RQF LEVEL 4

Oicture portraying the module

TRADE:

MODULE CODE: GENAFA402

TEACHER'S GUIDE

Module name: FUNDAMENTAL MATHEMATICAL ANALYSIS

MODULE NAME: GENAFA402 FUNDAMENTAL MATHEMATICAL ANALYSIS

2022

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Introduction

This general module describes the knowledge, skills and attitude required to apply basic mathematical analysis. The ability to do basic mathematical analysis is absolutely vital to successfully passing any field. At the end of this module, the trainee of Level Four will be able to solve algebraically and graphically linear or quadratic equations and inequalities. He/she will also be able to determine analyse algebraic functions, and to apply fundamentals of differentiation. As Algebra and fundamentals of differentiation are tools of different field.

Therefore, this module will be useful to trainees as a means of analysis and improving their understanding of Mathematics and he/she will be prepared to perform well in any fields that require some knowledge of mathematics especially algebra and fundamentals of differentiation as well as working in daily mathematical logic and problem solving, financial and economics in hospitality sector for an effective performance in critical thinking, and so on.

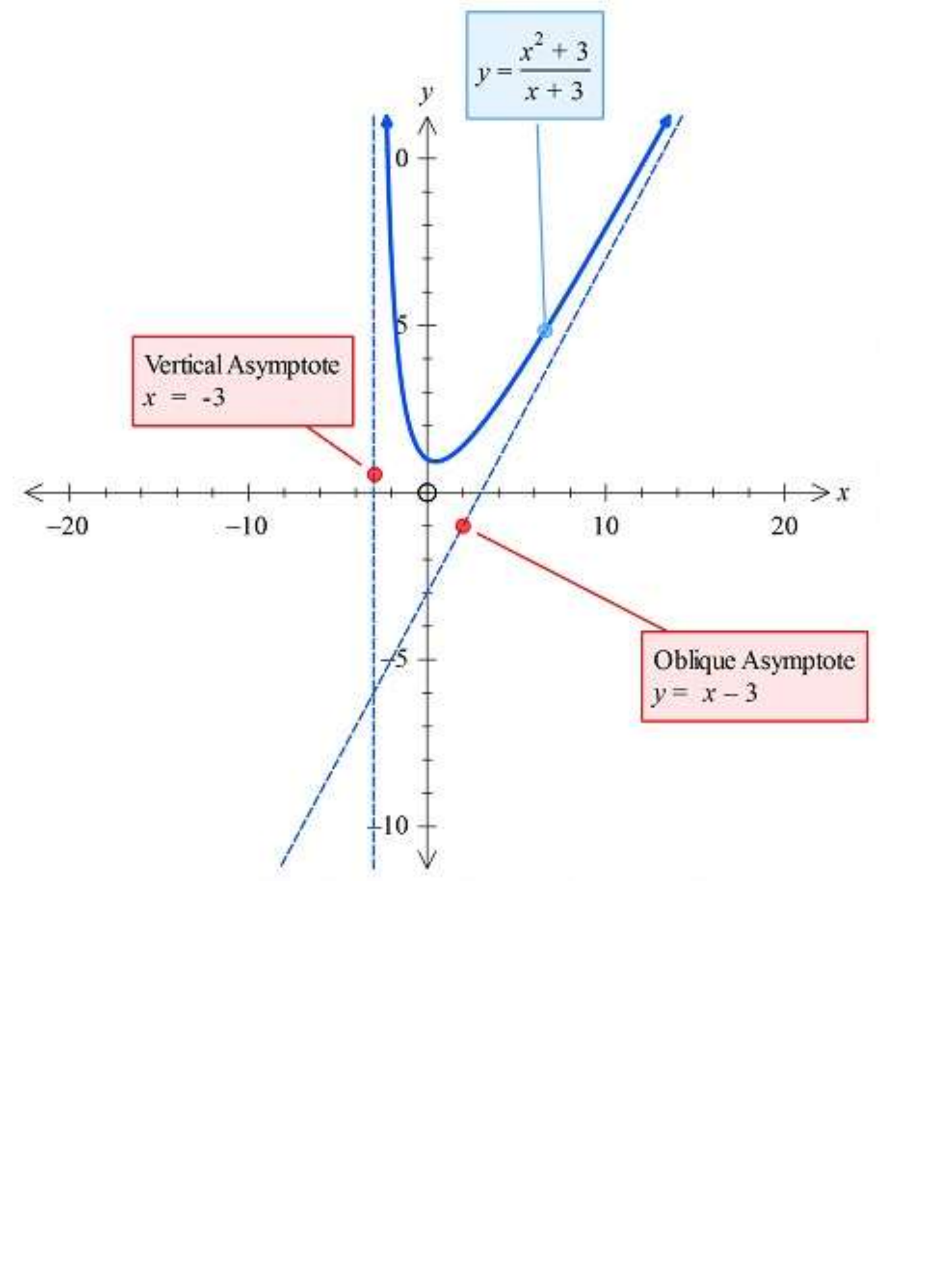
Module Code and Title:

GENAM401: FUNDAMENTAL MATHEMATICAL ANALYSIS

Learning Units:

1. Determine and analyses numerical function
2. Apply fundamentals of differentiation
3. Apply exponential functions
4. Apply natural logarithmic functions

Learning Unit 1: Determine and analyze numerical functions



STRUCTURE OF LEARNING UNIT

Learning outcomes:

1.1. Determine the domain and range of algebraic function.

1.2. Identify the symmetry of algebraic function.

1.3. Determine limits of a function.

1.4. Determine the asymptotes to the rational and polynomial functions.

Learning outcome 1.1. Determine the domain and range of algebraic function.

Duration: 5 hrs

Learning outcome 1.1 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Define correctly the terms: "Domain and Range" of a function as applied in basic mathematical analysis.

2. Describe clearly how to determine the domain of a function as applied in basic mathematical analysis.

3. Describe clearly how to determine the range of a function as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper**	**Geometric instruments (Ruler, T-square)** **Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on algebraic function

1.1.1 : Definitions of terminologies

- **Definitions:**
  - ✓ Existence condition
  - ✓ Domain of definition of a function
  - ✓ Range of a function

numerical function: **For any two subsets A and B of the real line, numerical function is a rule that assigns exactly one element y in set B to each element x in set A.**

x is the independent variable, $f(x)$ is the image of x under function f, $y = f(x)$ is the dependent variable.

We write $f : A \longrightarrow B : x \longmapsto f(x)$

Examples of numerical functions include:

$f: \mathbb{N} \to \mathbb{Z}: x \mapsto y = 2x - 1$; h: $\mathbb{R} \to \mathbb{R}: x \mapsto y = \sqrt{x}$

g: $\mathbb{R} \to \mathbb{R}: x \mapsto y = \frac{x}{x^2+1}$; t: $\mathbb{R} \to \mathbb{R}: x \mapsto y = \frac{x}{\sqrt{x^3 - 27}}$

The domain of a numerical function f

Is the largest set of real numbers for which the function is defined? We write D_f

Thus $D_f = \{x \in \mathbb{R} ; f(x) \in \mathbb{R}\}$.

The set of all values $f(x)$ is called the range of the numerical function f, it is denoted by Imf.

Range of a function

We have already discussed the domain of a function $f(x)$ i.e. the values of x for which $f(x)$ is defined. Next we consider the values $f(x)$ we get as x varies over the domain. This is, not surprisingly, called the range of $f(x)$.

Thus $\text{Imf} = \{f(x); x \in D_f\}$.

The **graph** consists of all points (x, y), where x is in the domain of f and where $y = f(x)$.

Theoretical learning Activity

- ✓ Discuss on how to find the domain and range of algebraic function?

Practical learning Activity

- ✓ By using symbols, describe how to apply the domain and range of algebraic function?

Points to Remember (Take home message)

1. The rational and irrational numbers together make up the set of **real numbers** denoted by $\mathbb{R}$. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all subsets of .
2. A **rational number** is one which can be expressed in the form $\frac{q}{p}$ where p and q are integers and $q \neq 0$.
3. If a and b are any two real numbers, then either $a < b$ or $b < a$ or $a = b$.

1.1.2. Determination of Range and Domain of functions

- **Calculations**

✓ Domain of definition of a function
✓ Range of a function

1. A polynomial

A polynomial function is any function that can be written in the form.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and n is a positive integer (the **degree** of the polynomial).

Note that the domain of every polynomial function is the entire real line.

Further, recognise that the graph of $f(x) = ax + b$ is a **straight line**, and the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$, is a **parabola**.

The following are examples of polynomials:

$f: \mathbb{R} \to \mathbb{R}: x \mapsto y = 2x^3 + \frac{1}{2}x + 3$; $D_f = \mathbb{R}$,

$g: \mathbb{R} \to \mathbb{R}: x \mapsto y = 5x^4 - 3x^2 + x - 5$; $D_f = \mathbb{R}$.

2. A rational function

Any function that can be written in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with no common factors other than 1, is called a **rational function**.

Notice that since $P(x)$ and $Q(x)$ are polynomials, they are both defined for all x, and so, the rational function $y = \frac{P(x)}{Q(x)}$ is defined for all x, for which $Q(x) \neq 0$.

Thus, $D_f = \{x \in \mathbb{R}; Q(x) \neq 0\}$ or $D_f = \mathbb{R} - \{x \in \mathbb{R}; Q(x) = 0\}$.

