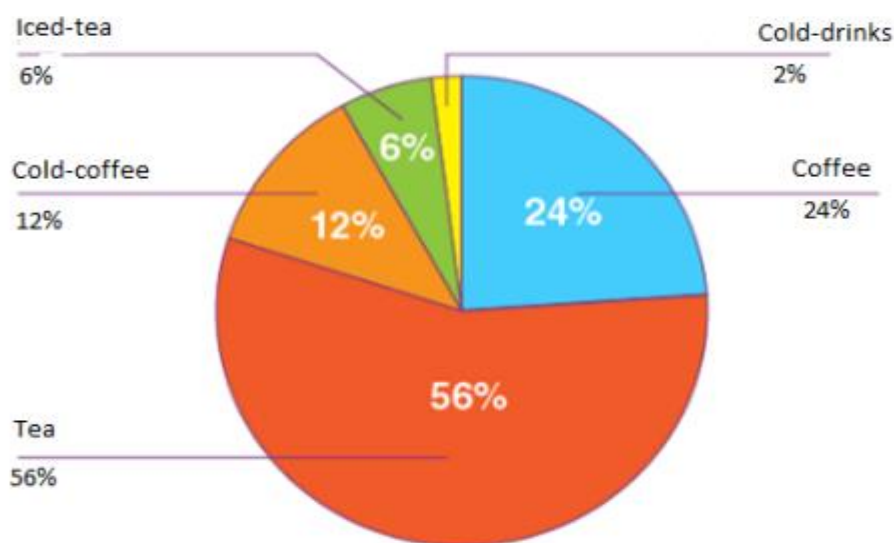


Figure 26: Pie-chart

Example 2: A person spends his time on different activities daily (in hours):

Activities	Office work	exercises	Traveling	Watching shows	Sleeping	Miscellaneous
Number of hours	9	1	2	3	7	2

- Find the central angle and percentage for each activity.
- Draw a pie chart for this information

c. Use the pie chart to comment on these findings.

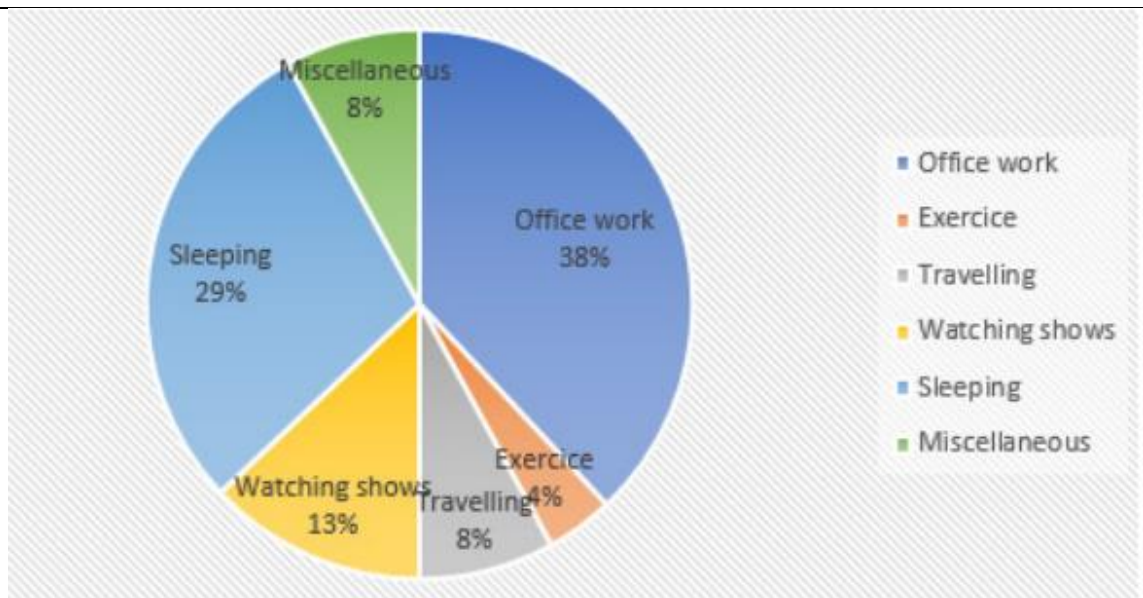
Solution

a. Central angles are calculated by using the formula: $\text{Degrees} = \frac{f}{n} \times 360^\circ$

and percentages calculated by using the formula: $\% = \frac{f}{n} \times 100\%$.

Activities	Number of hours	Central angles	Percentages
Office work	9	$\frac{9}{24} \times 360^\circ = 135^\circ$	$\frac{9}{24} \times 100 = 37.5\% \approx 38\%$
Exercise	1	$\frac{1}{24} \times 360^\circ = 15^\circ$	$\frac{1}{24} \times 100 = 4.16\% \approx 4\%$
Traveling	2	$\frac{2}{24} \times 360^\circ = 30^\circ$	$\frac{2}{24} \times 100 = 8.33\% \approx 8\%$
Watching shows	3	$\frac{3}{24} \times 360^\circ = 45^\circ$	$\frac{3}{24} \times 100 = 12.5\% \approx 13\%$
Sleeping	7	$\frac{7}{24} \times 360^\circ = 105^\circ$	$\frac{7}{24} \times 100 = 29.16\% \approx 29\%$
Miscellaneous	2	$\frac{2}{24} \times 360^\circ = 30^\circ$	$\frac{2}{24} \times 100 = 8.33\% \approx 8\%$
Total	24	360	100

b. Using a protractor, graph each section and write its name and corresponding percentage, as shown in the Figure below



4.1.2. Vertical bar charts and Horizontal bar charts

Bar-chart or bar-graph: is a graph consisting of rectangular bars whose lengths are proportional to the frequencies in a data distribution.

- **Steps in drawing a bar chart**

To draw a bar chart first

- ✓ identify what quantities to represent on the horizontal and vertical axis.
- ✓ Determine the suitable width and uniform gap between the bars. The bars should have the same width and uniform gap between them.
- ✓ Draw the bar chart and shade them with a visible color.

Note: In a vertical bar chart frequency is represented on the vertical axis and on horizontal bar chart frequency is represented on horizontal axis.

Example of vertical bar chart: The graph below shows the sizes of sweaters worn by 30 level 5 students in a certain school. Observe it and interpret it by answering the questions below:

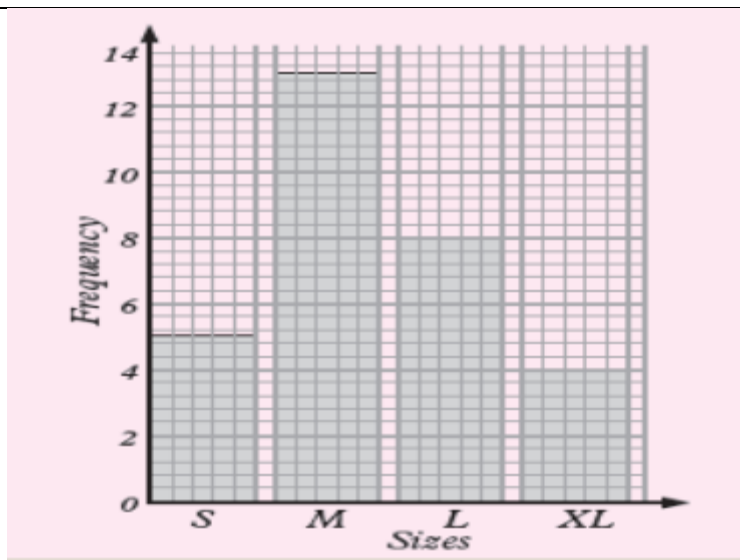


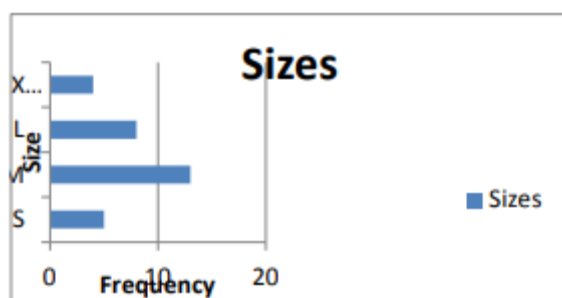
Figure 27: Bar-chart

- How many students are with small size?
- How many students with medium size, large size and extra-large size are there?
- present the given data as horizontal bar chart.

Solution:

- 5 Students
- Students with medium size are 13
Students with large size are 8
Students with extra-large size are 4

a.



It is clearly that the frequencies are in horizontal axis. (REB, 2023)

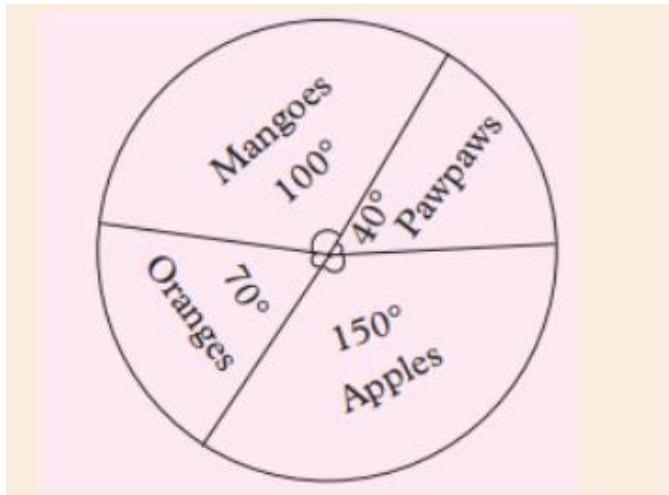


Activity 2: Guided Practice



Task 51:

1. After selling fruits in a market, Aisha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.



- a. Find the total cost of mangoes and paw paws if a mango sells at 30 FRW and pawpaw at 160 FRW each.
 - b. Which types of fruit remained the most?
 - c. Draw a frequency table to display the information on the pie chart
2. In a vertical bar chart, the sales of five different products (A, B, C, D, and E) in thousands are as follows: A: 50, B: 75, C: 120, D: 90, E: 60. How would the chart look if the data were presented in a horizontal bar chart instead, and what differences would you expect in the visualization of the data?
 3. A vertical bar chart shows the number of students enrolled in five different courses (Math, English, History, Science, and Art) as follows: Math: 200, English: 180, History: 150, Science: 220, and Art: 130. If the data were displayed as a horizontal bar chart, which course would have the longest bar, and how would the comparison between Science and Math change in the horizontal format?

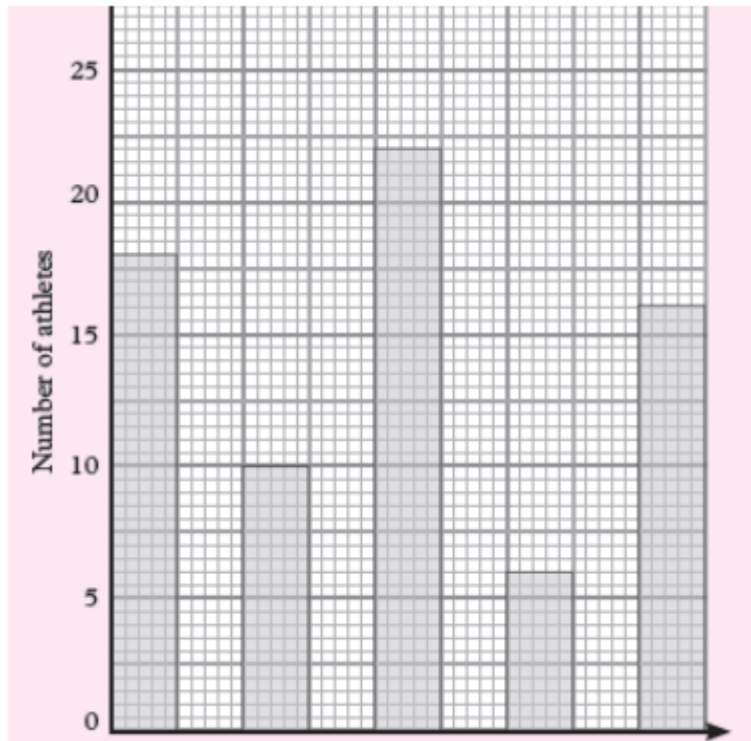


Activity 3: Application



Task 52:

The bar graph shows the number of athletes who represented five African countries in an international championship.



The countries which are on the horizontal axis are Rwanda, Nigeria, Tanzania, Egypt and South Africa.

- What was the total number of athletes representing the five countries?
- What was the smallest number of athletes representing one country?
- What was the most number of athletes representing a country?
- Represent the information on the graph on a frequency table.
- present the given data as horizontal bar chart

Topic 4.2: Measures of central tendency



Activity 1: Problem Solving



Task 53:

In Junior Secondary, you were introduced to statistics. You learnt about measures of central tendencies

- Discuss in groups the meaning of measures of central tendencies.
- What are they?
- Where can we apply them?

Key Facts 4.2: Measures of central tendency

- **Mean**

Mean is the average of a data set.

- ✓ **How to find mean**

The most used measure of central tendency is called mean (or the average). Here the main of interest is to learn how to calculate the mean when the data set is raw data.

The following steps are used to calculate the mean:

Step 1: Add the numbers Step

2: Count how many numbers there are in the data set Step

3: Find the mean by dividing the sum of the data values by the number of data values

Mathematically, mean is calculated as follows: $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$, where n is the number of

observations in the dataset, x_i are observations. Or $\bar{x} = \frac{\sum_{i=1}^n f x_i}{\sum f}$, Here the mean can

also be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values.

- **Median**

The median is the middle value in a set of ranked observations. It is also defined as the middle value in a list of values arranged in either ascending or descending orders.

How to find median

- ✓ Rank the given data sets (in increasing or decreasing order)
- ✓ Find the middle term for the ranked data set that obtained in step 1.
- ✓ The value of this term represents the median. In general form, calculating the median depends on the number of observations (even or odd) in the data set, therefore applying the above steps requires a general formula. Consider the ranked data $x_1, x_2, x_3, \dots, x_n$ the formula for calculating the median for the two cases (even and odd) is given by: If n is odd, Median = $x_{\left(\frac{n+1}{2}\right)}$, or median is given

by $\left(\frac{n+1}{2}\right)^{th}$ number which is located on this position If n is even, Median

$$\frac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}, \text{ or Median is given by } \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2}, \text{ then the median is a half}$$

of the sum of number located on those two positions.