Examples:

Determine the domain of each of the following numerical functions:

a) $f(x) = \frac{x+1}{2x-4}$

b) $f(x) = \frac{2x+3}{x^3 + 2x^2 - x - 2}$

c) $f(x) = \frac{x}{x^2+1}$

Solutions

a) $D_f = \mathbb{R} - \{x \in \mathbb{R};\ 2x - 4 = 0\}$

$= \mathbb{R} - \{2\}$

$=]-\infty, 2[\cup]2, +\infty[$

b) $Df = \mathbb{R} - \{x \in \mathbb{R};\ x^3 + 2x^2 - x - 2 = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ x^2(x+2) - (x+2) = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ (x^2 - 1)(x+2) = 0\}$

$= \mathbb{R} - \{x \in \mathbb{R};\ (x-1)(x+1)(x+2) = 0\}$

$= \mathbb{R} - \{1; -1; -2\}$

$=]-\infty, -2[\cup]-2, -1[\cup]-1, 1[\cup]1, +\infty[$

c) $D_f = \mathbb{R} - \{x \in \mathbb{R};\ x^2 + 1 = 0\}$

$= \mathbb{R} - \varnothing$: $x^2 + 1 \neq 0$ for all $x \in \mathbb{R}$

$= \mathbb{R}$

$=]-\infty, +\infty[$

3. An irrational function

Any function that can be written as $f(x) = \sqrt[n]{P(x)}$, where $n \in \mathbb{N} - \{0; 1\}$ and $P(x)$ is a polynomial is called an irrational function.

We have the following:

	n:odd	n:even
$P(x)$ is a polynomial	$D_f = \mathbb{R}$	$D_f = \{x \in \mathbb{R};\ P(x) \geq 0\}$
$P(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials	$D_f = \{x \in \mathbb{R};\ D(x) \neq 0\}$	$D_f = \{x \in \mathbb{R};\ \dfrac{N(x)}{D(x)} \geq 0\}$

Examples

Find the domain of each of the following numerical functions:

a) $f(x) = \sqrt[3]{\dfrac{x+1}{x^2-3x+2}}$

b) $f(x) = \sqrt[3]{x^3-3x+2}$

c) $f(x) = \sqrt{x^2-3x-10}$

d) $f(x) = \sqrt[4]{x^2-2x+3}$

e) $f(x) = \sqrt{-4x^2-3x-5}$

f) $f(x) = \sqrt{\dfrac{-x^2-2x+3}{2x+8}}$

g) $f(x) = \sqrt{\dfrac{-x^2-x-2}{9-x^2}}$

Solution

a) $D_f = \{x \in \mathbb{R};\ x^2 - 3x + 2 \neq 0\}$
$= \mathbb{R} - \{1, 2\}$
$=]-\infty, 1[\text{u}]1, 2[\text{u}]2, +\infty[$

b) $D_f = \mathbb{R}$

c) $D_f = \{x \in \mathbb{R};\ x^2 - 3x - 10 \geq 0\}$
$=]-\infty, -2] \text{ u } [5, +\infty[$
(from the sign of $x^2 - 3x - 10$:

x	$-\infty$	-2		5	$+\infty$
$x^2-3x-10$	$+$	0	$-$	0	$+$

d) $D_f = \{x \in \mathbb{R};\ x^2 - 2x + 3 \geq 0\}$
$=]-\infty, +\infty[$ ($x^2 - 2x + 3 \geq 0$ for all $x \in \mathbb{R}$ since $\Delta = (-2)^2 - 4(1)(3) < 0$)

e) $D_f = \{x \in \mathbb{R};\ -4x^2 + 3x - 5 \geq 0\}$
$= \varnothing$; $-4x^2 + 3x - 5 < 0$ for all $x \in \mathbb{R}$

f)

x	$-\infty \quad -4 \quad -3 \quad 1 \quad +\infty$
$-x^2+2x+3$	$- \quad - \quad 0 \quad + \quad 0 \; - \; - \; -$
$2x+8$	$- \quad 0 \quad + \quad + \quad +$
$\dfrac{-x^2-2x+3}{2x+8}$	$+ \quad \Vert \quad - \quad 0 \quad + \quad 0 \quad -$

$D_f = \{x \in \mathbb{R};\ \dfrac{-x^2-2x+3}{2x+8} \geq 0\}$

$=]-\infty, -4[\cup [-3, 1]$

g)

x	$-\infty \quad -3 \quad -1 \quad 2 \quad 3 \quad +\infty$
$-x^2+x+2$	$- \quad - \quad -0 \quad + \quad 0 \; - \; - \; - \quad -$
$9-x^2$	$- - - \; 0 \quad + \quad + \; + \; + \; 0 \; - - -$
$\dfrac{-x^2+x+2}{9-x^2}$	$+ \quad \Vert \; - \quad 0 \; + \quad 0 \; - \; \Vert \; - \; +$

$D_f = \{x \in \mathbb{R};\ \dfrac{-x^2+x+2}{9-x^2} \geq 0\}$

$D_f =]-\infty, -3[\text{u} [-1, 2] \text{u}] 3, +\infty[$

Examples for finding the range of functions:

(a) f(x) = x. The domain is $\mathbb{R}$ i.e. all numbers and the range is also $\mathbb{R}$.

(b) f(x) = x^2. The domain is once again $\mathbb{R}$, but the range is all positive numbers as $x^2 \geq 0$ i.e.[0,∞] .

(c) g(x) = sin (x). The domain is $\mathbb{R}$, but the range is given by [-1, 1] as $-1 \leq \sin(x) \leq 1$.

(d) h(t) = $\sqrt{t}$. Remember that this is the positive square root. The domain is [0,∞] as is the range.

Exercises

Find the ranges of the following functions:

1. f(x) = 3 – 2x

2. f(x) = $3x^2$ - 2

Solutions:

(a) f(x) = 3 – 2x. the domain is $\mathbb{R}$.

To find the range, let f be a value in the range then

$$3 - 2x = f \Rightarrow$$

$$x = (3 - f) / 2.$$

This shows than no matter what value f we choose we can find x such that f(x) = f, hence the range is also $\mathbb{R}$.

(b) f(x) = $3x^2 - 2$. The domain is once again $\mathbb{R}$, but the range is all $f \geq -2$ as given $f \geq -2$ then $f = 3x^2 - 2 \Rightarrow x = \sqrt{f + 2/3}$ gives f(x) = f i.e. the range is [-2 , ∞) .

Theoretical learning Activity

- ✓ Distinguish range and domain accordingly?

Practical learning Activity

- ✓ Determine the domain and range of the given function: $y = -\sqrt{2x + 3}$

Points to Remember (Take home message)

- ✓ Domain of a function is the set of all real numbers for which the expression of the function is defined as a real number.
- ✓ Let f: A → B be a function. The range of f, denoted by Im(f) is the image of A under f, that is, Im (f) = f [A]. The range consists of all possible values the function f can have.

Learning outcome 1.1: formative assessment

Determine the domain and range of the given function: $y = \frac{x^2+x-2}{x^2-x-2}$

Learning outcome 1.2. Identify the symmetry of algebraic function.

Duration: 5 hrs

Learning outcome 1.2 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Define correctly the terms: "Even function and Odd function" as applied in basic mathematical analysis.

2. Describe clearly how to identify a function as Even as applied in basic mathematical analysis.

3. Describe clearly how to identify a function as Odd as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper**	**Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on how to identify symmetry of algebraic function.

1.2.1: Definition of terminologies

- **Definitions:**
  - ✓ Even function
  - ✓ Odd function

Let f be an algebraic function whose domain is Df.

f is even if and only if:

$: (\forall x \in D_f); -x \in D_f \text{ and } f(-x) = f(x);$

f is odd if and only if:

$: (\forall x \in D_f); -x \in D_f \text{ and } f(-x) = -f(x);$

Theoretical learning Activity

✓ Differentiate the even from odd function?

Practical learning Activity

✓ State whether the following function is even or odd: $f(x) = x^2 - 1$

Points to Remember (Take home message)

- ✓ Let f be a function of in we say that f is even if $\forall x \in Dom(f)$, $(-x) \in Dom(f)$; $f(-x) = f(x)$
- ✓ We say that a function f is odd if $\forall x \in Dom(f)$, $(-x) \in Dom(f)$; $f(-x) = -f(x)$

1.2.2: Identification of functions

- **Even function**
- **Odd function**

Examples

1. Determine whether f is odd or even in each of the following cases:

a) $f(x) = 2x^2 + 1$

b) $f(x) = \cos x$

c) $f(x)\ x^3 + 2x$

d) $f(x) = \sin x$

Solution

a) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = 2(-x)^2 + 1$

$= 2x^2 + 1$

$= f(x)$

Therefore, f is even.

b) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = \cos(-x)$

$= \cos x$

$= f(-x)$

Therefore, f is even.

c) $D_f = \mathbb{R}$;

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(x) = (-x)^3 + 2(-x)$

$= -x^3 - 2x$

$= -(x^3 + 2x)$

$= -f(x)$

Therefore, f is odd.

d) $D_f = \mathbb{R}$

$(\forall x \in \mathbb{R}), -x \in \mathbb{R}$

and $f(-x) = \sin(-x)$

$= -\sin x$

$= -f(x)$

Therefore, f is odd.

Theoretical learning Activity

- ✓ What do you understand by even and odd functions?

Practical learning Activity

State whether the following function is even or odd: $g(x) = \dfrac{x^3 + x}{5}$

Points to Remember (Take home message)

- $f: \mathbb{R} \to \mathbb{R}: x \to f(x) = x^2$ is an even function because $f(-x) = (-x)^2 = x^2 = f(x)$
- $f: \mathbb{R} \to \mathbb{R}: x \to f(x) = x^3$ is odd function because $f(-x) = (-x)^3 = -x^3 = -f(x)$

Learning outcome 1.2: Formative Assessment

Practical assessment

- ✓ Task to be performed:

State whether the following function is even or odd: $h(x) = \dfrac{x^3 + x + 2}{x^2}$

Learning outcome 1.3. Determine limits of a function.

Duration: 10 hrs

Learning outcome 2.3 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to determine the limits of functions as applied in basic mathematical analysis.

2. Describe correctly how to remove indeterminate cases as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference Books **Internet**	**Didactic materials such as manila paper**	**Hand-out notes**

Advance preparation:

. Refer to manual discussing on how to determine the limits of a function.

1.3.1: Determination of function limits

Introduction

Consider the functions f(x) = $\frac{x^2 - 4}{x - 1}$ and g(x) = $\frac{x^2 - 5}{x - 2}$. Notice that both functions are undefined at x = 2. But the two functions have quite different beahaviours in the vicinity of x = 2.

From numerical approach, for f(x) = $\frac{x^2-4}{x-1}$, we have :

1.

X	f(x)
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

2.

X	f(x)
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x – values get closer to 2, but are all less than 2. We use the notation x → 2^- to indicate that x approaches 2 from the left side. f(x) is getting closer and closer to 4. In view of this, we say that the limit of f(x) as x approaches 2 from the left is 4, written $\lim_{x\to 2^-} f(x) = 4$.

The second table suggests that as x gets closer and closer to 2 (with x > 2), f(x) is getting closer and closer to 4.

In view of this, we say that the limit of f(x) at x approaches 2 from the right is 4, written $\lim_{x\to 2^+} f(x) = 4^-$.

We call $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ one sided limits of f(x), if they are the same, then we summarize the results by saying that limit of f(x) as x approaches 2 is 4, *written* $\lim_{x\to 2} f(x) = 4$.

In general if

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$ we conclude that $\lim_{x\to a} f(x) = l$

If

$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ we conclude that $\lim_{x\to a} f(x)$ doesn't exist

<u>Operations on limits</u>

(1) For any constant c and any real number a, $\lim_{x \to a} c = c$.

(2) For any real number a, $\lim_{x \to a} x = a$.

(3) Suppose that $\lim_{x \to a} (x)$ and $\lim_{x \to a} g(x)$ both exist and let c be any constant.

The following then apply:

(i) $\lim_{x \to a} [c.f(x)] = c. \lim_{x \to a} f(x);$

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x),$

(iii) $\lim_{x \to a} [(x).g(x)] = [\lim_{x \to a} f(x)].[\lim_{x \to a} g(x)],$

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim_{x \to a} g(x) \neq 0.$

Observe that by applying part (iii) of rule (3), within $g(x) = f(x)$, we get that , whenever $\lim_{x \to a} f(x)$ exists,

$$\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x).f(x)]$$

$$= [\lim_{x \to a} f(x)].[\lim_{x \to a} f(x)]$$

$$= [\lim_{x \to a} f(x)]^2.$$

Likewise, for any positive integer n, $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$.

(4) For any polynomial $P(x)$, and any real number a, $\lim_{x \to a} P(x) = P(a)$

Proof:

Suppose $P(x)$ is the polynomial of degree $n \geq 0$, $P(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$.

Then, $\lim_{x \to a} P(x) = \lim_{x \to a} (c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0)$

$$= c_n \lim_{x \to a} x^n + c_{n-1} \lim_{x \to a} x^{n-1} + \ldots + c_1 \lim_{x \to a} x + \lim_{x \to a} c_0$$

$$= c_n a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0$$

$$= P(a).$$

(5) Suppose that $\lim_{x \to a} f(x) = L$, and n is any positive integer, $n \geq 2$.

Then, $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$, where for n even, we assume that $L>0$.

(6) For any real number a, we have:

(i) $\lim_{x \to a} \sin x = \sin a$,

(ii) $\lim_{x \to a} \cos x = \cos a$,

(iii) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a$, for $-1<a<1$,

(iv) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(v) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a$, $a \in \mathbb{R}$,

(vi) If $P(x)$ is a polynomial and $\lim_{x \to P(a)} f(x) = L$, then $\lim_{x \to a} f[P(x)] = L$.

- **Finite limits**

Examples

Find the following limits:

(a) $\lim_{x \to 2} (3x^2 - 5x + 4)$

(b) $\lim_{x \to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$

(c) $\lim_{x \to 2} \sqrt[5]{3x^2 - 2x}$

(d) $\lim_{x \to 0} \sin^{-1}\left(\dfrac{x+1}{2}\right)$

Solution

(a) $\lim_{x \to 2} (3x^2 - 5x + 4) = 3(2)^2 - 5(2) + 4$
$= 6$

(b) $\lim_{x \to 2} \frac{x^3 - 5x + 4}{x^2 - 2} = \frac{3^3 - 5(3) + 4}{3^2 - 2}$
$= \frac{16}{7}$

(c) $\lim_{x \to 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{3(2)^2 - 2(2)}$
$= \sqrt[5]{8}$

(d) $\lim_{x \to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \text{Sin}^{-1}\frac{1}{2}$
$= \frac{\pi}{6}.$

Exercises

Evaluate:

(1) $\lim_{x \to 1} \frac{x+1}{\sqrt{x^2}}$

(2) $\lim_{x \to \frac{\pi}{6}} (\sin x + \cos x)$

(3) $\lim_{x \to \frac{\pi}{2}} 2(\sin x - \cos x + \cos^2 x)$

(4) $\lim_{x \to \frac{\pi}{4}} \frac{\sin 2x + \tan x}{\cos 3x + 4}$

- **Infinite limits**

In general, any limit of the six types

$$\lim_{x \to a^-} f(x) = -\infty, \qquad \lim_{x \to a^-} f(x) = \infty,$$
$$\lim_{x \to a^+} f(x) = -\infty, \qquad \lim_{x \to a^+} f(x) = \infty,$$
$$\lim_{x \to a} f(x) = -\infty, \qquad \lim_{x \to a} f(x) = \infty,$$

Is called an **infinite limit.**

Examples

Evaluate:

(a) $\lim_{x \to 5} \frac{1}{(x-5)^2}$

(b) $\lim_{x \to -2^-} \frac{x+1}{(x-3)(x+2)}$ and $\lim_{x \to -2^+} \frac{x+1}{(x-3)(x+2)}$

Solution

(a) $\lim\limits_{x \to 5} \dfrac{1}{(x-5)^2} = \dfrac{1^+}{0^+} = +\infty.$

(b) $\lim\limits_{x \to -2^-} \dfrac{x+1}{(x-3)(x+2)} = \dfrac{-2+1}{(-2-3).0^-} = \dfrac{-1}{0^+} = -\infty$

$\lim\limits_{x \to -2^+} \dfrac{x+1}{(x-3)(x+2)} = \dfrac{-1}{0-} = +\infty$

$\lim\limits_{x \to -2} \dfrac{x+1}{(x-3)(x+2)}$ does not exist.

• **Limits at infinity**

In this section we are intended in examining the limiting behaviour of functions as x increases without bound (written $x \to +\infty$) or as x decreases without bound (written $x \to -\infty$).

Example1

Calculate, from numerical approach $\lim\limits_{x \to +\infty} \frac{1}{x}$

x	$\frac{1}{x}$
1	1
10	0.1
100	0.01
1 000	0.001
10 000	0.0001

Conclusion: $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$

Example2

Evaluate $\lim\limits_{x \to \infty} \sqrt{\dfrac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}}$.