- **Mode**

Mode is the most frequently occurring value in a set of values or (item with highest frequency).

Example 1: To understand the three statistical concepts, consider the following example: A Supermarket recently launched a new mint chocolate chip ice cream flavor. They want to compare customer traffic numbers to their store in the past seven days since the launch to understand whether their new offering intrigued customers. Here is the customer data from last week: Monday =92 customers, Tuesday =92 customers, Wednesday =121 customers, Thursday =120 customers, Friday = 132 customers, Saturday = 118 customers, and Sunday =128 customers. To make sense of this data, we can calculate the average:

✓ Find the sum by adding the customer data together, $92 + 92 + 118 + 120 + 121 + 128 + 132 = 803$

✓ Number of days is equal to 7.

✓ Mean or average is $\frac{803}{7} = 114.714$. This means that mean average of customers

in the past week is 115 customers. The mode is 92 customers because on Monday and Tuesday, 92 customers were received. To find the median, we need to arrange data as follows: 92, 92, 118, 120, 121, 128, 132. Then, the middle value is 120. Therefore, the median is 120. By using the formula, n is equal to 7 which is odd.

The median $= x_{\left(\frac{n+1}{2}\right)} = x_4$. The value at the fourth position in the ranked data above

is 120. Hence, median is 120.

Example 2: Calculate the mean of the pocket money of some 5 students who get 2500 FRW, 4000 FRW, 5500 FRW, 7500FRW and 3000 FRW. Sum all the pocket money of five students = $2500+4000+5500+7500+3000= 22500$ FRW. Divide the sum by the number of students = $22500 / 5= 4500$ FRW. The mean of the pocket money of 5 students is 4500 FRW.

Example 3: The table below shows the marks of 62 students in a test. Find the median

Marks	40	41	42	43	44	45	46	47	48	49	50
N of student s	2	4	6	9	10	12	8	7	2	1	1

Solution: Using the cumulative frequencies

Marks	Number of students	C.F(Cumulative F)
40	2	2
≤ 41	4	6
≤ 42	6	12
≤ 43	9	21
≤ 44	10	31
≤ 45	12	43

≤ 46	8	51
≤ 47	7	58
≤ 48	2	60
≤ 49	1	61
≤ 50	1	62

Because the number of frequency is even number, $Median = \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2} + 1\right)^{th}}{2}$

$$Median = \frac{\left(\frac{62}{2}\right)^{th} + \left(\frac{62}{2} + 1\right)^{th}}{2} = \frac{31^{st} + 32^{nd}}{2} = \frac{44 + 45}{2} = 44.5$$

Exercise

Find the average and median monthly salary (FRW) of all six secretaries each month earn (in thousands) 104, 340, 140, 185, 270, and 258 each, respectively. (REB, 2023)



Activity 2: Guided Practice



Task 54:

Find the mean, mode and median of the following data:

marks	15	25	35	45	55	65	75
Frequency	2	8	10	9	8	4	2



Activity 3: Application



Task 55

A school conducted a survey to determine the number of hour students spend studying each week. The data collected is grouped into classes representing different ranges of study hours, and the frequency represents the number of students in each range. Calculate the mean number of study hours to understand the average study time among the students.

Class	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[
Frequency	1	8	10	9	8	4	2

Topic 4.3: Measures of dispersion



Activity 1: Problem Solving



Task 56:

A fitness center wants to compare the consistency of workout durations among its members across two different classes: Yoga and Cardio. Over a week, they record the following workout durations (in minutes) for each class:

- Yoga Class: 45, 50, 55, 60, 50
- Cardio Class: 30, 45, 50, 70, 65

Using this data:

1. Find the **range** of workout durations for both classes.
2. Calculate the **variance** and **standard deviation** for each class to measure the spread of workout durations.
3. Determine the **coefficient of variation (CV)** for each class to assess the relative variability.
4. Based on your calculations, which class shows more consistent workout durations?

Key Facts 4.3: Measures of dispersion

- **Range**

A **measure of dispersion** is the degrees of spread of observation in data. The common measure of dispersion is range, variance and the standard deviation (the square root of the variance) and Coefficient of variance.

The range is defined as the difference between the largest value in the set of data and the smallest value in the set of data, $X_L - X_s$.

Example: What is the range of the following data?

4 8 1 6 6 2 9 3 6 9

Solution The largest score (X_L) is 9; the smallest score (X_s) is 1; the range is $X_L - X_s$
 $= 9 - 1 = 8$

The range is rarely used in scientific work as it is fairly insensitive.

- ✓ It depends on only two scores in the set of data, (X_L) and (X_S)
- ✓ Two very different sets of data can have the same range

For example: 1 1 1 1 9 and 2 3 4 1 6 9

- **Variance**

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

- ✓ For the population, the variance is denoted and defined by: $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$ **or**

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n} \text{ . where } x \text{ is individual value, } \bar{x} \text{ is the population mean, and } n \text{ is}$$

the population size and f is frequency

- ✓ And also you can calculate variance by using this formula: $\sigma^2 = \frac{\sum x^2}{n} - \bar{x}^2$ **or**

$$\sigma^2 = \frac{\sum fx^2}{n} - \bar{x}^2 \text{ . where } x \text{ is individual value, } \bar{x} \text{ is the population mean, and } n \text{ is}$$

the population size and f is frequency

- ✓ For the sample, the variance is denoted and defined by: $S^2 = \frac{\sum f(x - \bar{x})^2}{n - 1}$, where

x is individual value, $\bar{x}$ is the sample mean, and n is the sample size. where x is

individual value, $\bar{x}$ is the sample mean, and n is the sample size

How to find variance

To calculate the variance, follow these steps:

- ✓ Work out the mean (the simple average of the numbers)

- ✓ Then for each number: subtract the Mean and square the result (the squared difference).
- ✓ Then work out the average of those squared differences.

Example1: The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the variance of the heights.

Solution: $\bar{x} = \frac{\sum x}{n} = \frac{1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30}{6} = 1.39m$

First way by using $\sigma^2 = \frac{\sum x^2}{n} - \bar{x}^2$

$$\text{variance} = \frac{\sum x^2}{n} - \bar{x}^2 = \frac{1.42^2 + 1.35^2 + 1.37^2 + 1.50^2 + 1.38^2 + 1.30^2}{6} - 1.39^2$$

$$\sigma^2 = 0.00386m$$

Second way by using $\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}$

$$\sigma^2 = \frac{(1.42 - 1.39)^2 + (1.35 - 1.39)^2 + (1.37 - 1.39)^2 + (1.50 - 1.39)^2 + (1.38 - 1.39)^2 + (1.30 - 1.39)^2}{6}$$

$$\sigma^2 = 0.00386m$$

2. The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	20-29	30-39	40-49	50-59	60-69	70-79
Number of days in 60 days.	6	12	16	14	8	4

Find the mean and variance of the number of customers served lunch using this grouped data.

Solution

To find the mean from grouped data, first we determine the mid-interval values for all intervals;

First way, we use $\sigma^2 = \frac{\sum fx^2}{n} - \bar{x}^2$

Interval	Mid interval (x)	f	fx	fx ²
20-29	24.5	6	147	3601.5
30-39	34.5	12	414	14283
40-49	44.5	16	712	31684
50-59	54.5	14	763	41583.5
60-69	64.5	8	516	33282
70-79	74.5	4	298	22201
		$\Sigma = 60$	$\Sigma = 2850$	$\Sigma = 146635$

$$\bar{x} = \frac{\sum fx}{\sum x} = \frac{2850}{60} = 47.5$$

$$\sigma^2 = \frac{\sum fx^2}{n} - \bar{x}^2 = \frac{146635}{60} - (47.5)^2$$

$$\sigma^2 = 187.67$$

Second way for calculating variance we use $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$

Interval	Mid interval (x)	f	fx	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
20-29	24.5	6	147	-23	529	3174
30-39	34.5	12	414	-13	169	2028
40-49	44.5	16	712	-3	9	144
50-59	54.5	14	763	7	49	686
60-69	64.5	8	516	17	289	2312
70-79	74.5	4	298	27	729	2916
		$\Sigma = 60$	$\Sigma = 2850$			$\Sigma = 11260$

$$\bar{x} = \frac{\sum fx}{\sum x} = \frac{2850}{60} = 47.5$$

$$\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n} = \frac{11260}{60} = 187.67$$

- **Standard deviation**

The standard deviation, S , is a very important and useful measure of spread. It gives a measure of the deviations of the readings from the mean, $\bar{x}$. It is calculated using all the values in the distribution.

A most used measure of variation is called standard deviation denoted by (σ for the population and S for the sample). The numerical value of this measure helps us to know how the values of the dataset corresponding to such measure are relatively closely around the mean.

Lower value of the standard deviation for a data set, means that the values are spread over a relatively smaller range around the mean. Larger value of the standard deviation for a data set means that the values are spread over a relatively smaller range around the mean.

How to find standard deviation

Take a square root of the variance. The population standard deviation is defined as: square root of the average of the squared differences from the population mean.

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}} \text{ or } \sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \text{ and also we can use } \sigma = \sqrt{\frac{\sum fx^2}{n} - \bar{x}^2} \text{ or}$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \text{ where } x \text{ is individual value, } \bar{x} \text{ is the population mean, and } n \text{ is the population size.}$$

The sample standard deviation is defined as: square root of the average of the squared

differences from the sample mean. $S = \sqrt{\frac{\sum f(x - \bar{x})^2}{n-1}}$ or $S = \sqrt{\frac{\sum (x - \bar{x})^2}{n-1}}$, where x is individual value, $\bar{x}$ is the sample mean, and n is the sample size.

Example 1: The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the standard deviation of the heights.

Solution: $\bar{x} = \frac{\sum x}{n} = \frac{1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30}{6} = 1.39m$

First way by using $\sigma^2 = \frac{\sum x^2}{n} - \bar{x}^2$

$$\sigma^2 = \frac{\sum x^2}{n} - \bar{x}^2 = \frac{1.42^2 + 1.35^2 + 1.37^2 + 1.50^2 + 1.38^2 + 1.30^2}{6} - 1.39^2$$

$$\sigma^2 = 0.00386$$

σ = square root of variance

$$\sigma = \sqrt{0.00386} = 0.062$$

Example 2. The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	20-29	30-39	40-49	50-59	60-69	70-79
Number of days in 60 days.	6	12	16	14	8	4

Find the standard deviation of the number of customers served lunch using this grouped data.

Solution: first we have to calculate the variance we use $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$

Interval	Mid interval (x)	f	f $\bar{x}$	x - $\bar{x}$	(x - $\bar{x}$) ²	f(x - $\bar{x}$) ²
20-29	24.5	6	147	-23	529	3174

30-39	34.5	12	414	-13	169	2028
40-49	44.5	16	712	-3	9	144
50-59	54.5	14	763	7	49	686
60-69	64.5	8	516	17	289	2312
70-79	74.5	4	298	27	729	2916
		$\Sigma = 60$	$\Sigma = 2850$			$\Sigma = 11260$

$$\bar{x} = \frac{\sum fx}{\sum x} = \frac{2850}{60} = 47.5$$

$$\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n} = \frac{11260}{60} = 187.67$$

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$$

$$\sigma = \sqrt{187.67} = 13.67$$

- Coefficient of variation**

A coefficient of variation (CV) is one of well-known measures that used to compare the variability of two different data sets that have different units of measurement.

Moreover, one disadvantage of the standard deviation that its being a measure of absolute variability and not of relative variability. The coefficient of variation, denoted by (CV), expresses standard deviation as a percentage of the mean and is computed as follows: For population data $CV = \frac{\sigma}{\bar{x}} \times 100\%$

For sample data $CV = \frac{s}{\bar{x}} \times 100\%$

For sample data $CV = \frac{s}{\bar{x}} \times 100\%$

Example 1. The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the coefficient of variation of the heights.

Solution: $\bar{x} = \frac{\sum x}{n} = \frac{1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30}{6} = 1.39m$

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{(1.42 - 1.39)^2 + (1.35 - 1.39)^2 + (1.37 - 1.39)^2 + (1.50 - 1.39)^2 + (1.38 - 1.39)^2 + (1.30 - 1.39)^2}{6}$$

$$\sigma^2 = 0.00386m$$

σ = square root of variance

$$\sigma = \sqrt{0.00386} = 0.062$$

$$CV = \frac{\sigma}{\bar{x}} \times 100\%$$

$$CV = \frac{0.062}{1.39} \times 100\% = 0.0446 \times 100\% = 4.5\%$$

$$CV = 4.5\%$$

Example 2. The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	20-29	30-39	40-49	50-59	60-69	70-79
Number of days in 60 days.	6	12	16	14	8	4

Find the coefficient of variance of the number of customers served lunch using this grouped data.

Solution: first we have to calculate mean, variance and standard deviation in order to

get the coefficient of variance we use $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$

Interval	Mid interval (x)	f	f $\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$	f $(x - \bar{x})^2$
----------	------------------	---	-------------	---------------	-------------------	---------------------

20-29	24.5	6	147	-23	529	3174
30-39	34.5	12	414	-13	169	2028
40-49	44.5	16	712	-3	9	144
50-59	54.5	14	763	7	49	686
60-69	64.5	8	516	17	289	2312
70-79	74.5	4	298	27	729	2916
		$\Sigma = 60$	$\Sigma = 2850$			$\Sigma = 11260$

$$\bar{x} = \frac{\sum fx}{\sum x} = \frac{2850}{60} = 47.5$$

$$\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n} = \frac{11260}{60} = 187.67$$

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$$

$$\sigma = \sqrt{187.67} = 13.67$$

$$CV = \frac{\sigma}{\bar{x}} \times 100\%$$

$$CV = \frac{13.67}{47.5} \times 100\% = 0.28779 \times 100\% = 28.78\%$$

Example 3. Two plants C and D of a factory show the following results about the number of workers and the wages paid to them.

	C	D
Number of workers	5000	6000
Average monthly salary	\$2500	\$2500
Standard deviation	9	10

Using coefficient of variation, find in which plant, C or D there is greater variability in individual wages. In which plant would you prefer to invest in?

Solution

To find which plant has greater variability, we need to find the coefficient of variation. The plant that has a higher coefficient of variation will have greater variability.