Solution Because the limit of the rational function inside the radical exists and is positive, we can write

$$\lim_{x \to \infty} \sqrt{\frac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}} = \sqrt{\lim_{x \to \infty} \frac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}} = \sqrt{\lim_{x \to \infty} \frac{2x^3}{6x^3}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}.$$

Theoretical learning Activity

✓ **Discuss on how to find the limit of a function?**

Practical learning Activity

✓ **Evaluate the following limits:** $\lim\limits_{x \to 2} 2x + 1$

Points to Remember (Take home message)

✓ 1. A neighbourhood of a real number is any interval that contains a real number a and some point below and above it.
2. If x is taking values sufficiently close to and greater than a, then we say that x tends to a from above and the limiting value is then what we call the **right-sided** limit. It is written as $\lim\limits_{x \to a^+} f(x) = \lim\limits_{\substack{x \to a \\ >}} f(x)$
3. If x is taking values sufficiently close to and less than a, then we say that x tends to a from below and the limiting value is then what we call the **left-sided** limit. It is written as $\lim\limits_{x \to a^-} f(x) = \lim\limits_{\substack{x \to a \\ <}} f(x)$
4. If the f(x) tends closer to a value L as x approaches the value a from either side, then L is the limit of f(x) as x approaches . We use the following notation: $\lim\limits_{x \to a} f(x) = L$

1.3.2 : Remove of indeterminate cases

- **Indeterminate cases**

✓ **Indeterminate form** $\frac{0}{0}$

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation by:

- factoring
- rationalizing
- using trigonometric transformation

Examples

1. Calculate:

(a) $\lim\limits_{x \to 1} \dfrac{x^2-1}{1-x}$ (b) $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}$

(c) $\lim\limits_{x \to 3} \dfrac{3x+9}{x^2-9}$

Solution

(a) $\lim\limits_{x \to 1} (x^2-1) = 1^2 - 1 = 0$ and $\lim\limits_{x \to 1} (1-x) = 0$

We have : $\lim\limits_{x \to 1} \dfrac{x^2-1}{1-x} = \lim\limits_{x \to 1} \dfrac{(x-1)(x+1)}{-(x-1)}$

$= \lim\limits_{x \to 1} [-(x+1)]$

$= -(1+1) = -2.$

(b) $\lim\limits_{x \to 1} (x^2-4) = 0$ and $\lim\limits_{x \to 1} (x-2) = 0$

Since the expression in the numerator factors,

$\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x \to 2} (x+2) = 4$

$\lim\limits_{x \to 3} \dfrac{3x+9}{x^2-9} = \dfrac{3(-3)+9}{(3)2-9} = \dfrac{0}{0}$ indeterminate form.

$\lim\limits_{x \to 3} \dfrac{3x+9}{x^2-9} = \lim\limits_{x \to 3} \dfrac{3(x+3)}{(x-3)(x+3)}$

$= \lim\limits_{x \to 3} \dfrac{3}{x-3}$

$= \dfrac{3}{-6}$

$= -\dfrac{1}{2}.$

2. Evaluate:

(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$ (b) $\lim\limits_{x \to 0} \dfrac{2x}{3-\sqrt{x+9}}$

Solution

(a) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x} \quad = \dfrac{\sqrt{2}-\sqrt{2}}{0} = \dfrac{0}{0}$ indeterminate form.

$$\lim_{x \to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2})}$$

$$= \lim_{x \to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{x+2}+\sqrt{2})}$$

$$= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

(b) $\lim\limits_{x \to 0} \dfrac{2x}{3-\sqrt{x+9}} \quad = \dfrac{0}{3-3} = \dfrac{0}{0}$ indeterminate form

$$\lim_{x \to 0} \frac{2x\,(3+\sqrt{x+9})}{(3-\sqrt{x+9})(3+\sqrt{x+9})}$$

$$= \lim_{x \to 0} \frac{2x\,(3+\sqrt{x+9})}{9-(x+9)}$$

$$= \lim_{x \to 0} \frac{2x\,(3+\sqrt{x+9})}{-x}$$

$$= -2 \lim_{x \to 0} (3+\sqrt{x+9})$$

$$= -2(6) = -12.$$

Notice that sometimes, it may be necessary to use a change of variable to simplify the problem.

✓ **Indeterminate form** $\infty - \infty$

If the evaluation of a limit leads to:

$(+\infty) - (+\infty)$; $\qquad (-\infty) - (-\infty)$;

$(+\infty) + (-\infty)$; $\qquad (-\infty) + (+\infty)$.

then the limit is an indeterminate form.

To remove the indetermination $\infty - \infty$, multiply and divide the expression by the conjugate.

Example

Calculate: $\lim\limits_{x \to +\infty} (x - \sqrt{x^2 + 4x - 1})$

Solution

$$\lim_{x \to +\infty} \sqrt{x^2 + 4x - 1} =$$

$$= \lim_{x \to +\infty} (x - \sqrt{x^2}) \qquad = \lim_{x \to +\infty} (x - |x|)$$

$$= \lim_{x \to +\infty} (x - x)$$

$$= (+\infty) - (+\infty) \qquad = \infty - \infty\text{: indeterminate form.}$$

$$\lim_{x \to +\infty} (x - \sqrt{x^2 + 4x - 1}) = \lim_{x \to +\infty} \frac{(x - \sqrt{x^2 + 4x - 1})(x + \sqrt{x^2 + 4x - 1})}{(x + \sqrt{x^2 + 4x - 1})}$$

$$= \lim_{x \to +\infty} \frac{x^2 - (x^2 + 4x - 1)}{x + \sqrt{x^2 + 4x - 1}} = \lim_{x \to +\infty} \frac{-4x}{x + x} = -2.$$

✓ **Indeterminate form $\frac{\infty}{\infty}$**

Example1 **Evaluate** $\lim\limits_{x \to +\infty} \frac{x+1}{x^2 + 3x + 1}$

Solution: $\lim\limits_{x \to +\infty} \frac{x+1}{x^2 + 3x + 1} = \frac{x\left(1 + \frac{1}{x}\right)}{x^2\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} \Rightarrow \lim\limits_{x \to +\infty} \frac{x+1}{x^2 + 3x + 1} = \lim\limits_{x \to +\infty} \frac{1}{x} \frac{\left(1 + \frac{1}{x}\right)}{\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} = \frac{1}{\infty} = 0$

✓ Indeterminate form $0 \times \infty$

Example1 **Evaluate** $\lim\limits_{x \to 0^+} x \ln x$

Solution $\lim\limits_{x \to 0^+} x \ln x = 0 \times \infty$ **which is indeterminate form.**

Theoretical learning Activity

- ✓ Discuss on what do you understand by **"indefinite forms"**

Practical learning Activity

- ✓ Evaluate the limit: $\lim_{x \to \pm\infty} \sqrt{x^2 + x + 2} - \sqrt{x^2 - x + 3}$

Points to Remember (Take home message)

> Suppose that $f(x) \to 0$ and $g(x)$ as $x \to a$. Then the limit of the quotient $\frac{f(x)}{g(x)}$ as $x \to a$ is said to give an indeterminate form, sometimes denoted by 00 . It may be that the limit of $\frac{f(x)}{g(x)}$ can be found by some methods such as factor method, rationalisation method, l'Hôspital's rule, etc...
> Similarly, if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then the limit of $\frac{f(x)}{g(x)}$ gives an indeterminate form, denoted by $\frac{\infty}{\infty}$. Also, if $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, then the limit of the product $f(x)g(x)$ gives an indeterminate form $0 \mathbf{x} \infty$.

Learning outcome 1.3: Formative Assessment

Practical assessment

- ✓ Task to be performed:

Find the following limits, if they exist.

1. $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 3x + 2}$ 2. $\lim_{x \to 3} \frac{\sqrt{25 - x^2} - 4}{x-3}$ **3.** $\lim_{z \to 0} \frac{\sqrt{2z + 4} - 2}{z}$

Learning outcome 1.4. Determine the asymptotes to the rational and polynomial functions.

Duration: 10 hrs

Learning outcome 2.4. Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly on the definitions of different types of asymptotes as applied in basic mathematical analysis.

2. Describe appropriately how to determine different types of asymptotes of a function as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet**	**Didactic materials such as manila paper** **Geometric instruments** **(Ruler, T-square)**	**Handouts on worked examples**

Advance preparation:

. Refer to manual discussing on how to determine the asymptotes to the rational and polynomial functions.

CONTENT

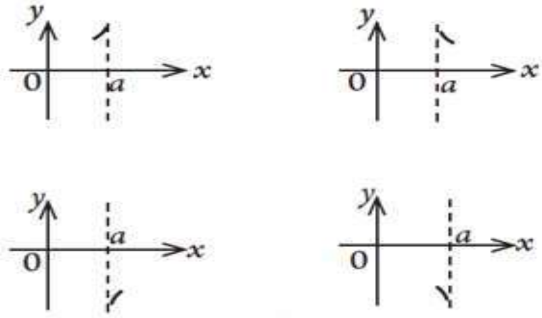
1.4.1: Definitions of asymptotes

• **<u>Definitions</u>**

A line (l) is an asymptote to a curve if the distance from a point P to the line (l) tends to zero as P tends to infinity along some unbounded part of the curve.

✓ **Boundaries of domain of definition**

✓ **Vertical asymptotes**

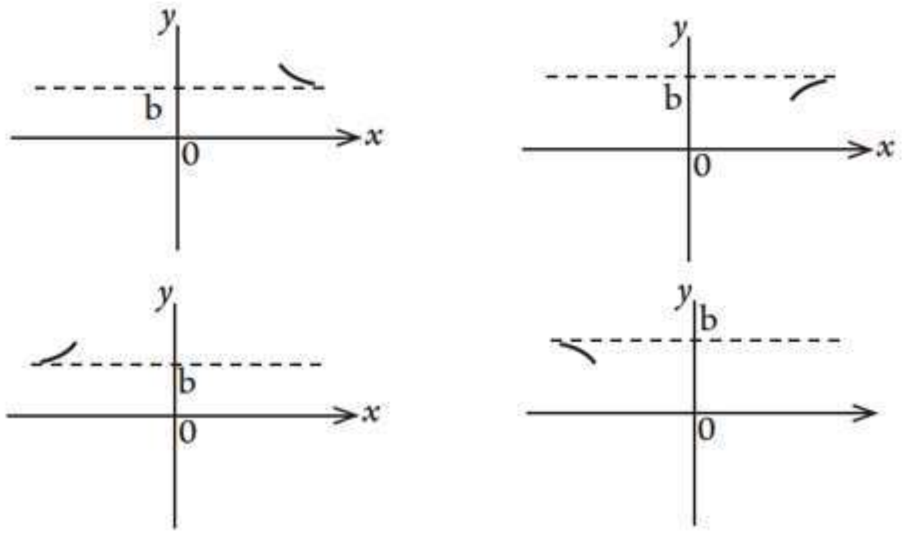


line $x = a$, where $a \in \mathbb{R}$, is said to be vertical asymptote to curve $y = f(x)$ if and only if $\lim_{x \to a} f(x) = \infty$.