Coefficient of variation for plant C:

Using coefficient of variation formula,

$$CV = \frac{\sigma}{\bar{x}} \times 100\%$$

$$CV = \frac{9}{2500} \times 100\% = 0.36$$

Now, Coefficient of variation for plant D:

$$CV = \frac{10}{2500} \times 100\% = 0.4$$

Plant C has $CV = 0.36$ and plant D has $CV = 0.4$. Hence plant D has greater variability in individual wages. I would prefer to invest in plant C as it has lower coefficient (of variation) because it provides the most optimal risk-to-reward ratio with low volatility but high returns. (REB, 2023)

Exercises:

Find the mean, variance, standard deviation and coefficient of variance of the following data: 5,4,5,5,4,5,4,4,5 and 3



Activity 2: Guided Practice



Task 57:

By using the formulas like $\bar{x} = \frac{\sum x}{n}$, $\sigma^2 = \frac{\sum f(x - \bar{x})^2}{n}$, $\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$ and

$$CV = \frac{\sigma}{\bar{x}} \times 100\%.$$

- Find the range, mean, variance, standard deviation and coefficient of variance of the following data:
 - 1,3,2,1, 2,5,4,0,2 and 6
 - 3,2,1,5,4,6,0,4,7 and 8
- A small business tracks the monthly sales of their products for the last six months. The sales (in thousands of dollars) are as follows: 50, 60, 70, 65, 55, and 75. How can you

calculate the range, variance, standard deviation, and coefficient of variation for these sales figures to assess the consistency of the business's performance over time?

3. A researcher is studying the test scores of 10 students in two different schools. The scores for School A are: 88, 92, 85, 90, 95, 87, 91, 89, 94, and 90. The scores for School B are: 75, 80, 70, 78, 82, 76, 81, 79, 77, and 73. How can you use range, variance, standard deviation, and coefficient of variation to compare the consistency and spread of test scores between the two schools?



Activity 3: Application



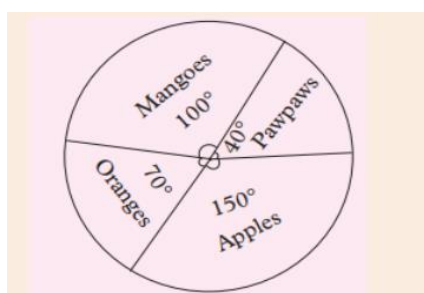
Task 58:

A music school has budgeted to purchase three musical instruments. They plan to purchase a piano costing \$3,000, a guitar costing \$550, and a drum set costing \$600. The mean cost for a piano is \$4,000 with a standard deviation of \$2,500. The mean cost for a guitar is \$500 with a standard deviation of \$200. The mean cost for drums is \$700 with a standard deviation of \$100. Which cost is the lowest, when compared to other instruments of the same type? Which cost is the highest when compared to other instruments of the same type. Justify your answer.



Formative Assessment

1. After selling fruits in a market, Aisha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.



- a. Find the total cost of mangoes and paw paws if a mango sells at 30 FRW and pawpaw at 160 FRW each.
- b. Which types of fruit remained the most?
- c. Draw a frequency table to display the information on the pie chart

2. Find the mean, variance, standard deviation and coefficient of variance of the following data: a. 1, 5, 6, 7, 6, 4, 2, 6 and 3 b. 8, 7, 6, 8, 6, 5, 6, 4 and 1
3. Calculate the variance, standard deviation, and coefficient of the variation of the following data:

Class	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[
Frequency	1	8	10	9	8	4	2

4. The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	24.5	34.5	44.5	54.5	64.5	74.5
Number of days in 60 days.	6	12	12	18	8	4

5. Find the mean, mode and median of the number of customers served lunch using this grouped data.
- a. The mean of the following frequency distribution is 3.66, find the value of A.

X_i	1	2	3	4	5	6
Frequency	3	9	A	11	8	7

- b. The mean of the numbers 12, 18, 21, c, 13 is 17. Find the value of c



Points to Remember

- Pie chart
- Bar chart
- Mode
- Mean
- Median
- Range
- Variance
- Standard deviation
- Coefficient of variation



Self-Reflection

- a. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Describe the components of pie charts (e.g., sectors, proportions) and bar charts (e.g., axes, bars, scales).					
Explain the principles of designing effective pie charts and bar charts (e.g., proportionate scaling, labeling, and appropriate use cases).					
Explain the concepts of mean, median, and mode, including their formulas and when each is most appropriate.					
Identify the strengths and limitations of mean, median, and mode in different data contexts					
Define the variance, standard deviation and the coefficient of variation					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Analyze and critically interpret data and infer conclusion					
Explain the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					
Analyze and compare information represented in pie charts and bar charts to extract meaningful insights.					
Construct clear and visually accurate pie charts and bar charts using manual or digital tools.					
Calculate and interpret mean, median, and mode from various data sets.					
Compare and justify the use of different measures of central tendency in analyzing specific datasets					
Determine the measures of dispersion					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
of a given statistical series.					
Apply and explain the standard deviation as the more convenient measure of the variability in the interpretation of data					
Express the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					
Think logically when analyzing and comparing information represented in pie charts and bar charts to extract meaningful insights.					
Commit to ensuring accuracy and clarity when creating data visualizations to support ethical and transparent reporting.					
Think logically when choosing the					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
appropriate measure of central tendency to ensure accurate data Representation					
Commit to using the most suitable measure to provide fair and meaningful insights while avoiding misleading conclusions					
Appreciate the importance of measures of dispersion in the interpretation of data					
Show concern on how to use the standard deviation as a measure of variability of data.					
Think logically when expressing the coefficient of variation as a measure of spread of a set of data as a proportion of its mean					

b. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	1.	1.
2.	2.	2.
3.	3.	3.

UNIT 5: FUNDAMENTALS OF PROBABILITIES

Playing Cards	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Spades:													
Diamonds:													
Hearts:													
Clubs:													

Unit summary

This unit provides you with the knowledge, skills and attitudes required to Apply fundamentals of probabilities required to Apply Basic Mathematical Analysis, Statistics and Probability. It covers the Counting techniques, Probabilities and Conditional probability.

Self-Assessment: Unit 1

Consider the deck of 52 playing card and Referring to the unit illustration above and a picture of cards above, answer the following questions:

1. How many possibilities do you have for the cards to be chosen?
2. How many possibilities do you have for the kings to be chosen?
3. How many possibilities do you have for the aces of hearts to be chosen?
4. Fill out the below self-assessment. Think about yourself: do you think you can do this? How well? Read the statements across the top. Assess your level of knowledge, skills and attitudes under this unit.
 - a. There is no right or wrong way to answer this assessment. It is for your own reference and self-reflection on the knowledge, skills and attitudes acquired during the learning process
 - b. Think about yourself: do you think you have the knowledge, skills or attitudes to do the task? How well?
 - c. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.
5. At the end of this unit, you will assess yourself again.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Recognize and describe the relationships between sets using Venn diagrams.					
Identify the structure and components of a tree diagram.					
Explain the multiplication principle as a method to					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
determine the total number of outcomes.					
Define permutations and differentiate between ordered and unordered arrangements.					
Distinguish between permutations and combinations.					
Understand key terms (e.g., random experiment, sample space, events, complementary event, etc.) in probability.					
Recognize formulas and methods for calculating probabilities of various types of events.					
Explain the concept of conditional probability and its dependence on prior events.					
Identify criteria for events to be considered independent.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Understand how tree diagrams integrate probabilities at each branch.					
Construct and interpret Venn diagrams to solve set-related problems					
Create and use tree diagrams to represent sequential events and calculate probabilities					
Apply the multiplication principle in scenarios involving multiple stages or choices					
Calculate permutations for given scenarios, including those with repetition or restrictions.					
Compute combinations to solve problems involving selection without regard to order.					
Use precise terminology to describe and analyze probabilistic scenarios					
Solve probability problems involving					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
simple, complementary, mutually exclusive, and inclusive events					
Calculate conditional probabilities using appropriate formulas or data representations					
Verify the independence of events and calculate probabilities for independent scenarios.					
Use tree diagrams to calculate the overall probability of compound events					
Appreciate the clarity and visual utility of organizing set relationships.					
Value the systematic approach of tree diagrams in solving probability problems.					
Be consistent and persistent when using systematic rules to simplify complex counting problems					
Appreciate the importance of order in					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
arranging items in real-life contexts.					
Think logically when Computing combinations to solve problems involving selection without regard to order.					
Cultivate an appreciation for the rigor and clarity of mathematical language					
Be consistent and persistent when solving probability problems using systematic calculations.					
Think logically when considering the influence of one event on another					
Appreciate the simplicity and power of independence in probability models					
Value the clarity and efficiency of visualizing probabilities in tree diagrams.					



Key Competencies:

Knowledge	Skills	Attitudes
1. Recognize and describe the relationships between sets using Venn diagrams.	1. Construct and interpret Venn diagrams to solve set-related problems	1. Appreciate the clarity and visual utility of organizing set relationships.
2. Identify the structure and components of a tree diagram.	2. Create and use tree diagrams to represent sequential events and calculate probabilities	2. Value the systematic approach of tree diagrams in solving probability problems.
3. Explain the multiplication principle as a method to determine the total number of outcomes.	3. Apply the multiplication principle in scenarios involving multiple stages or choices	3. Be consistent and persistent when using systematic rules to simplify complex counting problems
4. Define permutations and differentiate between ordered and unordered arrangements.	4. Calculate permutations for given scenarios, including those with repetition or restrictions.	4. Appreciate the importance of order in arranging items in real-life contexts.
5. Distinguish between permutations and combinations.	5. Compute combinations to solve problems involving selection without regard to order.	5. Think logically when Computing combinations to solve problems involving selection without regard to order .
6. Understand key terms (e.g., random experiment, sample space, events, complementary event, etc.) in probability.	6. Use terminologies to describe and analyze probabilistic scenarios	6. Cultivate an appreciation for the rigor and clarity of mathematical language
7. Recognize formulas and methods for calculating probabilities of various types of events.	7. Solve probability problems involving simple, complementary, mutually exclusive, and inclusive events	7. Be consistent and persistent when solving probability problems using systematic calculations.

Knowledge	Skills	Attitudes
8. Explain the concept of conditional probability and its dependence on prior events.	8. Calculate conditional probabilities using appropriate formulas or data representations	8. Think logically when considering the influence of one event on another
9. Identify criteria for events to be considered independent.	9. Verify the independence of events and calculate probabilities for independent scenarios.	9. Appreciate the simplicity and power of independence in probability models
10. Understand how tree diagrams integrate probabilities at each branch.	10. Use tree diagrams to calculate the overall probability of compound events	10. Value the clarity and efficiency of visualizing probabilities in tree diagrams.



Discovery activity:

In a survey of 50 people about which Hotels they patronize among Hilltop, Serena, and Lemigo. We find that 15 people eat at Hilltop, 30 people eat at Serena, 19 people eat at Lemigo 8 people eat at Hilltop and Serena, 12 people eat at Hilltop and Lemigo, 7 people eat at Serena and Lemigo. 5 people eat at Hilltop, Serena, and Lemigo.



Task 59

1. What is the chance that a person selected at random eats only at Hilltop?
2. How many eat at Hilltop and Serena, but not at Lemigo?
3. How many people don't eat at any of these three hotels?
4. What is the probability that a person selected at random do not eat at any of the hotels mentioned?

Topic 5.1: Counting techniques

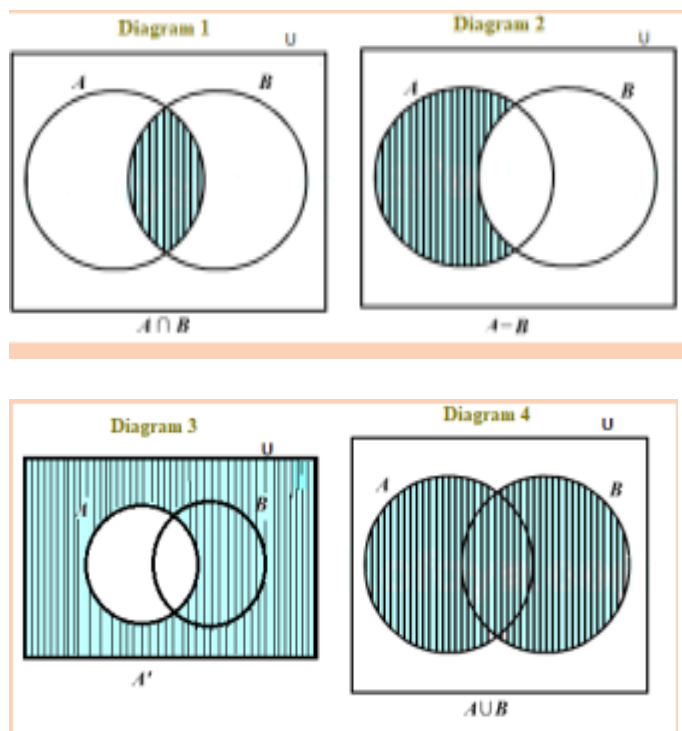


Activity 1: Problem Solving



Task 60:

Consider a class of students as a universal set. Set A is the set of all students who were present in the Auditing class, while Set B is the set of all students who were present in the Taxation class. It is obviously that there were students who were present in both classes as well as those who were present in one class but not in the other class. The shaded part in the Venn diagrams below shows different scenarios on the presence of the students in class.



Observe the diagrams and identify which one to represent the following:

1. All students who were absent in the Auditing class
2. All students who were present in at least one of the two classes.
3. All the students who were present for both Auditing as well as Taxation classes.
4. All the students who have attended only the Auditing class and not the Taxation class

Key Facts 5.1: Counting techniques

- **Venn diagram.**

- ✓ **Counting techniques**

Combinatory, also known as combinatorial analysis, is the area of mathematics concerned with counting strategies to calculate the ways in which objects can be arranged to satisfy given conditions.

✚ **A Venn Diagram** is an illustration that shows logical relationships between two or more sets (grouping items) using closed geometrical figures. Commonly, Venn diagrams show how given items are similar and different. The capital letter outside the circle denotes the name of the set while the small letters inside the circle denote the elements of the set. When drawing Venn diagrams, some important facts like “**intersection**”, “**union**” and “**complement**” should be well considered and represented.

Examples: 1. The following diagram is a Venn diagram representing set A, B and C

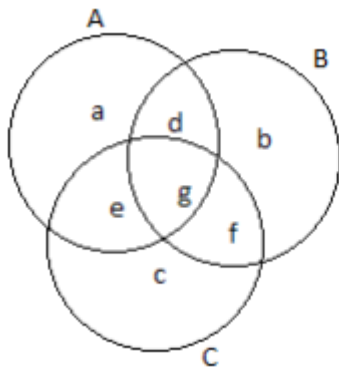


Figure 28: Venn diagram

The above figure is a representation of a Venn diagram where each of the circles A, B and C represents a set of elements.

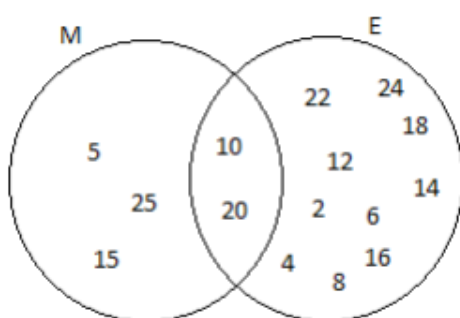
- Set A has the elements a, d, e and g
- Set B has the elements b, d, g and f.
- Set C has the elements e, g, f and c.
- Both A and B have the elements d and g.

- e. Both B and C have the elements g and f.
- f. Both C and A have the elements e and g.
- g. A, B and C all have the element g.

1. Present a Venn diagram showing the correlation between set contains even number from 1 to 25 and the set contains numbers of multiples 5 from 1 to 25.

Solutions

The following Venn diagram shows the correlation between set contains even numbers from 1 to 25 and the other set contains numbers in the multiples of 5 from 1 to 25. Set M represent multiple of 5 from 1 to 25 while set E represent even number from 1 to 25

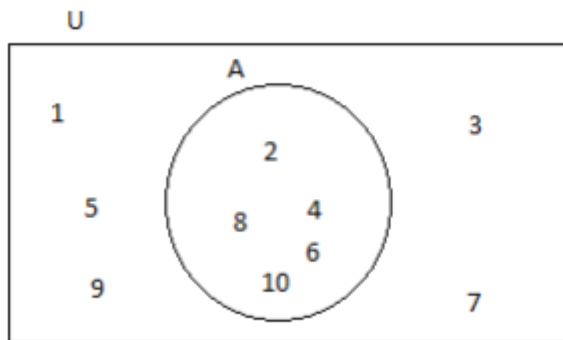


In the above Venn diagram, the elements 10 and 20 are both even and multiple of 5 from 1 to 25.

Universal set

The universal set is the set of all elements or members of all related sets in each scenario. It consists of all the elements of its subsets, including its own elements. It is usually denoted by the symbol U. A universal set can be either a finite or infinite set. The universal set “U” is represented by a rectangle and its subsets “A, B, C, ...” represented by circles.

For example, 2:



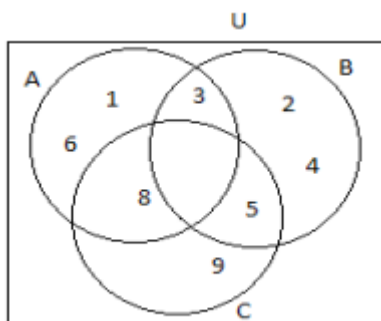
The above diagram shows the universal set U with the elements {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and the subset of this universal set is the set $A = \{2, 4, 6, 8, 10\}$.

Example 3. Given the three sets named A , B , and C . The elements of all sets are defined as: $A = \{1, 3, 6, 8\}$, $B = \{2, 3, 4, 5\}$, $C = \{5, 8, 9\}$

- Represent the information on Venn diagram
- Find out the universal set
- Comment on your findings by showing the nature of each element in the universal set.

Solutions:

- The given information is presented on Venn diagram as follows:



- By definition, a universal set includes all elements of the given sets. Therefore, a Universal set for sets A , B and C is $U = \{1, 2, 3, 4, 5, 6, 8, 9\}$

- c. From this example, the elements of sets A, B and C are altogether available in Universal set “U”. Also, there is no repeated element in the universal set as all the elements are unique.

Operations on sets of numbers

From the above activity, two or more sets can be represented using one Venn diagram and from the representations, different sets can be determined. These consist of ways or operations whereby sets are combined to obtain other sets of interest.

The operations on sets are:

- ✓ Intersection of sets
- ✓ Union of sets - Simple difference of sets
- ✓ Symmetric difference of sets
- ✓ Complement of sets

a. The intersection of sets

The common elements which appear in two or more sets form the intersection of sets. The symbol used to denote the intersection of sets is $\cap$. The intersection of sets A and B is denoted by $A \cap B$ and consists of those elements which belong to A and B that is $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Example: Given that set A = {the first letters of the alphabet} and set B = {all the vowels};

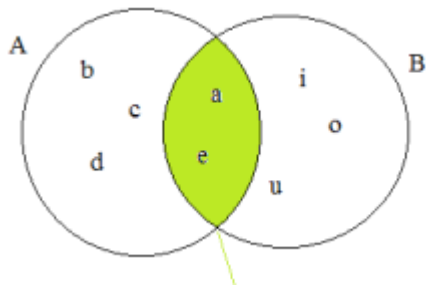
- i. List the elements of each set.
- ii. Find $A \cap B$
- iii. Draw a Venn diagram to represent set $A \cap B$

Solution:

i. $A = \{a, b, c, d, e\}$, $B = \{a, e, i, o, u\}$

ii. $A \cap B = \{a, b, c, d, e\} \cap \{a, e, i, o, u\} = \{a, e\}$

iii The Venn diagram for intersection of sets A and b is as shown below



b. The union of sets

Elements of two or more sets can be put together to form a set. The set formed is known as the union of sets. The symbol for the union of sets is $\cup$. The union of two set A and B, is denoted by $A \cup B$ and consists of all the elements which are members of either A or B or both A and B that is $A \cup B = \{x / x \in A \text{ or } x \in B\}$

Example: Given the following sets $A = \{a, b, c, d, e, f\}$ and $B = \{a, b, c, h, i\}$

a. Find the number of elements of the following sets A, B and $A \cup B$:

i. $n(A)$

ii. $n(B)$

iii. $n(A \cup B)$

b. Draw Venn diagrams to represent $A \cup B$

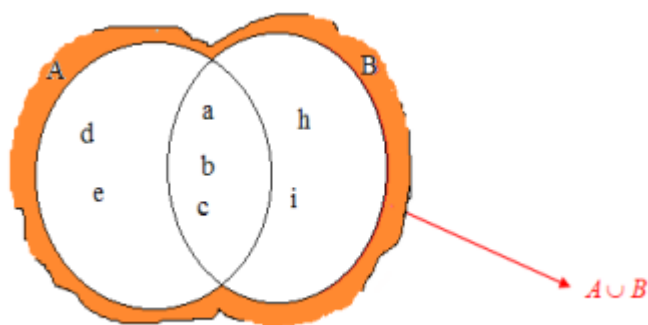
Solutions:

a. i. $n(A) = 6$

ii. $n(B) = 5$

iii. $A \cup B = \{a, b, c, d, e, f\} \cup \{a, b, c, h, i\} = \{a, b, c, d, e, f, h, i\}$
 $n(A \cup B) = 8$

b. The Venn diagram for union of sets A and b is as shown below



c. Difference and Symmetric difference of sets

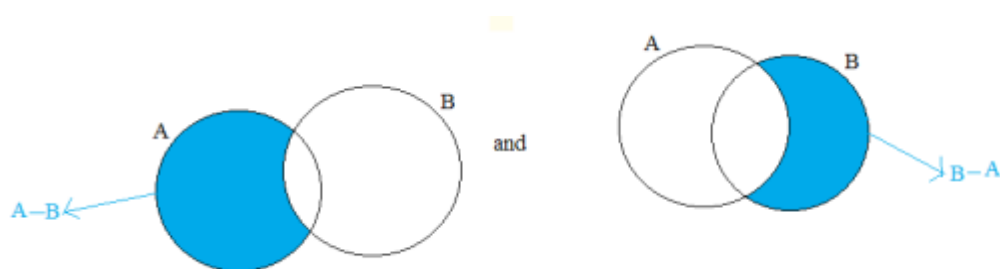
1. Difference of sets

Difference between sets A and B written as $A - B$ or $A \setminus B$ is the set of the elements of set A which are not in set B. It means that $A - B = \{x / x \in A, x \notin B\}$

Likewise, $B - A$ or $B \setminus A$ is difference between sets B and A . This is the set of elements that are in set B and not in set A. It means that

$$B - A = \{x / x \in B, x \notin A\}$$

$A - B$ and $B - A$ can be shown using a Venn diagram as follows:

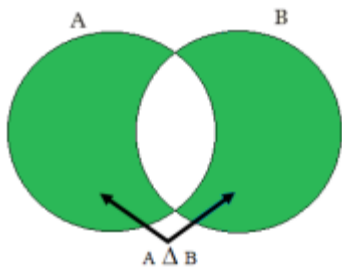


a. Symmetric difference of sets

The union of sets $A - B$ and $B - A$ is known as the symmetric difference between sets A and B. It is written in symbols as $A \Delta B$ to mean

$$A \Delta B = (A - B) \cup (B - A) \text{ or } A \Delta B = (A \cup B) - (A \cap B)$$

$A \Delta B$ can be shown using a Venn diagram as follows:



Example: Given that $A = \{3,4,5,6,7,8\}$ and $B = \{2,4,8,12\}$, find:

- a. $A - B$
- b. $B - A$
- c. $A \Delta B$

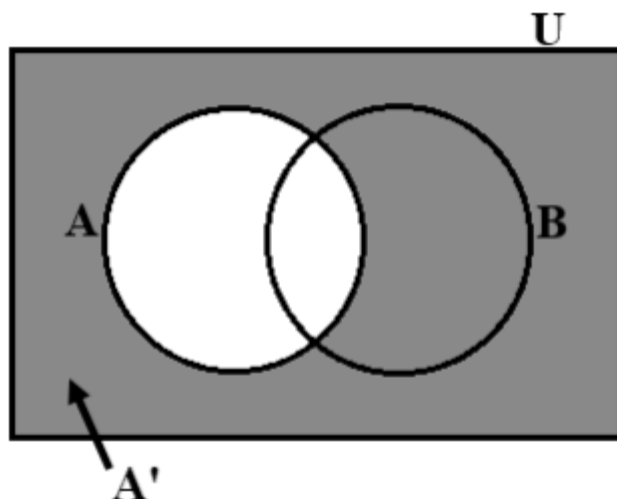
Solutions:

- a. $A - B = \{3,5,6,7\}$
- b. $B - A = \{2,12\}$
- c. $A \Delta B = \{2,3,5,6,7,12\}$

2. The complement of a set

Complement of a set is the set of all elements in the universal set that are not members of a given set. The symbol for the universal set is U .

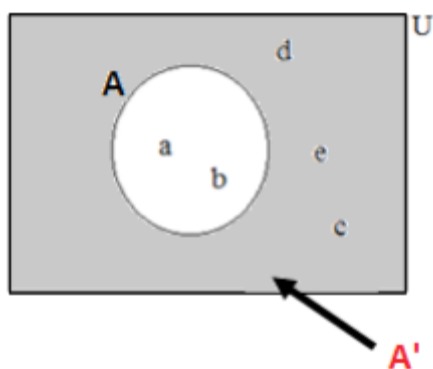
The complement of A is denoted by A' or $\bar{A}$ and consist of all those elements in the universal set which do not belong to A that is $A' = \{x / x \in U, x \notin A\}$



Example Given that $U = \{a, b, c, d, e\}$ and $A = \{a, b\}$, find A' where $A' = U \setminus A$ is the complement of A.

Solution: $U = \{a, b, c, d, e\}$, $A = \{a, b\}$ and $A' = \{c, d, e\}$

This can be represented on a Venn diagram as shown by the diagram.



Exercises:

- Consider these two sets $A = \{2, 4, 6, 8, 10\}$ and $B = \{2, 3, 5, 7\}$. Represent them in a Venn diagram
- Let U be the universal set containing all the natural numbers between 0 and 11. Hence, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let P be the set containing all prime numbers between 0 and 11. Thus, $P = \{2, 3, 5, 7\}$. Let E be the set containing all the even numbers between 0 and 11. Hence, $E = \{2, 4, 6, 8, 10\}$.
 - Represent the above situations using a Venn diagram
 - Comment on the Venn diagram in (a)
- Consider the sets $A = \{1, 2, 3, 5, 6, 8\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 3, 5, 7\}$. Find and draw the Venn diagrams for: a. $B \cap C$ b. $A \cap C$ c. $B \cap A$ d. $B \cup C$ e. $B \cap A$.
- Given U , {letters of the word elephantiasis}, Set $A = \{\text{all vowels}\}$, Set $B = \{\text{first five letters of the English alphabet}\}$. Find:
 - A
 - B
 - $A - B$
 - $B - A$
 - $A \Delta B$
 - A'
- If $U = \{a, e, i, o, u, c, d\}$, $X = \{a, b, e\}$ and $Y = \{c, d, e\}$, find: a. $(X \cap Y)'$ b. $(X \cup Y)'$

(REB, 2022)

- **Tree diagram**

A tree diagram is a diagram with a structure of branching connecting lines representing a relationship. It can be used to find the number of possible outcomes of experiments where each experiment occurs in a finite number of ways. For example, when you toss a coin, the outcome is either head or tail. A second toss would also give head or tail. We represent this as:

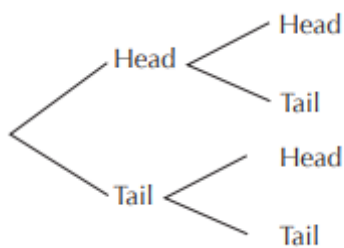
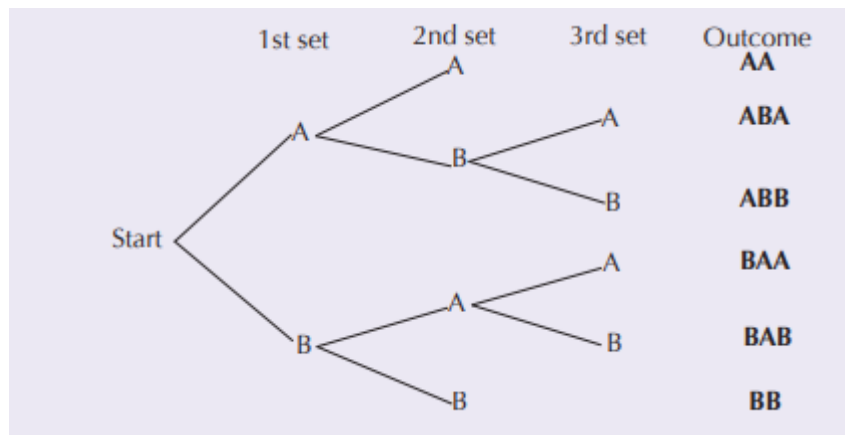


Figure 29: Tree-diagram

If H is head and T is tail, the possible outcomes are $\{HH, HT, TH, TT\} = 4$.