More exactly, if at least one of the following is verified:

$$\lim_{x \to a+} f(x) = +\infty; \quad \lim_{x \to a-} f(x) = +\infty; \quad \lim_{x \to a-} f(x) = -\infty; \quad \lim_{x \to a} f(x) = -\infty.$$

✓ **Horizontal asymptotes**

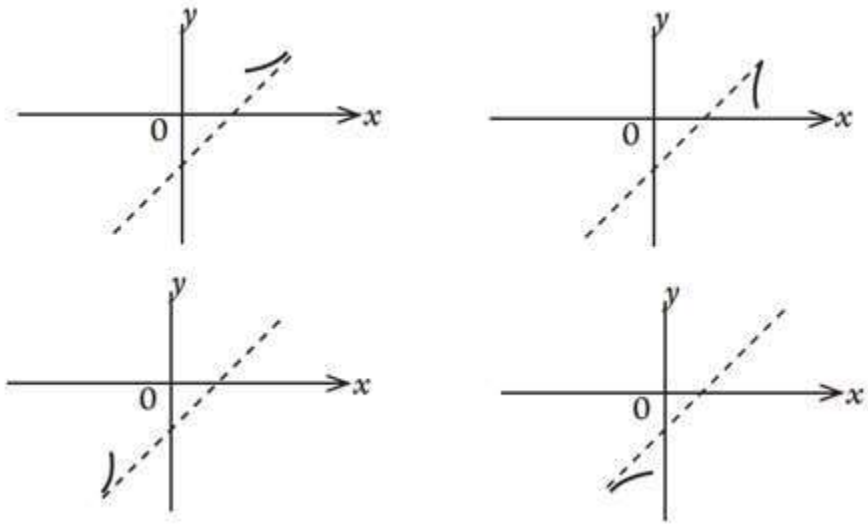


Line $y = b$, where $b \in \mathbb{R}$, is said to be horizontal asymptote to curve $y = f(x)$ if and only if $\lim_{x \to \infty} f(x) = b$.

More exactly,

$$\lim_{x \to -\infty} f(x) = b \text{ or } \lim_{x \to +\infty} f(x) = b, \ b \in \mathbb{R}.$$

✓ **Oblic asymptotes**



Line $y = m\,x + p$, where m and p are real numbers and $m \neq 0$ is said to be oblique asymptote of the graph of function $y = f(x)$ if and only if:

$$m = \lim_{x \to \infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \to \infty} [f(x) - mx].$$

Theoretical learning Activity

- ✓ Describe the types of asymptotes?

Practical learning Activity

- ✓ What is an asymptote? Carry out research and find out the meaning. Also, find out the types of asymptotes.

Points to Remember (Take home message)

A line **L** is an asymptote to a curve if the distance from a point P of the curve to the line L tends to zero as P tends to infinity along some unbounded part of the curve. We have three kinds of asymptotes: vertical asymptote, horizontal asymptote and oblique asymptote

CONTENT

1.4.2 : Determination of asymptotes

- **Calculations**

✓ Vertical asymptote

✓ Horizontal asymptote

✓ Oblique asymptote

Example 1:

Find the vertical asymptotes of function f(x) = $\dfrac{x^3}{1-x^2}$ and the position of the graph with respect to the vertical asymptotes.

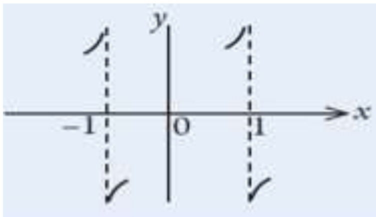
Solution

Vertical asymptotes: x = -1 and x = 1.

To determine the position of the curve with respect to the vertical asymptotes, we consider the sign of f(x) in the vicinity of x = -1 and x = 1.

x	$-\infty \quad -1 \quad 0 \quad 1 \quad +\infty$
x^3	$- \; - \; - \; - \; - \; 0 \; + \; + \; + \; +$
$1-x^2$	$- \; - \; - \; 0 \; + \; + \; 0 \; - \; - \; -$
$f(x)$	$+ \; \| \; - \; 0 \; + \; \| \; -$

$\lim\limits_{x\to -1^-} f(x) = +\infty$; $\lim\limits_{x\to -1^+} f(x) = -\infty$; $\lim\limits_{x\to 1^-} f(x) = +\infty$; $\lim\limits_{x\to 1^+} f(x) = -\infty$.



Example 2:

Find the equations of vertical and horizontal asymptotes of function f(x) = $\dfrac{x^2+3x}{4-x^2}$

Solution

Vertical asymptotes: x = -2 and x = 2

x	$-\infty \quad -3 \quad -2 \quad 0 \quad 2 \quad -\infty$
x^2+3x	$+ \; + \; 0 \; - \; - \; - \; 0 \; + \; + \; + \; +$
$4-x^2$	$- \; - \; - \; - \; 0 \; + \; + \; + \; 0 \; - \; - \; - \; -$
$f(x)$	$- \; 0 \; + \; \| \; - \; 0 \; + \; \| \; -$

$\lim\limits_{x\to -2^-} f(x) = +\infty$;

$\lim\limits_{x\to -2^+} f(x) = -\infty$;

$\lim\limits_{x\to 2^-} f(x) = +\infty$;

$\lim\limits_{x\to 2^+} f(x) = \infty$.

Horizontal asymptote:

y = -1 ($\lim_{x \to \infty} f(x) = -1$)

Example 3:

Find the asymptotes of the function f(x) = $\frac{x^2-x-2}{x-2}$

Solution

Let y = mx + p be the equation of oblique asymptote:

$$\text{Then } m = \lim_{x \to \infty} \frac{f(x)}{x}$$

$$= \lim_{x \to \infty} \frac{x^2 - x - 2}{x^2 + 2x} = 1$$

$$P = \lim_{x \to \infty} [f(x) - mx]$$

$$= \lim_{x \to \infty} \left(\frac{x^2 - x - 2}{x^2 + 2x} - x\right)$$

$$= \lim_{x \to \infty} \frac{x^2 - x - 2 - x^2 - 2x}{x + 2}$$

$$= \lim_{x \to \infty} \frac{-3x - 2}{x + 2} = -3$$

Therefore, the oblique asymptote : $y = x - 3$

Practice:

Lines $x = 2$ and $y = 3$ are asymptotes of function $f(x) = \frac{ax + 5}{bx + 4}$. Find a and b.

Line $x + 3 = 0$ is asymptote of function $f(x) = \frac{3x + 5}{2x + a}$. Find the value of a.

Find the vertical and horizontal asymptotes of the following functions:

(a) f(x) = $\frac{x^2-x-6}{x^2-x-20}$

(b) g(x) = $\frac{x+1}{(x+3)(x+5)}$ (c) h(x) = $\frac{(x+1)^2}{x^2+4x+3}$

Theoretical learning Activity

- ✓ Describe the types of asymptotes?

Practical learning Activity

- ✓ Graph the following and find its asymptotes y = $\frac{x^2-x-2}{x-2}$

Points to Remember (Take home message)

An asymptote can be in a negative direction, the curve can approach from any side (such as from above or below for a horizontal asymptote), or may actually cross over (possibly many times), and even move away and back again.
The important point is that:
The distance between the curve and the asymptote tends to zero as they head to infinity (or - infinity)

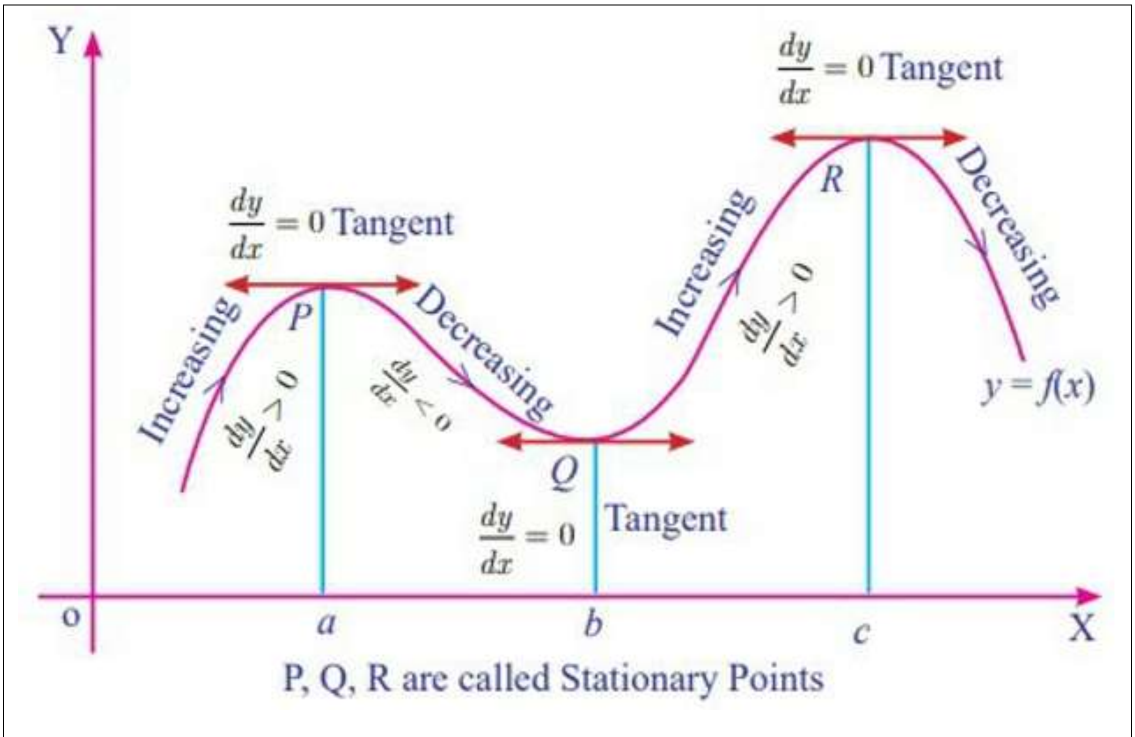
Learning outcome 1.4: Formative Assessment

1. **Find the asymptotes of $\frac{(x^2-3x)}{(2x-2)}$ and sketch the graph.**

2. Graph the following and find their asymptotes.

   1. y = $\frac{x+2}{x^2+1}$ 2. y = $\frac{x^3-8}{x^2+5x+6}$

Learning Unit 2: Apply fundamentals of differentiation



STRUCTURE OF LEARNING UNIT

Learning outcomes:

- **2.1. Determine derivative of a function.**
- **2.2. Interpret derivative of a function.**
- **2.3. Apply derivative.**
- **2.4. Sketch graph of a function.**

Learning outcome 2.1. Determine derivative of a function.

Duration: 5 hrs

Learning outcome 2.1. Objectives: By the end of the learning outcome, the trainees will be able to: 1. Define correctly the term: “Derivative” of a function as applied in basic mathematical analysis. 2. Describe clearly how to determine the derivatives of different types of functions as applied in basic mathematical analysis.

Resources		
Equipment	**Tools**	**Materials**
Reference books **Internet**	**Didactic materials such as manila paper**	**Geometric instruments (Ruler, T-square)** **Handouts on worked examples**

Advance preparation: . Refer to manual describing fundamentals of differentiation

CONTENT

2.1.1: Definition of derivative

- **Definition of derivative**

The **derivative of f(x)** is the function f '(x) given by f '(x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists.

The process of computing a derivative is called **differentiation.**

Further, f is differentiable on interval I if f is differentiable at every point in I.

Theoretical learning Activity

- ✓ Find out the meaning of the term derivative in mathematics?

Practical learning Activity

- ✓ What is the derivative of f(x) = $2x - 5$

Points to Remember (Take home message)

The derivative of a function, also known as slope of a function, or derived function or simply the derivative, is defined as f '(x) = $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$

CONTENT

2.1.2. Determination of derivatives

• <u>**Calculation of derivatives**</u>

✓ Derivative of function at a given point

The derivative of function $y = f(x)$ at $x = a$ is defined as $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists.

We say that f is differentiable at $x = a$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

By calculating the derivative in this way, we say that f is differentiated using definition or **first principle.**

1. Calculate the derivative of f(x) = $3x^2 + 2x - 1$ at x = 1.

Solution

We have: f ' (1) = $\lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h \to 0} \frac{[3\,(1+h)^2+2\,(1+h)-1]-4}{h}$$

$$= \lim_{h \to 0} \frac{3h^2+8h}{h}$$

$$= \lim_{h \to 0} (3h + 8) = 11\,.$$

Find the derivative of $f(x) = x^2 - 5x + 3$ at an unspecified value of x. Then evaluate $f'(0)$, $f'(1)$

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) + 3 - x^2 + 5x - 3}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 5)}{h}$$

$$= \lim_{h \to 0} (2x + h - 5) = (2x - 5)$$

$f'(0) = 2(0) - 5 = -5$

$f'(1) = 2(1) - 5 = -3$

Properties:

If $y = f(x)$, then

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

The expression $\frac{d}{dx}$ is called a differential operator and tells us that the derivative is with respect to x (that is by considering x as a variable).

(1) For any constant c, $\frac{d}{dx}\, c = 0$.

(2) $\frac{d}{dx}\, x = 1$.

(3) For any integer $n > 0$, $\frac{d}{dx} x^n = n\, x^{n-1}$. The formula is generalised for any real number r, $\frac{d}{dx}(x^r) = r\, x^{r-1}$.

(4) If f(x) and g(x) are differentiable at x and c is any constant, then.

(i) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$;

(ii) $\frac{d}{dx}[c\, f(x)] = c\, f'(x)$.

✓ **Derivative of a polynomial function**

If f(x) and g(x) are differentiable at x and c is any constant, then.

(i) $\frac{d}{dx}\ [f(x) \pm g(x)]$ = f ' (x) $\pm$ g ' (x) ;

(ii) $\frac{d}{dx}\ [c\ f(x)] = c\ f\ '\ (x)$.

Example

Find the derivative of f **(**x**)** = **2** + **3**x + **6**x**²** + **21**x**³**

Solution

f **'(**x**)** = **0** + **3**+**12**x + **63**x**²**

Product rule

Suppose that f and g are differentiable functions at x. Then

$\frac{d}{dx}[f(x).g(x)] = f'(x).g(x) + f(x).g'(x).$

Example

Find f ' (x) if f(x) = $(2x^4 - 3x + 5)\ (x^2 - \sqrt{x}\ +\ \frac{2}{x})$:

Solution

f ' (x) = $(8x^3 - 3)\ (x^2 - \sqrt{x}\ +\ \frac{2}{x}) + (2x^4 - 3x + 5)(2x - \frac{1}{2\sqrt{x}} - \frac{2}{x^2})$.

✓ **Derivative of a rational function**

Quotient rule

Suppose that f and g are differentiable at x and $g(x) \neq 0$.

Then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$.

Example

Find f ' (x) if f(x) $= \frac{x^2-2}{x^2+1}$.

Solution

$$f\ '\ (x) = \frac{(x^2-2)'\ (x^2+1) - (\ x^2+1)'\ (x^2-2)}{(x^2+1)^2}$$

$$= \frac{2x\ (x^2+1) - 2x\ (x^2-2)}{(x^2+1)^2}$$

$= \frac{6x}{(x^2+1)^2}$.

Exercises

Find the derivative of each function:

1. f (x) = $(x^2 + 3)(x^2 - 3x + 1)$. **2.** f(x) = $\frac{3x-2}{5x+1}$.

3. f(x) = $\frac{(x+1)(x-2)}{x^2 - 5x + 1}$. **4.** f(x) = $\frac{6x - \frac{2}{x}}{x^2 + \sqrt{x}}$.

Trigonometric formulas

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x, \qquad \frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

✓ **Derivative of an irrational function**

Formula

f(x) = $\sqrt[n]{u} \rightarrow f'(x) = \frac{u'}{n\sqrt[n]{u^{n-1}}}$

Example

f(x) = $\sqrt{x^2 - 3x} \rightarrow$ f ' = $\frac{2x-3}{2\sqrt{x^2 - x3}}$ f(x) = $\sqrt[3]{x^2+1} \rightarrow f'(x) = \frac{2x}{3\sqrt[3]{(x^2+1)^2}}$

f(x) = $\sqrt[3]{(x^2-3x)^2} \rightarrow f'(x) = \frac{2 \cdot (2x-3) \cdot (x^2-3x)}{3 \cdot \sqrt[3]{(x^2-3x)^4}}$

✓ **Successive derivatives**

Let $y = f(x)$

$$y' = \frac{dy}{dx} : \text{the first derivative of } y$$

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2} : \text{the second derivative of } y$$

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^3y}{dx^3} : \text{the third order derivative of } y$$

$$y^{(n)} = \frac{d^ny}{dx^n} : \text{the derivative of order n.}$$

Example

Let $y = 3x^4 - 2x^2 + 1$.

Compute as many derivatives as possible

Solution

$y' = 12x^3 - 4x$; $y'' = 36x^2 - 4$; $y''' = 72x$; $y^{(4)} = 72$; $y^{(5)} = 0$; $y^{(n)} = 0 \quad for\, n \geq 5$

Theoretical learning Activity

- ✓ Discuss on the first principle of derivative?

Practical learning Activity

- ✓ Find the derivative of the following function using the definition of the derivative: g(t) = $\sqrt{t}$

Points to Remember (Take home message)

- ✓ **The derivative of a function f(x), also known as slope of a function, or derived function or simply the derivative, is defined as**

  $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- ✓ From the first principles, the derivative functions of: **f(x) = x²**

  Solution

  $f(x) = x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x+0 = 2x$$

Learning outcome 2.1: Formative Assessment

Determine from first principles the derivative of the following functions:

(a) 2 **(b)** $x + x^3$ **(c)** $x^3 + 2x + 3$

Learning outcome 2.2. Interpret derivative of a function.