Example: Akimana and Bisangwa play a game of tennis in which the first person to win two sets becomes the winner. In how many different ways can this be done?



Thus the number of possible outcomes of the game is 6.

Exercise: A coin is tossed 3 times, by using tree diagram what are the possible outcomes?

Contingency table

This is a method of presenting the frequencies of the outcomes from an experiment in which the observations in the sample are categorised according to two criteria. Each cell of the table gives the number of occurrences for a particular combination of categories. An example of contingency table in which individuals are categorised by gender and performance is shown below.

	Male	Female	Total
Pass	32	43	75
Fail	8	15	23
	40	58	98

A final column giving the row sums and a final row giving the column sums may be added. Then the sum of the final column and the sum of the final row both equal the number in the sample.

If the sample is categorized according to three or more criteria, the information can be presented similarly in a number of such tables. (Ndorimana, 2016)

- **Multiplication principle**

This is also known as product rule or product principle or multiplication principle. Suppose that an experiment (operation) is to be performed in two successive ways. If there are n_1 different ways of performing the first operation and for each of these there are n_2 different ways of performing a second independent operation, then the two operations in succession can be performed in $n_1 \times n_2$ different ways. More generally, if the operation is composed of k successive steps which may be performed in $n_1, n_2, n_3, \dots, n_k$ distinct ways, respectively, then the operation may be performed in $n_1 \times n_2 \times n_3 \times \dots \times n_k$ distinct ways

Examples:

1. There are 3 different paths from Eric to Jane and 2 different paths from Jane to John. How many different pathways could be taken from Eric to John via Jane?

2. Solution: The number of different pathways is $3 \times 2 = 6$
3. Habimana has 4 shirts, 3 pair of trousers and 2 pairs of shoes. He chooses a shirt, a pair of trousers and a pair of shoes to wear every day. Find the maxim number of days he does not need to repeat his clothing.
4. Solution: The number of all possible different clothing is $4 \times 3 \times 2 = 24$. Therefore, the maximum number of days Habimana does not need to repeat his clothing is 24.
5. Find how many ways the first three places (with no ties) can be filled in a race with 5 contestants
6. Solution: The first place can be filled in 5 ways, since any contestant can come first.
7. When the first place has been filled, there are 4 more contestants to choose from for the second place. Hence the second place can be filled in 5×4 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 3 contestants, and the total number of ways is $5 \times 4 \times 3 = 60$ (Ndorimana, 2016)

- **Arrangements and Permutations**

Activity Imagine that you are a photographer. You want to take a photograph of a group of say 7 people. In how many different ways can they be arranged in a single row?

Factorial notation

The factorial notation of n integers denoted by $n!$ is the product of the first consecutive integers and is read as 'factorial n '. Thus, $n! = n \times (n - 1) \times (n - 2) \times \dots \times 4 \times 3 \times 2 \times 1$.

Thus, for example $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Note: $0! = 1$

Activity: Discuss in groups of 5: in how many different ways can a group of 3 students be arranged to sit in a row? Share your findings with the rest of the class.

Arrangements without repetition

Suppose that two of the three different pictures, A, B and C are to be hung, in line, on a wall of three places. The pictures can be hung in different orders: A B, A C, B A, B C,

CA, C B Each of these orders is a particular arrangement of the pictures and is called a permutation.

Thus, a permutation is an ordered arrangement of the items in a set. Here there are 6

permutations, or $\frac{3!}{(3-2)!} = \frac{3 \times 2 \times 1}{1} = 6$

The number of arrangements of r objects, taken from n unlike objects, can be considered as the number of ways of filling r places in order by the n given objects and

is given by $P_r^n = \frac{n!}{(n-r)!}$.

Note: A different notation can be used of permutation with no repetition:

$$P_r^n = P_{(n,r)} = {}_n P_r = {}^n P_r$$

Condition: $n \geq r$

Example 1: Find the number of permutations of 6 objects taken 3 at time.

Solution: The first object can be put in any of 6 places; following this, the second object can be put in any of 5 remaining places; following this, the last object can be put in any of 4 remaining places. Thus, by the product principle, there are $6 \times 5 \times 4 = 120$ possible permutations of 6 objects taken 3 at time. Or

$$P_3^6 = \frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120.$$

Example 2: Find how many ways the first three places (with no ties) can be filled in a race with 15 contestants.

Solution: The first place can be filled in 15 ways, since any contestant can come first. When the first place has been filled, there are 14 more contestants to choose from for the second place. Hence the first two places can be filled in 15×14 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 13 contestants, and the total number of ways is $15 \times 14 \times 13 = 2730$. Alternatively, from the formula

$$P_3^{15} = \frac{15!}{(15-3)!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 15 \times 14 \times 13 = 2730$$

Note: If $n = r$, this requires us to arrange n different objects in n different places and

the number of different ways is: $p_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$.

Example: How many different ways can 10 students be arranged in a class of 10 places? **Solution:** The number of different ways is $10! = 3,628,800$ ways

Arrangements with repetition

If repetition is allowed, each place can be filled by the objects in n different ways. For example, to the six arrangements of A, B, C without repetitions (AB, AC, BA, BC, CA, and CB) are added the three with repetitions (AA, BB, and CC), for a total of 9, which is equal to 3^2 . Thus, the number of arrangements of r objects, taken from n unlike objects where each of which may be repeated any number of times is n^r .

Example 1: Find how many three letter codes can be formed from an alphabet of 26 letters (assuming that letters can be repeated).

Solution: The first letter of the code may be any of the 26 in the alphabet. The same is true for the second letter and for the third letter. Therefore, the total number of codes is $26 \times 26 \times 26 = 26^3 = 1,757,626$

Example 2: Three schools have teams of six or more runners in a cross country race. In how many ways can the first six places be taken by the three schools, if there are no dead heats?

Solution: The first place can be taken by any one of the three schools. When the first runner has come in, the second place can be taken by any of the three schools. Continuing the argument for the third, fourth, fifth and sixth, the first six places may be taken by the three schools in $3^6 = 729$ ways.

Arrangement in a line

The number of permutation in a line is $n!$

Example: Find the number of permutations of 4 objects in a line (all taken at a time).

Solution: The first object can be arranged in any of the 4 different places. The second object can be arranged in three remaining places and the third object can be arranged in 2 remaining places and finally the last object can be arranged in 1 remaining place. Thus, by the product principle, the number of permutations of the 4 objects is $4 \times 3 \times 2 \times 1 = 24$. Therefore, the number of permutations of 4 objects is $4! = 24$.

Arrangement in a circle

Observe the arrangement of four letters A, B, C and D on a circle as shown below.

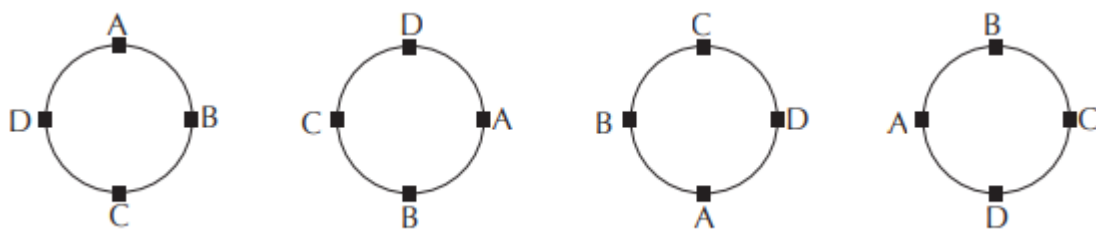


Figure 30: Arrangement in a circle

We can say that for all the arrangements:

- A is on the opposite side of C and its neighbours are B and D, and
- B is on the opposite side of D and its neighbours are A and C. Thus all of the four arrangements are the same. In arranging 4 different objects on a circle, 4 are indistinguishable. Therefore, the number of different ways of arranging 4 objects on a circle is $\frac{4!}{4} = 3!$.

In general, the number of different ways of arranging n objects on a circle is $(n-1)!$

Example: In how many different ways can 5 people be arranged in a circle?

Solution: The number of different ways of arranging 5 people is $(5-1)! = 4! = 24$.

Conditional arrangements

Sometimes arrangements may have certain restrictions which should be dealt with first.

Example 1. How many arrangements of the letters in the word GROUP start with a vowel? **Solution:** Since we have two vowels, there are only 2 possibilities to choose the first letter. When the first has been chosen, the second can be any one of the remaining 4 letters (including the vowel not chosen), and 3 for the third, 2 for the fourth and 1 for the last. Therefore, the number of arrangements in the word GROUP starting with a vowel is $2 \times 4 \times 3 \times 2 \times 1 = 2(4!) = 48$

Example 2. Four boys and two girls are to sit in a row. The two girls, Mukaneza and Mukakarangwa insist on sitting together. In how many different ways can the six students be arranged?

Solution: As Mukaneza and Mukakarangwa must sit together we first treat them as one unit i.e. we have only five items to arrange, four boys and a pair of girls. This is done in $5!$ ways. In any one of these arrangements, the two girls can be arranged two ways among themselves i.e. $2!$ Thus, the total number of arrangements is $2! \times 5! = 240$.

Arrangement with indistinguishable elements

In arranging objects, some sets may contain certain elements that are indistinguishable from each other. Below is how we find the number of arrangements.

The number of ways of arranging n objects in a line, of which p are alike, is $\frac{n!}{p!}$.

Example: How many different words of 5 letters (not necessarily sensible), can be formed from the letters of the words

(a) MATHS

(b) POPPY

Solution (a) All the letters are distinguishable, so the number of different words is $5! = 120$

(a) The 3 'p's are indistinguishable, so the number of different words is $\frac{5!}{3!} = 5 \times 4 = 20$.

In general, The number of ways of arranging in a line n objects of which p of one type are alike, q of a second type are alike, r of a third type are alike, and so on, is $\frac{n!}{p!q!r!}$.

Example 1: In how many different ways can 4 identical red balls, 3 identical green balls and 2 yellow balls be arranged in a row?

Solution: The number of ways is $\frac{9!}{4!3!2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 6 \times 2} = 9 \times 4 \times 7 \times 5 = 1260$

Exercises: 1. Six people want to take a photo. Find how many different ways they can stand on a line.

2. How many different ways can 11 people be arranged at a round table?

3. How many 3-digit numbers can be made from the figures 1, 2, 3, 4, 5 when

- a. repetitions are allowed
- b. Repetitions are not allowed?

4. Find the number of ways of arranging five different books in a row in a bookshelf.

5. There are ten teams in the local football competition. In how many ways can the first four places in the premiership table be filled?

6. In how many different ways can five identical blue balls, two identical red balls and a yellow ball be arranged in a row?

7. In how many ways can three books be distributed among ten people if

- a. each person may receive any number of books
- b. no person may be given more than one book (Ndorimana, 2016)

- **Combination**

Definitions and properties

Combination is a selection of objects where the order is not taken into account

- the order in which the objects are arranged is not important. When considering the number of combinations of r objects, the order in which they are placed is not important. Note that ABC, ACB, BAC, BCA, CAB, CBA are different arrangements, but they represent the same combination of letters. Denoting the number of

combinations of three letters from the seven letters A, B, C, D, E, F, G, by 7C_3 then

$${}^7C_3 \times 3! = {}^7P_3$$

$${}^7C_3 = {}^7P_3 / 3! = \frac{7!}{3!4!} = 35$$

In general, the number of combinations of r objects from n unlike objects is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \text{ Condition } n \geq r$$

Note: Other notations that can be used are ${}_nC_r = C_r^n = C_{(n,r)} = {}^nC_r = \binom{n}{r}$.

Some properties are

$$1) \binom{n}{r} = \binom{n}{n-r}$$

$$2) \binom{n}{n} = \binom{n}{0} = 1$$

$$3) \binom{n}{1} = n$$

There is a relationship between combinations and arrangements:

$$\binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!r!}$$

Example: How many different committees of 3 people can be chosen from a group of 12 people? Solution The number of committees is

$$\binom{12}{3} = \frac{12!}{(12-3)!3!} = \frac{12!}{9!3!} = \frac{12 \times 11 \times 10 \times 9!}{9! \times 6} = 2 \times 11 \times 10 = 220$$

Conditional combination

Sometimes we are given a condition that must be taken into account in a combination.

Example 1: A committee of 5 is to be chosen from 12 men and 8 women. In how many ways can this be done if there are to be 3 men and 2 women on the committee?

Solution: The number of ways of choosing the men is $\binom{12}{3} = \frac{12!}{(12-3)!3!} = 220$

The number of ways of choosing the women is $\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{8!}{6!2!} = \frac{8 \times 7 \times 6!}{6!2!} = 28$

Therefore, the total number of ways of choosing the committee (given by the product rule) is

$$220 \times 28 = 6160$$

Example 2: A group consists of 4 boys and 7 girls. In how many ways can a team of three be selected if it is to contain:

- a) 1 boy and 2 girls
- b) Girls only
- c) Boys only
- d) At least 2 boys
- e) At least one member of each gender?

Solutions: a. $\binom{4}{1} \times \binom{7}{2} = 4 \times 21 = 84$ b. $\binom{7}{3} = 35$ c. $\binom{4}{3} = 4$

$$\begin{aligned} \text{a. } \binom{4}{2} \times \binom{7}{1} + \binom{4}{3} \times \binom{7}{0} &= \frac{4!}{(4-2)!2!} \times \frac{7!}{(7-1)!1!} + \frac{4!}{(4-3)!3!} \times \frac{7!}{(7-0)!0!} \\ &= 6 \times 7 + 4 \times 1 = 42 + 4 = 46 \end{aligned}$$

$$\text{b. } \binom{4}{2} \times \binom{7}{1} + \binom{4}{1} \times \binom{7}{2} = 126$$

Example 3: A football team of 11 is to be chosen from 15 players. How many different teams can be selected if:

- a. there is no condition (no restriction)?
- b. the captain was chosen before and must be in the team?

- c. the captain was chosen before must be in the team and one of the other players is injured and cannot play?