Duration: 5 hrs

Learning outcome 2.2 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly on Geometric interpretation of a derivative of a function at a point as applied in basic mathematical analysis.

2. Describe properly the kinematical meaning of a derivative as applied in basic mathematical analysis.

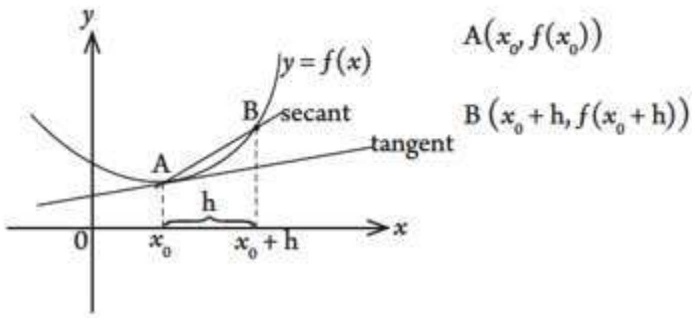
Resources

Equipment	Tools	Materials

Reference books **Internet**	**Didactic materials such as manila paper**	**Handouts on worked examples** **Geometric instruments**
Advance preparation: . Refer to manual describing interpretation of derivative		

2.2.1 : Geometric interpretation of derivative of a function at a point

• **Geometric interpretation of derivative at a point**



The gradient of the secant line AB is $\frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$.

As $h \to 0$, B moves on the graph $y = f(x)$ towards A, the secant line approaches the tangent line to the graph $y = f(x)$ at x_0. Thus, the gradient (the slope) of the tangent is $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$. Therefore, the equation of the tangent to curve $y = f(x)$ at point $(x_0, f(x_0))$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

Since, the normal is the perpendicular to the tangent, the equations of the normal to curve $y = f(x)$ at point $(x_0, f(x_0))$ is $y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$.

Examples

Find the equation of the tangent to graph:

(a) $y = x^2 + 1$ at $x = 1$

(b) $y = \frac{2}{x}$ at $x = 2$

Solution

(a) $x = 1$; $y = 1^2 + 1 = 2$

$y' = 2x$; at $x = 1$, $y' = 2$

The equation of the tangent is

$y - 2 = 2(x - 1)$; $y = 2x$.

(b) $x = 2$; $y = 1$

$y' = -\frac{2}{x^2}$; $at\ x = 2,\ y' = -\frac{1}{2}$.

The equation of the tangent is

$y - 1 = -\frac{1}{2}(x - 2);\ y = -\frac{1}{2}x + 2$.

Exercises

1. Find the equation of the tangent line to the graph of function $f(x) = 4 - 4x + \frac{2}{x}$

2. Find the equation of the normal to the graph of function

$f(x) = (x^4 - 3x^2 + 2x)(x^3 - 2x + 3)$ at the origin.

Theoretical learning Activity

- ✓ **Discuss on the interpretation of a derivative of a certain function at any given point?**

Practical learning Activity

- ✓ Find from first principles, the slope of the tangent to the following functions at a given value of x: $f(x) = 2x^2 + 3$ at $x = 2$

Points to Remember (Take home message)

.The slope (gradient) of the tangent to a curve of f(x) is defined as the slope of
the curve f(x), and is the instantaneous rate of change in y with respect to x.

• Finding the slope using the limit method is said to be using first principles.

• A chord (secant) of curve is a straight line segment which joins any two
points on the curve.

• A tangent is straight line which touches curve at point.

.The slope of the tangent at the point x = a is defined as the slope of the curve at the point where x= a, and is the instantaneous rate of change in y with respect to x at that point.

2.2.2: Kinematical meaning of a derivative

• <u>**Kinematical meaning of a derivative**</u>

Velocity

Suppose that the function $f(t)$ gives the position at time t of an object moving along a straight line. That is, $f(t)$ gives the displacement from a fixed reference point, so that $f(t) < 0$ means that the object is located $|f(t)|$ away from the reference point, but in the negative direction. Then, for two times a and b (where $a < b$), $f(b) - f(a)$ gives the signed distance between the positions $f(a)$ and $f(b)$.

The average velocity is given by

$$v = \frac{f(b) - f(a)}{b - a}.$$

If $f(t)$ represents the position of an object with respect to some fixed point at time t as it moves on a straight line, then the instantaneous velocity at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$ provided the limit exists.

Example

The height of a falling object, t seconds after being dropped from a height of 64 meters is given by $f(t) = 64 - 16t^2$. Find the average velocity between times t = 1 and t = 2, and find the instantaneous velocity at time t = 2.

Solution

Average velocity = $\frac{f(2)-f(1)}{2-1} = \frac{0-48}{1}$ = - 48m/s .

Instantaneous velocity:

$f'(t) = -32t$

$t = 2$, $f'(2) = -32(2) = -64$ m/s .

If s(t), v(t) and a(t) are position, velocity and acceleration at time t respectively, we have $v = \frac{ds}{dt}$ *and* $a = \frac{d^2s}{dt^2}$ *or* $a = \frac{dv}{dt}$.

✓ A body moves along the x-axis so that at time t seconds $x(t) = t^3 + 3t^2 - 9t$. Find:
(a) the position and velocity of the body at t = 0. 1, 2
(b) where and when the body comes to rest
(c) the maximum speed of the body in the first 1 second of motion
(d) the maximum velocity of the body in the first 1 second of motion
(e) the total distance travelled by the body in the first 2 seconds of motion.

✓ **Solution**

(a) $x(t) = t^3 + 3t^2 - 9t$; $v(t) = 3t^2 + 6t - 9 = 3(t^2 + 2t - 3) = 3(t + 3)(t - 1)$

When t = 0, x = 0 and v = –9; when t = 1, x = –5 and v = 0; when t = 2, x = 2 and v = 5 .

At t = 0, the body is at the origin with velocity of $-9ms^{-1}$.

At t = 1, the body is 5m to the left of 0 with velocity of $0ms^{-1}$.

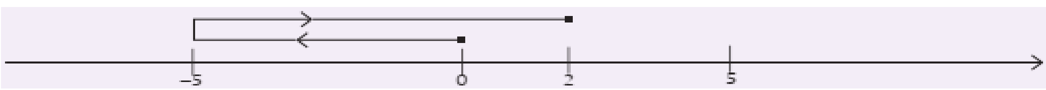
At t = 2, the body is 2m to the right of 0 with velocity of 15 ms^{-1}.

(b) The body is at rest when v = 0. This occurs when t = 1 (t ≥ 0). At this time the body is 5m to the left of the origin.

(c) The velocity is increasing in the interval [0, 1] since $v'(t) = 6t + 6 > 0$. v(0) = –9 and v(1) = 0. Therefore the maximum speed in the first 1 second is $9ms^{-1}$.

(d) From part (c), the maximum velocity is $0ms^{-1}$.

(e) The following diagram illustrates the position of the body from t = 0 to t = 2.



From the diagram the total distance travelled is 12 m.

Exercises

1. The height of a falling object t seconds after being released is given by, $s(t) = 640 - 20t - 16t^2$. Find the acceleration at time t.
2. Use the position function to find the velocity at time t:
   (a) $s(t) = t^2 - \sin 2t;\ t = 0.$ (b) $s(t)= \frac{\cos t}{t};\ t = \pi.$
   (c) $s(t) = 4 + 3\sin t;\ t = \pi.$

Indeterminate forms and Hospital's rule

Suppose that f and g are differentiable on the interval $]a, b[$, except possibly at some fixed point $c \in]a, b[$ and that $g'(x) \neq 0$ on $]a, b[$, except possibly at c.

Suppose further that $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ has the indetermination $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and that $\lim\limits_{x \to c} \frac{f'x)}{g'(x)} = L$ (or ∞).

Then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \lim\limits_{x \to c} \frac{f'x)}{g'(x)}$.

The conclusion holds if $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ is replaced with any of the limits

$$\lim_{x \to c^+} \frac{f(x)}{g(x)}, \quad \lim_{x \to c^-} \frac{f(x)}{g(x)} \quad \text{or} \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)}.$$

Example

Evaluate: $\lim\limits_{x \to 0} \frac{1-\cos x}{\sin x}$.

Solution

$$\lim_{x \to 0} \frac{1-\cos x}{\sin x} = \frac{1-\cos 0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}; \text{ indeterminate form.}$$

$$\lim_{x \to 0} \frac{1-\cos x}{\sin x} = \lim_{x \to 0} \frac{(1-\cos x)'}{(\sin x)'} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0.$$

Remark: Before applying Hospital's rule, make sure you have indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theoretical learning Activity

- ✓ Discuss on what is Kinematical meaning of a derivative?

Practical learning Activity

- ✓ A particle is moving along the x-axis such that its position, x(t) metres to the right of the origin at time t seconds, is given by $x(t) = t^3 - 9t^2 + 24t - 18$. Describe the particle motion during the first five seconds and calculate the distance travelled in that time.

Points to Remember (Take home message)

If the function y = f(x) is represented by a curve, then $f'(x) = \frac{dy}{dx}$ is the slope function; it is the rate of change of y with respect to x. Since $f''(x) = \frac{d^2y}{dx^2}$ is the derivative of the slope function, it is the rate of change of slope and is related to a concept called convexity (bending) of a curve. If x = t is time and if y = s(t) is displacement function of moving object, then $s'(t) = \frac{ds}{dt}$ is the velocity function. The derivative of velocity i.e. the second derivative of the displacement function is $s''(t)$ or $s''(t)$ or $\frac{d^2s}{dt^2}$; it is the rate of change of the velocity function, which is, the acceleration function.