Solution:

- a. The number of different teams possible is:

$$\binom{15}{11} = \frac{15!}{(15-11)!11!} = \frac{15 \times 14 \times 13 \times 12 \times 11!}{4!11!} = \frac{15 \times 14 \times 13 \times 12}{24} = 1365$$

- b. Since the captain must be in the team, the selection is to have the other 10 players from the 14 available to join the captain. The number of teams

possible is $\binom{14}{10} = \frac{14!}{(14-10)!10!} = \frac{14 \times 13 \times 12 \times 11 \times 10!}{24 \times 10!} = 1001$

- c. Since the captain must be in the team and one player cannot play because of injury, the selection is to now to choose the other ten players from the 13 available to join the captain. The number of teams possible is

$$\binom{13}{10} = \frac{13!}{(13-10)!10!} = \frac{13 \times 12 \times 11 \times 10!}{3!10!} = \frac{13 \times 12 \times 11 \times}{6} = 286 \text{ . (Ndorimana, 2016)}$$



Activity 2: Guided Practice



Task 61:

- By using this formula $\binom{n}{r} = \frac{n!}{(n-r)!r!}$, perform the following task
 - in how many ways can a committee of 5 students be chosen from a class of 30 students? Share your findings with the rest of the class.
 - If $(n + 2)! = 20n!$ find n.
 - Evaluate: a. C_2^4 b. C_3^9 c. C_4^{16}
- In how many ways can a team of 2 men and 3 women be selected in a group of 6 men and 7 women? (Ndorimana, 2016)



Activity 3: Application



Task 62:

A basketball team of 6 is to be chosen from 11 available players. In how many ways can this be done if:

- there are no restrictions?
- 3 of the players are automatic selections?
- 3 of the players are automatic selections and 2 other players are injured and cannot play?

Topic 5.2: Probabilities



Activity 1: Problem Solving



Task 63:

Consider the letters of the word “PROBABILITY”

- How many letters are in this word?
- How many vowels are in this word?
- What is the ratio of numbers of vowels to the total number of letters?
- How many consonants are in this word?
- What is the ratio of numbers of consonants to the total number of letters?

Key Facts 5.2: Probabilities

- Definition of terminologies**

- ✓ **Probability** is random variable that can be happen or not, i.e. is the chance that something will happen. Using the following examples, we can see how the concept of probability can be illustrated in various contexts

Let us consider a game of playing cards. In a pack of deck of 52 playing cards, cards are divided into four suits of 13 cards each. If any player selects a card by random (simple random sampling: chance of picking any card in the pack is always equal), then each card has the same chance or same probability of being selected.

Second example, assume that a coin is tossed, it may show Head (H-face with logo) or tail (T-face with another symbol), all these are mentioned below:



We cannot say beforehand whether it will show head up or tail up. The result will depend on chance. The same, a card drawn from a well shuffled pack of 52 cards can be red or black. This also depends on chance. All these phenomena are called probabilistic, means that can be occurred depending on the chance/ uncertainty. The theory of probability is concerned with this type of phenomena.

Probability is a concept which numerically measures the degree of uncertainty and therefore certainty of occurrence of events.

- **Random experiments and Events**

- ✓ **A random experiment** is an experiment whose outcome cannot be predicted or determined in advance. These are some example of experiments: Rolling (tossing) a die is an example of a random experiment and probability is the study of such random experiments. When we roll a die, we know that the set of possible outcomes is $S = \{1, 2, 3, 4, 5, 6\}$, called the sample space. We have no idea exactly which of the elements of S will appear in any toss but we know intuitively that each of these outcomes is equally likely. That is, a '2' is no more likely to appear than a '1', which is non more likely to appear than a '3', and so on. If this experiment is performed on n separate occasions and 't' is the number of times a '2' appears, we

know from observation that the ratio $\frac{t}{n}$ becomes close to $\frac{1}{6}$ as n increases. We

can define **probability theory** as the study of the chance of an event happening.

Example 1. What is the chance of getting a 4 when a die is rolled?

Solution: Number of ways it can happen: 1 (there is only 1 face with a 4 on it)

Total number of outcomes: 6 (there are 6 faces altogether)

So the probability = $\frac{1}{6}$

Example 2. There are 5 marbles in a bag: 4 are blue, and 1 is red. What is the probability that a blue marble gets picked?

Solution: Number of ways it can happen: 4 (there are 4 blues) Total number of outcomes: 5 (there are 5 marbles in total)

So the probability = $\frac{4}{5} = 0.8$

- **Sample space and events**

The set S of all possible outcomes of a given experiment is called **the sample space**. A particular outcome, i.e., an element of S , is called **a sample point**. Any subset of the sample space is called **an event**. The event $\{a\}$ consisting of a single element of S is called a simple event.

- ✓ **Experiment or trial:** an action where the result is uncertain. Tossing a coin, throwing dice, seeing what fruits people prefer are all examples of experiments.
- ✓ **Sample space:** all the possible outcomes of an experiment

Choosing a card from a deck There are 52 cards in a deck (not including Jokers) So the sample space is all 52 possible cards: {Ace of Hearts, 2 of Hearts, and so on...}

- **Event**

Event: a single result of an experiment

- a. Getting a Tail when tossing a coin is an event
- b. Rolling a "5" is an event.

An event can include one or more possible outcomes:

- c. Choosing a “King” from a deck of cards (any of the 4 Kings) is an event
- d. Rolling an “even number” (2, 4 or 6) is also an event
- e. Anne wants to see how many times a “double” comes up when throwing 2 dice.

Each time Anne throws the 2 dice is an experiment. It is an experiment because the result is uncertain. The event Anne is looking for is a “double”, where both dice have the same number. It is made up of these 6 sample points: (1,1), (2,2), (3,3), (4,4), (5,5) and (6,6) or $S = \{(1,1), (2,2), \dots\}$

The sample space is all possible outcomes (36 sample points): (1,1), (1,2), (1,3), (1,4) ... (6,3), (6,4), (6,5), (6,6)

To understand well the sample space, we have to draw table:

D ₁ \ D ₂	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- **Complementary event:** The complement of a set say A is the set written A' or $\bar{A}$ of an element in the universal set but not in the set A i.e. $A' = U - A$ and we can write $A \cup A' = U$. we did examples at the key fact called Venn diagram

Note: Universal set is the same as sample space in probability theory

- **Probability of event under an equally likely event**

The **probability of an event** under the assumption of equally likely outcomes is defined as the ratio of the number of favourable outcomes for the event to the total number of possible outcomes in the sample space.

- **Inclusive events:** Inclusive events can occur simultaneously. That is, there is an overlap between the events, meaning they can both happen at the same time.

For example: Rolling a die and considering the events:

A: Getting an even number (2,4,6)

B: Getting a number greater than 3 (4, 5, 6)

Here there is intersection between A and B that is the reason why we say set A and B are inclusive events.

$$A \cap B = \{4, 6\}$$

- **Disjoint sets** are sets with no elements in common. If A and B are disjoint sets, then $A \cap B = \emptyset$. Also A and B are said to be **mutually exclusive**.

Mutually exclusive: Exclusive events cannot occur simultaneously. That is, there is no overlap between the events; if one occurs, the other cannot.

Example: Tossing a coin and considering the events:

A: Getting heads

B: Getting tails

Here, $A \cap B = \emptyset$, meaning A and B cannot both occur. (REB, 2023)

- **Key Facts 5.2.2: Calculation of probabilities**

✓ **Complementary event**

If E is an event, then E' is the event which occurs when E does not occur. Event E and E' are said to be complementary events.

Theorem: $P(E') = 1 - P(E)$ or $P(E) = 1 - P(E')$

Proof: $P(E') = \frac{n(E')}{n(S)} = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|}$ Thus $P(E') = 1 - P(E)$ and clearly $P(E) = 1 - P(E')$.

Also $P(E) + P(E') = 1$.

An ordinary die of 6 sides is rolled once. Determine the probability of:

- obtaining 5
- not obtaining 5
- obtaining 3 or 4
- not obtaining 3 or 4

Solution: The sample space of possible outcomes is $S = \{1, 2, 3, 4, 5, 6\}$

a. $P(5) = \frac{1}{6}$

b. $P(\text{not a } 5) = 1 - \frac{1}{6} = \frac{5}{6}$

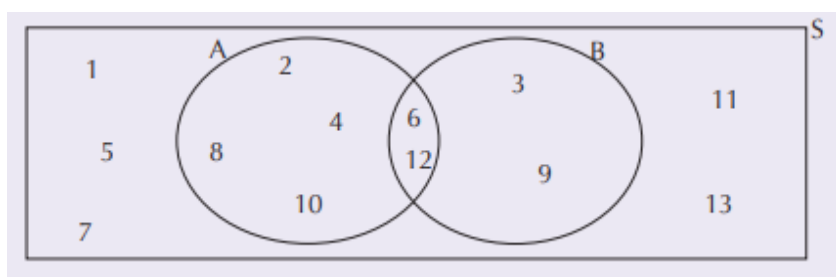
c. $P(3 \text{ or } 4) = \frac{2}{6} = \frac{1}{3}$

d. $P(\text{not } 3 \text{ or } 4) = 1 - \frac{1}{3} = \frac{2}{3}$

Consider two different events A and B, which may occur when an experiment is performed. The event $A \cup B$ is the event which occurs if A or B or both A and B occurs, i.e. at least one of A or B occurs. The event $A \cap B$ is the event which occurs when both A and B occurs. The event $A - B$ is the event which occurs when A occurs and B does not occur. The event A' is the event which occurs when A does not occur.

Example: An integer is chosen at random from the set $S = \{x | x \in \mathbb{Z}^+, 0 < x < 14\}$. Let A be the event of choosing a multiple of 2 and B the event of choosing a multiple of 3. Find: (a) $P(A \cup B)$ (b) $P(A \cap B)$ (c) $P(A - B)$

Solution:



From the diagram (a) $P(A \cup B) = \frac{8}{13}$ (b) $P(A \cap B) = \frac{2}{13}$ (c) $P(A - B) = \frac{4}{13}$

- **Probability of an event under equal assumptions**

The probability of an event E, denoted by $P(E)$ or $P_r(E)$, is a measure of the possibility of the event occurring as the result of an experiment. If the sample space S is finite and the possible outcomes are all equally likely, then the probability of the

event E is equal to $\frac{|E|}{|S|} = \frac{n(E)}{n(S)}$ where $|E|$ and $|S|$ denote the number of elements in E and S respectively.

$P(E) = \frac{|E|}{|S|} = \frac{n(E)}{n(S)}$ Where $|E|$ is number of favourable outcomes and $|S|$ total number of possible outcomes The probability that a randomly selected element from a finite population belongs to a certain category is equal to the proportion of the population belonging to that category.

Theorem: Suppose that an experiment has only a finite number of equally likely outcomes. If E is an event, then $0 \leq P(E) \leq 1$.

Proof: Since E is a subset of S, the set of equally likely outcome then $0 \leq |E| \leq |S|$.

Hence $0 \leq \frac{|E|}{|S|} \leq 1$ or $0 \leq P(E) \leq 1$

Note: if $E = S$, then clearly $|E| = |S|$ and $P(E) = P(S) = 1$ (the event is certain to occur), and if $E = \emptyset$, then $|E| = 0$ and $P(E) = P(\emptyset) = 0$ (the event cannot occur).

Example: A coin is tossed two times. Find the probability of obtaining

(a) $A = \{\text{two heads}\}$

(b) $B = \{\text{one head and one tail}\}$

Solution The sample space is $S = \{HH, HT, TH, TT\}$,

$A = \{HH\}$, $B = \{HT, TH\}$

(a) $P(A) = \frac{|A|}{|S|} = \frac{1}{4}$ (b) $P(B) = \frac{|B|}{|S|} = \frac{2}{4} = \frac{1}{2}$

Example 2. A letter is chosen from the letters of the word "MATHEMATICS". What is the probability that the letter chosen is an "A"?

Solution: Since two of the eleven letters are "A", the probability of choosing a letter

"A" is $\frac{2}{11}$. (Stirzaker, 2003)

- **Permutations and combinations in probability theory**

Permutations and combinations can be used to find probabilities of various events particularly when large sample sizes occur. In everything we do, we have to use the

formula $P(E) = \frac{n(E)}{n(S)}$

Example 1. A bag contains 6 blue balls, 5 green balls and 4 red balls. Three balls are selected at random without replacement. Find the probability that

- a. they are all blue
- b. 2 are blue and 1 is green
- c. there is one of each colour.

Solution The number of all possible outcomes is $\binom{15}{3} = 455$

- a. The number of ways of obtaining 3 blue balls is $\binom{6}{3} \times \binom{5}{0} \times \binom{4}{0} = 20 \times 1 \times 1 = 20$

$$\text{Thus, } P(\text{all are blue}) = \frac{20}{455} = \frac{4}{91}$$

- b. The probability of obtaining 2 blue balls and one green ball is:

$$\binom{6}{2} \times \binom{5}{1} \times \binom{4}{0} = 75$$

$$\text{Thus, } P(2 \text{ blue and } 1 \text{ green}) = \frac{75}{455} = \frac{15}{91}$$

- c. The probability of obtaining 1 blue ball, 1 green ball and 1 red ball is:

$$\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} = 120$$

$$\text{Thus, } P(1 \text{ blue, } 1 \text{ green and } 1 \text{ red}) = \frac{120}{455} = \frac{24}{91}$$

Example 2: If 4 people A, B, C, D sit in a row on a bench, what is the probability that A and B sit next to each other?

Solution The number of ways of arranging 4 people in a row is $|S| = 4! = 24$. The number of ways of arranging 4 people so that A and B are next to each other is $|X| = 2 \times 3! = 12$. Thus, $P(\text{A and B sit next to each other})$ is $\frac{|X|}{|S|} = \frac{12}{24} = \frac{1}{2}$.

Example 3. If 5 cards are selected at random from an ordinary deck of 52 cards, find the probability that exactly 2 of them are aces.

Solution The number of ways of selecting 2 aces from the 4 aces is $\binom{4}{2}$. The number

of ways of selecting 3 non-aces from the 48 non-aces is $\binom{48}{3}$. Therefore the number

of ways of selecting 5 cards of which exactly 2 are aces is $|A| = \binom{4}{2} \times \binom{48}{3}$. The

number of ways of selecting 5 cards from 52 is $|S| = \binom{52}{5}$. Thus the required

$$\text{probability is } \frac{|A|}{|S|} = \frac{\binom{4}{2} \times \binom{48}{3}}{\binom{52}{5}} = \frac{12453120}{311875200} = 0.0399 \text{ (Ndorimana, 2016)}$$

- **Finite probability spaces**

Let $S = \{a_1, a_2, a_3, \dots, a_n\}$ be a finite sample space. A finite probability space obtained by assigning to each point $a_r \in S$ a real number p_r called the probability of a_r , satisfying the following:

(a) $p_r \geq 0$ for all integers r $1 \leq r \leq n$

(b) $\sum_{r=1}^n p_r = 1$

If A is any event, then the probability $P(A)$ is defined to be the sum of the probabilities of the sample points in A.

Example: A coin is weighted such that heads is three times as likely to appear as tails.

Find $P(T)$ and $P(H)$. **Solution:** Let $P(T) = p$, then $P(H) = 3p$. However, $P(T) + P(H) = 1$.

Therefore $4p = 1$ or $p = \frac{1}{4}$. Thus $P(T) = \frac{1}{4}$ and $P(H) = \frac{3}{4}$.

Additional law of probability, Mutual exclusive, Inclusive and exhaustive

For any event A and B from a sample space E, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

This is known as the **addition law of probability** from which we deduce that if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$. If E_i and E_j are mutually exclusive we denote this by $E_i \cap E_j = \phi$ that is the compound event $E_i \cap E_j$ is an impossible event and so will never occur.

On Venn diagram A and B shown as **mutually exclusive (disjoint sets)** events and shown as no mutually exclusive.

Mutually exclusive (disjoint) events A and B

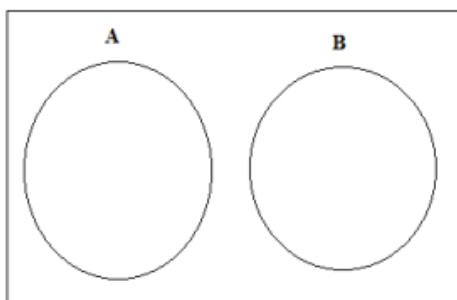


Figure 31: Mutually exclusive of events A and B

Inclusive events A and B

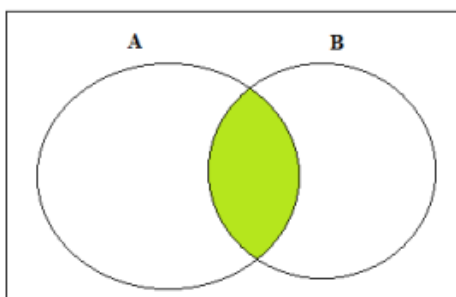


Figure 32: Inclusive events A and B

Suppose that $E_1, E_2, E_3, \dots, E_n$ are n events and that in a single trial only one of these events can occur. The occurrence of any event, E_i , excludes the occurrence of all other events. Such events are mutually exclusive. Generally, For mutually exclusive events,

the addition law of probability applies:

$$P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } \dots \text{ or } E_n) = P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = p(\cup_i^n E_i) =$$

$$P(E_1) + P(E_2) + \dots + P(E_n) = \sum_{i=1}^j P(E_i)$$

(For inclusive events the addition law of probability applies:

$$P(E_1 \text{ or } E_2 \text{ or } E_3 \text{ or } \dots \text{ or } E_n) = P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = p(\cup_i^n E_i) =$$

$$\sum_{i=1}^n P(E_i) - \sum_{j>i=1}^n P(E_i \cap E_j) + \sum_{k>j>i=1}^n P(E_i \cap E_j \cap E_k) + (-1)^n P(E_1 \cap E_2 \cap \dots \cap E_n)$$

Example 1: A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find $P(A)$, $P(B)$, $P(A \cap B)$ and $P(A \cup B)$.

Solution: $P(A) = P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$, $P(B) = P(\text{a spade}) = \frac{13}{52} = \frac{1}{4}$, $P(A \cap B) = P(\text{the ace of spades}) = \frac{1}{52}$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Example 2: A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find $P(A)$, $P(B)$ and $P(A \cup B)$.

Solution: $P(A) = \frac{5}{10} = \frac{1}{2}$, $P(B) = \frac{3}{10}$ and since the marble cannot be both red and blue,

A and B are mutually exclusive so $P(A \cup B) = P(A) + P(B) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$.

Example 3. Suppose a student is selected at random from 100 students where 30 are taking Auditing, 20 are taking Taxation, and 10 are taking Auditing and Taxation. Find the probability that the student is taking Auditing or Taxation.

Solution: Let M be the event "students taking Auditing" and C the event "students taking Taxation" then, $P(M) = \frac{30}{100} = \frac{3}{10}$, $P(C) = \frac{20}{100} = \frac{1}{5}$ and $P(M \cap C) = \frac{10}{100} = \frac{1}{10}$.

Thus, by the addition principle

$$P(M \text{ or } C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{10} + \frac{1}{5} - \frac{1}{10} = \frac{3}{10}.$$

Example 4. In a competition prepared by Rwanda revenue in which there are no dead heats, the probability that John wins is 0.3, the probability that Mike wins is 0.2 and the probability that Putin wins is 0.4. Find the probability that:

- a. John or Mike wins
- b. John or Putin or Mike wins,
- c. Someone else wins.

Since only one person wins, the events are mutually exclusive.

- a. $P(\text{John or Mike win}) = 0.3 + 0.2 = 0.5$
- b. $P(\text{John or Putin or Mike win}) = 0.3 + 0.2 + 0.4 = 0.9$
- c. $P(\text{Someone else wins}) = 1 - 0.9 = 0.1$ (REB, 2023)



Activity 2: Guided Practice



Task 64:

By using the finite probability property work out, the following questions:

1. An unbiased cubic die is thrown. Find the probability that the number showing is
 - a. Even;
 - b. prime;
 - c. less than 4
2. A die is loaded in such a way that $P(1) = P(3) = \frac{1}{12}$, $P(2) = P(6) = \frac{1}{8}$ and $P(4) = \frac{1}{2}$. Find the probability that the number appearing
 - a. odd
 - b. even
 - c. prime
 - d. not 3.



Activity 3: Application



Task 65

1. A fair die is rolled, what is the probability of getting an even number or prime number?
2. Events A and B are such that they are both mutually exclusive and exhaustive. Find the relation between these two events.
3. In a class of a certain school, there are 12 girls and 20 boys. If a teacher wants to choose one student to answer the asked question
 - a. What is the probability that the chosen student is a girl?
 - b. What is the probability that the chosen student is a boy?
4. On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06. What is the probability of a person driving while intoxicated or having a driving accident?
5. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability that:
 - a. both cards are spades
 - b. at least one card is a spade

Topic 5.3: Conditional probability



Activity 1: Problem Solving



Task 66:

Suppose that you have a deck of cards; then draw a card from that deck, not replacing it, and then draw a second card.

- a. What is the sample space for each event?
- b. Suppose you select successively two cards, what is the probability of selecting two red cards?

- c. Explain if there is any relationship (Independence or dependence) between those two events considering the sample space. Does the selection of the first card affect the selection of the second card?

Key Facts 5.3: Conditional probability

- **Independence, Dependence and conditional probability**

- ✓ **Independent events**

If probability of event B is not affected by the occurrence of event A, events A and B are said to be independent and $P(A \cap B) = P(A) \times P(B)$. This rule is the simplest form of the **multiplication law of probability**.

Example: A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution: Let A be the event: "a 4 is obtained on the first throw", then $P(A) = \frac{1}{6}$

That is $A = \{4\}$ B be the event: "an odd number is obtained on the second throw".

That is $B = \{1, 3, 5\}$ Since the result on the second throw is not affected by the result on the first throw, A and B are independent events. There are 3 odd numbers, then $P(B) = \frac{3}{6} = \frac{1}{2}$.

Therefore, $P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Example 2. Given that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = x$. Find x if A and B are

- independent
- mutually exclusive.

Solution

(a) If the events are independent, then $P(A \cap B) = P(A) \times P(B) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{2}{5} - \frac{2}{15} = \frac{5+6-2}{15} = \frac{9}{15} = \frac{3}{5} \text{ Thus, } x = \frac{3}{5}$$

b) If the events are mutually exclusive, then $A \cap B = \emptyset$ which gives $P(A \cap B) = 0$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

$$x = \frac{1}{3} + \frac{2}{5} = \frac{5+6}{15} = \frac{11}{15} \text{ Thus, } x = \frac{11}{15}$$

Example 3. A factory runs two machines, A and B. Machine A operates for 80% of the time while machine B operates for 60% of the time and at least one machine operates for 92% of the time. Do these machines operate independently?

Solution: The data does not give any clues. However, we are given $P(A) = 0.8$, $P(B) = 0.6$ and

$$P(A \cup B) = 0.92. \text{ Now } P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.6 - 0.92 = 0.48$$

and we have $P(A) \times P(B) = 0.8 \times 0.6 = 0.48$.

Since $P(A \cap B) = 0.48 = P(A) \times P(B)$, then, these machines do operate independently.

Example 4. Find the probability that in 3 throws of a fair die, the 3 numbers are all even.

Solution: The probability that the first number is even is $\frac{1}{2}$. The probability that

the second number is even is $\frac{1}{2}$. The probability that the third number is even is

$\frac{1}{2}$. Since these events are independent, the probability that the 3 numbers are all

even is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. This rule is the simplest form of the multiplication law of

probability. The extension to events which are not independent will be considered in the next section.

Exercises:

1. If A and B are any two events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{12}$, and $P(A \cap B) = \frac{1}{4}$. Find

$P(A \cup B)$.

2. Events A and B are such $P(A - B) = 0.3$, $P(B - A) = 0.4$ and $P(A' \cap B') = 0.1$.

Find a) $P(A \cap B)$ b) $P(A)$ c) $P(B)$

3. If A and B are independent events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{6}$, find

a. $P(A \cap B)$; b) $P(A \cup B)$; c) $P(A \cap B')$; d) $P(A' \cup B')$

4. die is biased so that the probability of throwing a six is $\frac{1}{3}$. If the die is thrown twice, find the probability of obtaining a) two sixes b) at least one six.

5. An unbiased die is thrown three times. Find the probability of obtaining a) 3 sixes; b) exactly 2 sixes; c) at least one six. (Ndorimana, 2016)

- **Dependence and conditional probability**

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent**. Suppose A is an event in a sample space S with $P(A) > 0$. The probability that an event B occurs once A has occurred, written as $P(B/A)$ is called the conditional probability of B given A and is defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$
 From this result, we have general statement of the

multiplication law: $P(A \cap B) = P(B/A) \times P(A)$.

This shows us that the probability that two events will both occur is the product of the probability that one will occur and the conditional probability that the other will occur given that the first has occurred. We can also write $P(A \cap B) = P(B) \times P(A/B)$.

Notice: If A and B are independent, then the probability of B is not affected by the occurrence of A and so $P(B/A) = P(B)$ giving $P(A \cap B) = P(B) \times P(A)$.

Example 1. Suppose a card is drawn from a deck and not replaced, and then the second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Solution: The probability of selecting an ace on the first draw is $\frac{4}{52}$. But since that card is not replaced, the probability of selecting a king on the second card is $\frac{4}{51}$, since there are 51 cards remaining. The outcomes of the first draw has affected the outcome of the second. By multiplication rule, the probability of both events occurring is: $\frac{4}{52} \times \frac{4}{51} = \frac{16}{2652} = \frac{4}{663} = 0.006$.

Example 2. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Solution: Let A be the event: “the number is a 4”, then $A = \{4\}$

B be the event: “the number is greater than 2”, then $B = \{3, 4, 5, 6\}$ and $P(B) = \frac{4}{6} = \frac{2}{3}$

But

$A \cap B = \{4\}$ and $P(A \cap B) = \frac{1}{6}$ Therefore, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$P(A|B) = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{6} \times \frac{3}{2} = \frac{1}{4}$$

Example 3. At a middle school, 18% of all students performed competition on Taxation and auditing, and 32% of all students performed competition on Taxation. What is the probability that a student who performed competition on Taxation also performed auditing?

Solution: Let A be a set of students who performed competition on Taxation and B a set of students who performed on auditing then the set of students who performed both competitions is $A \cap B$. We have $P(A) = 32\% = 0.32$,

$P(A \cap B) = 18\% = 0.18$. We need the probability of B known that A has occurred.

$$\text{Therefore, } P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.18}{0.32} = 0.5625 = 56.25\%$$

Example 4. A coin is tossed twice in succession. Let A be the event that the first toss is heads and let B be the event that the second toss is heads. Find:

(a) $P(A)$, (b) $P(B)$, (c) $P(B \cap A)$, (d) $P(B|A)$.

Solution: (a) $P(A) = \frac{1}{2}$ (b) $P(B) = \frac{1}{2}$

(c) $P(B \cap A) = P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(d) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$ (REB, 2015)

Contingency table

Contingency table (or **Two-Way table**) provides a different way of calculating probabilities. It helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another. Below, the contingency table shows the favorite leisure activities for 50 adults, 20 men and 30 women. Because entries in the table are frequency counts, the table is a **frequency table**.

	Dance	Sport	TV	Total
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

Entries in the total row and total column are called **marginal frequencies** or the **marginal distribution**. Entries in the body of the table are called **joint frequencies**.

Example: Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in last year	No speeding violation in last year	Total
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

Calculate the following probabilities using the table:

- P (person is a car phone user)
- P (person had no violation in the last year)
- P (person had no violation in the last year AND was a car phone user)
- P (person is a car phone user OR person had no violation in the last year)
- P (person is a car phone user GIVEN person had a violation in the last year)
- P (person had no violation last year GIVEN person was not a car phone user)

Solution

- P (person is a car phone user = number of car phone user / total number in

$$\text{study} = \frac{305}{755}$$

- P (person had no violation in the last year) = number that had no violation / total

$$\text{number in study} = \frac{685}{755}$$

- P (person had no violation in the last year AND was a car phone user) = $\frac{280}{755}$

- P (person is a car phone user OR person had no violation in the last year) =

$$\frac{305}{755} + \frac{685}{755} - \frac{280}{755} = \frac{710}{755}$$

- The sample space is reduced to the number of persons who had a violation.

Then $P(\text{person is a car phone user GIVEN person had a violation in the last year})$

$$= \frac{25}{70}$$

- f. The sample space is reduced to the number of persons who were not car phone users.

Then $P(\text{person had no violation last year GIVEN person was not a car phone user}) = \frac{405}{450}$ (REB, 2023)

- **Probability by tree diagram**

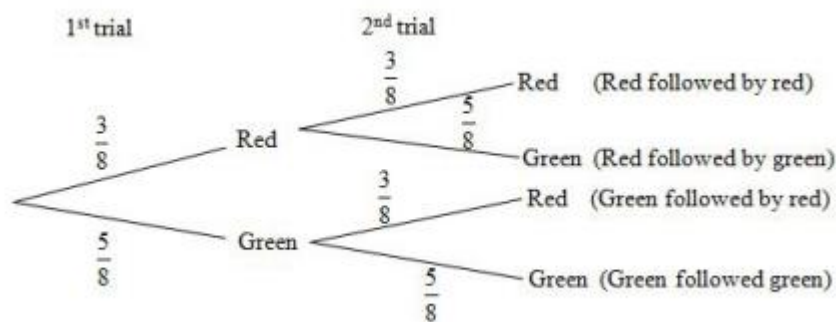
A **tree diagram** is a tool in the fields of probability, and statistics that helps calculate the number of possible **outcomes** of an event or problem, and to cite those potential outcomes in an organized way.

It can be used to show the probabilities of certain outcomes occurring when two or more **trials** take place in succession. The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring. For each trial the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, A and B, of each trial. Successive trials are events which are performed one after the other; all of which are mutually exclusive.

Example: 1. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability the ball drawn will be a) Red followed by green, b) Red and green in any order, c) Of the same color.

Solution: Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is $\frac{3}{8}$ and probability of choosing a green ball is $\frac{5}{8}$

and since after the 1st trial the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial. Draw a tree diagram showing the probabilities of each outcome of the two trials.



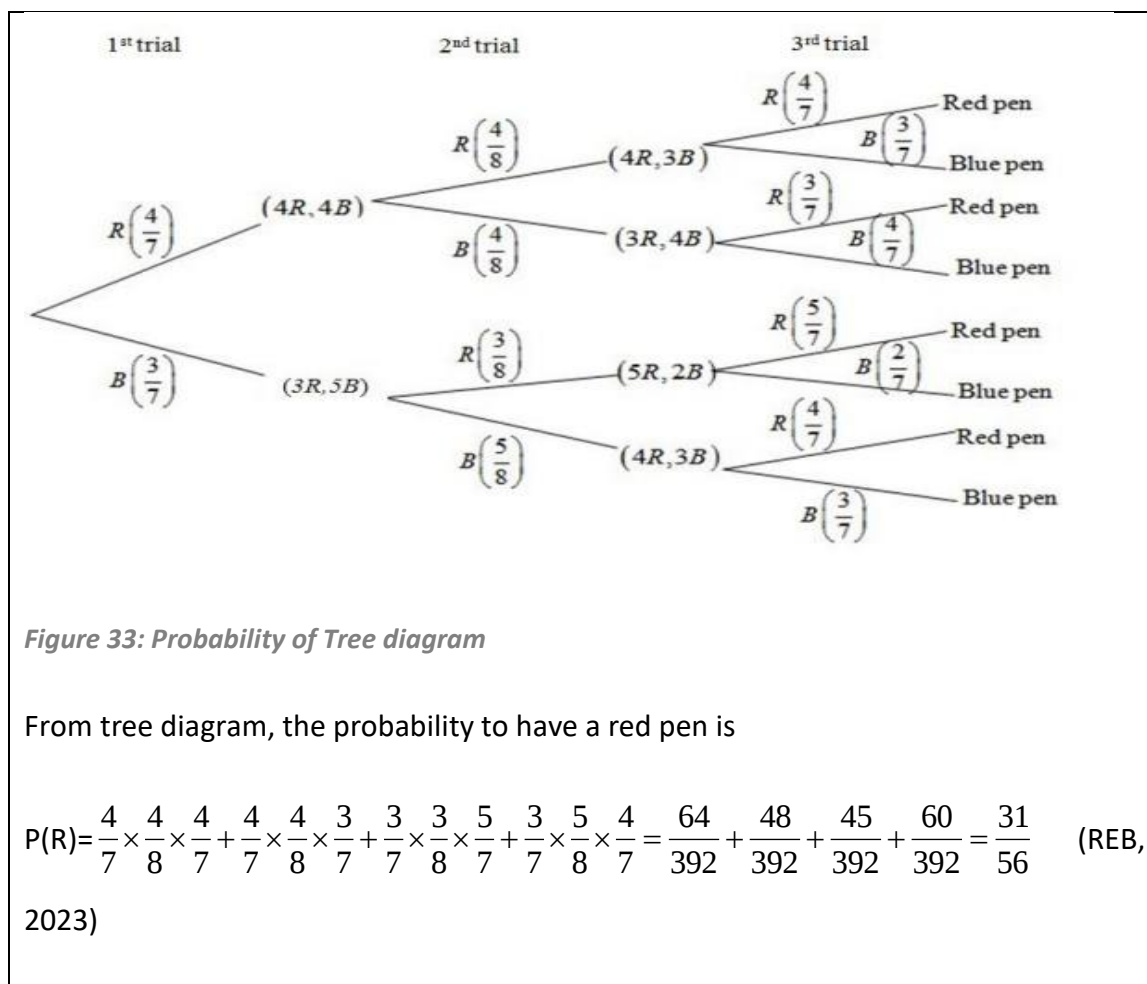
a. $P(\text{Red followed by green}) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

b. $P(\text{Red and green in any order}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{15}{32}$

c. $P(\text{Both of the same color}) = \frac{3}{8} \times \frac{3}{8} + \frac{5}{8} \times \frac{5}{8} = \frac{17}{32}$

Example 2. A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.

Solution: Tree diagram is given below:



Activity 2: Guided Practice



Task 67

By using one the independent events and conditional probabilities find the following questions:

1. A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?
2. A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.

3. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that: both of them will be selected, only one of them will be selected, none of them will be selected?
4. The world-wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.
5. Calculate the probability of a 6 being rolled by a die if it is already known that the result is even.
6. A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?



Activity 3: Application



Task 68:

1. A pair of fair dice is tossed. If the sum is 6, find the probability that one of the dice is a 2.
2. A bag contains 5 red marbles and 3 blue marbles. If 4 marbles are withdrawn from the bag one at a time, find the probability that 3 red marbles are followed by 1 blue marble if
 - a. each marble is replaced before another is withdrawn
 - b. the marbles are not replaced after they are withdrawn.
3. The probability of the weather being fine, raining or snowing respectively are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$. The probabilities that a student arrives on time for school under each of these

conditions are $\frac{3}{4}$, $\frac{2}{5}$ and $\frac{3}{10}$, respectively. What is the probability that (a) the student

arrives at school on time on any given day? (b) if the student is late, it was raining?

4. A tin contains 4 red and 6 blue marbles. Three marbles are withdrawn from the tin one after the other (without replacement). Find the probability that
 - a. the first two is red and the third is blue
 - b. two of the marbles are red and the other is blue.



Formative Assessment

1. Consider a class of 40 students where 15 students don't like pepper. Find the probability that a student selected at random likes pepper
2. A survey involving 120 people about their preferred breakfast showed that:
 - 55 eat eggs for breakfast.
 - 40 drink juice for breakfast.
 - 25 eat both eggs and drink juice for breakfast.
 - a. Represent the information on a Venn diagram.
 - b. Calculate the following probabilities.
 - i. A person selected at random takes only one type for breakfast.
 - ii. A person selected at random takes neither eggs nor juice for breakfast
3. In a survey of 150 Rwandan people about which newspapers they read, 83 read the New Times, 58 read the Imvaho Nshya. 36 read neither of those two papers. Represent the data on the Venn diagram and find the chance that a person selected at random reads both papers.
4. Using a tree diagram, determine all the possible outcomes that can be obtained when a coin is tossed twice and thrice.
5.
 - i. Find the number of words can be organized with the letters of the word MATHEMATICS by regrouping them.
 - ii. In how many ways can we permute the letters of word MISSISSIPPI?

6. How many ways did 2 green and 2 black balls come out of a bag containing 7 green and 8 black balls?
7. Find the numeral of ways in which a 6-member of cabinet can be set up from 8 gentlemen and 4 ladies so that the cabinet consists of at least 3 ladies.
8. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.
 - a. What is the total possible number of hands?
 - b. In how many of these hands are there:
 - i. Kings?
 - ii. Clubs and 3 Hearts?
 - iii. All Hearts?
 - iv. All the same colour?
 - v. Four of the same kind?
 - vi. Aces and two Kings
9. Find the sample space for the experiment of rolling a die:
 - (a) once and (b) 2 times
10. A coin is tossed twice.
 - a. Represent the outcomes on a tree diagram.
 - b. Determine the following probabilities.
 - i. Getting H followed by T
 - ii. Getting two heads
 - iii. Getting head and tail irrespective of order.
11. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability that the ball drawn will be
 - a. Red followed by green,
 - b. Red and green in any order,
 - c. Of the same color.
12. A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw

13. Three different machines in a factory have different probabilities of breaking down during a shift as shown in table below:

Machine	Probability of breaking
A	$\frac{4}{15}$
B	$\frac{3}{10}$
C	$\frac{2}{11}$

Find:

- The probability that all machines will break down during one shift.
 - The probability that none of the machines will break down in a particular shift.
14. At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?
15. A die is tossed. Find the probability that the numbers obtained are 1 and 4 given that the number is greater than 2.
16. Two discs are selected one at a time without replacement from a box containing 5 red and 3 blue discs. Find the probability that:
- the discs are of the same colour
 - if the discs are of the same colour, both are red.
17. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of:
- Three boys being chosen,
 - Exactly two boys and a girl being chosen.
 - Exactly two girls and a boy being chosen,
 - Three girls being chosen.

18. A. Find n if ${}^nP_4 = 30 {}^nC_5$

B. Mary asks 200 students which of these types of music they listen to: pop, Jazz and classical.

Her results are:

- . 90 students listen to classical,
- . 123 students listen to pop,
- . 69 students listen to Jazz,
- . 53 students listen to both classical and pop,
- . 27 students listen pop and Jazz,
- . 34 students listen classical and jazz,
- . 15 students listen to all three.

Using the Venn diagram, write down the number of students who:

- i. Classic music only,
- ii. Do not listen to any 3 types of music.

C) In a group there are 3 men, and 2 women. Three persons are selected at random from this group. Find the probability that 1 man and 2 women or 2 men and 1 woman are selected?



Points to Remember

- Venn diagram
- Multiplication principle
- Permutation
- Combination
- Random experiment
- Sample space
- Events
- Complementary events
- Probability of events under equally likely
- Inclusive events
- Mutually exclusive events
- Conditional probability
- Independent events
- Probability by tree diagram.



Self-Reflection

- a. Read the statements across the top. Put a check in a column that best represents your level of knowledge, skills and attitudes.

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Recognize and describe the relationships between sets using Venn diagrams.					
Identify the structure and components of a tree diagram.					
Explain the multiplication principle as a method to determine the total number of outcomes.					
Define permutations and differentiate between ordered and unordered arrangements.					
Distinguish between permutations and combinations.					
Understand key terms (e.g., random experiment, sample space, events, complementary event, etc.) in probability.					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Recognize formulas and methods for calculating probabilities of various types of events.					
Explain the concept of conditional probability and its dependence on prior events.					
Identify criteria for events to be considered independent.					
Understand how tree diagrams integrate probabilities at each branch.					
Construct and interpret Venn diagrams to solve set-related problems					
Create and use tree diagrams to represent sequential events and calculate probabilities					
Apply the multiplication principle in scenarios involving multiple stages or choices					
Calculate permutations for given scenarios,					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
including those with repetition or restrictions.					
Compute combinations to solve problems involving selection without regard to order.					
Use precise terminology to describe and analyze probabilistic scenarios					
Solve probability problems involving simple, complementary, mutually exclusive, and inclusive events					
Calculate conditional probabilities using appropriate formulas or data representations					
Verify the independence of events and calculate probabilities for independent scenarios.					
Use tree diagrams to calculate the overall					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
probability of compound events					
Appreciate the clarity and visual utility of organizing set relationships.					
Value the systematic approach of tree diagrams in solving probability problems.					
Be consistent and persistent when using systematic rules to simplify complex counting problems					
Appreciate the importance of order in arranging items in real-life contexts.					
Think logically when Computing combinations to solve problems involving selection without regard to order .					
Cultivate an appreciation for the rigor and clarity of mathematical language					

My experience	I don't have any experience doing this.	I know a little about this.	I have some experience doing this.	I have a lot of experience with this.	I am confident in my ability to do this.
Knowledge, skills and attitudes					
Be consistent and persistent when solving probability problems using systematic calculations.					
Think logically when considering the influence of one event on another					
Appreciate the simplicity and power of independence in probability models					
Value the clarity and efficiency of visualizing probabilities in tree diagrams.					

b. Fill in the table above and share results with the trainer for further guidance.

Areas of strength	Areas for improvement	Actions to be taken to improve
1.	1.	1.
2.	2.	2.
3.	3.	3.

REFERENCE

1. Emmanuel N. & Pacifique I. (2017). *Advanced Mathematics for Rwanda Secondary Schools Learner's Book Senior Five*. Kigali: Fountain Publishers Rwanda Ltd. doi:978-9970-19-410-0.
2. Icyimpaye, E. N. (2017). *Advanced Mathematics for Rwanda Schools Learner's Book Senior Six*. Kigali: Publishers Rwanda Ltd. doi:ISBN: 978-9970-19-419-3
3. Ndorimana, I. I. (2016). *Mathematics for Rwandan Schools Senior 4 Student's Book*. Kigali: East African Educational Publishers Ltd. doi:978-9966-56-168-8
4. REB. (2015). *Subsidiary Mathematics Syllabus*. MINEDUC, Kigali, Rwanda.
5. REB. (2022). *Mathematics for Accounting senior four student's book*. Kigali, Rwanda: MINEDUC.
6. REB. (2023). *Mathematics for Accounting senior five student's book*. Kigali, Rwanda.
7. REB. (2023). *Mathematics for Accounting senior six student's book*. Kigali, Rwanda.
8. Shampiona, A. (2005). *Mathématiques 6*. Kigali.
9. Stirzaker, D. (2003). *Elementary Probability*. Cambridge, United States of America: Cambridge University Press.



April 2025