Learning outcome 2.2: Formative Assessment

1. A body moves along the x-axis so that its position is x(t) metres to the right of the origin at time t seconds.

a) If $x(t) = t^3 - 3t^2$ explain why the total distance travelled in the first three seconds of motion is not equal to the displacement in that time.

b) If $x(t) = t^3 - 3t^2 + 3t$ explain why the distance travelled in that first three seconds of motion is now equal to the displacement in that time.

Learning outcome 2.3. Apply derivative.

Duration: 5 hrs

Learning outcome 2.3 Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Describe clearly how to determine the equation of tangent line at a given point as applied in basic mathematical analysis.

2. Describe correctly how to determine the equation of normal line at a given point as applied in basic mathematical analysis.

3. Demonstrate appropriately the increasing and decreasing intervals for a function as applied in basic mathematical analysis.

4. Describe properly the maximum and minimum points of a function as applied in basic mathematical analysis.

5. Discuss clearly the concavity, inflection point on a graph as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference Books **Internet**	**Didactic materials such as manila paper**	**Hand-out notes** **Geometric instruments**

Advance preparation:

. Refer to manual discussing on application of derivative.

CONTENT

2.3.1: Determination of equation of tangent line at a given point

● **Tangent and normal at a point a of a function**

●Determination of equation of normal line at a given point

The equation of the tangent line at x = a can be expressed as:

$$\boxed{f(x) - f(a) = \mathbf{f'(a)}(x - a)}$$

The equation of the normal line at x = a can be expressed as:

$$f(x) - f(a) = \frac{-1}{f'(a)}(x - a)$$

Example 1: Find the equation of the tangent and normal lines of the function $f(x) = \sqrt{2x-1}$ at the point (5, 3).

Solution:

a) Equation of the Tangent Line.

Step 1: Find the slope of the function by solving for its first derivative.	$f(x) = \sqrt{2x-1}$ $f(x) = (2x-1)^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}(2x-1)^{\frac{-1}{2}}(2)$ $f'(x) = \frac{1}{\sqrt{2x-1}}$

Step 2: Knowing $f'(a)$, solve for the slope of the tangent at $a = 5$.	$f'(5) = \frac{1}{\sqrt{2(5)-1}}$ $f'(5) = \frac{1}{3}$
Step 3: Solve for $f(a)$.	$f(5) = \sqrt{2(5)-1}$ $f(5) = 3$
Step 4: Substitute found values into the equation of a tangent line.	$f(x) - f(a) = f'(a)(x-a)$ $f(x) - 3 = \frac{1}{3}(x-5)$

b) Equation of the Normal Line.

Step 1: Find the slope of the normal line $\frac{-1}{f'(a)}$.	Since $f'(5) = \frac{1}{3}$, then $\frac{-1}{f'(a)} = -3$
Step 2: Given the equation of a tangent line, swap slopes.	$f(x) - f(a) = \frac{-1}{f'(a)}(x-a)$ $f(x) - 3 = -3(x-5)$

Theoretical learning Activity

- ✓ Discuss on what is a tangent line at normal point?

Practical learning Activity

- ✓ Find the equation of the tangent to f(x) = x^2 + 2 at the point where x = 1.

Points to Remember (Take home message)

- ✓ **Tangent line to a curve of function**
  Consider a curve y = f(x) . If P is the point with x-coordinate a, then the slope of the tangent at this point is f′(a) . The equation of the tangent is by equating slopes $\frac{y-f(a)}{x-a} = f'(a)$ or $y - f(a) = f'(a)(x - a)$
- ✓ **Normal line to a curve of function**
  A normal to a curve is a line which is perpendicular to the tangent at the point of contact. Therefore, if the slope of the tangent at x = a is f′(a) , then the slope of a normal at x = a is $-\frac{1}{f'(a)}$ This comes from the fact that the product of gradients of two perpendicular lines is –1.
- ✓ **Note**: If a tangent touches y = f(x) at (a, b) then it has equation $\frac{y-b}{x-a} = f'(a)$ or $y - b = f'(a)(x - a)$
- ✓ Vertical and horizontal lines have equations of the form x = k and y = c respectively, where c and k are constants.

2.3.2: Increasing and decreasing intervals for a function

A function f is **strictly increasing** on an interval I if for every x_1, x_2 in I, with $x_1<$ $f(x_1)<f(x_2)$, that is $f(x)$ gets larger as x gets larger. For convenience, we place ascenc arrows; ↗ .

A function f is **strictly decreasing** on an interval I if for every x_1, x_2, in I , with $x_1<$ $f(x_1)>f(x_2)$, that is $f(x)$ gets smaller as x gets larger. For convenience, we place descenc arrows; ↘ .

Suppose that f is differentiable on an interval I:

(i) If $f'(x) > 0$ for all $x \in$ I, then f is increasing,

(ii) If $f'(x) < 0$, 0 for all $x \in$ I, then f is decreasing.

Example

Find the interval where $f(x) = 2x^3 + 9x^2 - 24x - 10$ is:

(a) increasing,

(b) decreasing.

Solution

$$f'(x) = 6x^2 + 18x - 24$$
$$= 6(x-1)(x+4)$$

x	$-\infty$		-4		1		$+\infty$
$f'(x)$	+	+	+ 0	− − −	0 +	+	+
		$\nearrow$		$\searrow$		$\nearrow$	

(a) f is increasing for $x \in \,]-\infty, -4[\, \cup \,]1, +\infty[$.

(b) f is decreasing for $x \in \,]-4, 1[$.

Note that the critical numbers (1 and −4) are the only possible locations for local extrema.

Theoretical learning Activity

- ✓ Describe what is an increasing and decreasing function?

Practical learning Activity

- ✓ Given the function $f(x) = 2x^3 + 9x^2 + 12x + 20$. Determine the interval where the graph of the function is increasing and where it is decreasing.

Points to Remember (Take home message)

- ✓ A real function f is **increasing** in or on an interval I if $f(x_1) \leq f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly increasing** if $f(x1) < f(x_2)$ whenever $x_1 < x_2$.

  A real function f is **decreasing** in or on an interval I if $f(x_1) \geq f(x_2)$ whenever x_1 and x_2 are in I with $x_1 < x_2$. Also, f is **strictly decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

2.3.3 : Maximum and minimum points of a function

• Maximum and minimum points of a function

First derivative test

Suppose that f is continuous on the interval $[a, b]$ and $c \in]a, b[$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in]a, c[$ and $f'(x) < 0$ for all $x \in]c, b[$ (that is f' changes the sign from positive to negative) then $f(c)$ is a local maximum.

(ii) If $f'(x) < 0$ for all $x \in]a, c[$ and $f'(x) > 0$ for all $x \in]c, b[$, then $f(c)$ is a local minimum.

(iii) If $f'(x)$ has the same sign on $]a, c[$ and $]c, b[$, then $f(c)$ is not a local extremum.

Example

Find the local extrema of function $f(x) = 2x^3 - 3x^2 - 12x + 13$, and state the nature of each of them. Find intervals when f is increasing, decreasing.

Solution

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1)$$

$f'(x) = 0$ if and only if $x = -1$ or $x = 2$

x	$-\infty$		-1		2		$+\infty$
$f'(x)$	+	+	0	− −	0	+	+
		↗		↘		↗	

f is increasing if $x \in]-\infty, -1[\cup]2, +\infty[$ f is decreasing if $x \in]-1, 2[$ maximum $(-1, 20)$, minimum $(2, -7)$.

Theoretical learning Activity

- ✓ Describe on what is maximum and minimum points of inflection?

Practical learning Activity

- ✓ Find the maximum profit that a company can make, if the profit function is given by $P(x) = 4 + 24x - 18x^2$

Points to Remember (Take home message)

- ✓ **Stationary point**

  This is a point on the graph $y = f(x)$ at which f is differentiable and $f'(x) = 0$. The term is also used for the number c such that $f'(c)$. The corresponding value $f(c)$ is a stationary value. A stationary point c can be classified as one of the following, depending on the behaviour of f in the neighbourhood of c:

  (i) A local maximum, if $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c,

  (ii) A local minimum, if $f'(x) < 0$ to the left of c and $f'(c) > 0$ to the right of c,

  (iii) Neither local maximum nor minimum, if (i) and (ii) are not satisfied.

  Note: Maximum and minimum values are termed as extreme values

2.3.4 : Concavity, inflection point on a graph

- <u>**Concavity, inflection point on a curve**</u>

For a function f that is differentiable on an interval I, the graph of f is:

(i) **Concave up** on I if $f''(x) > 0$ (that is f' is increasing on I): indicated by

(ii) **Concave down** on I if $f''(x) < 0$ (that is f' is decreasing on I): indicated by

Suppose that f is continuous on the interval $]a, b[$ and that the graph changes the concavity at a point $c \in]a, b[$.

Then the point $(c, f(c))$ is called an **inflection point** of f.

Second derivative test

Suppose that f is continuous on the interval $]a, b[$ and $f'(c) = 0$, for some number $c \in]a, b[$,

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum,

(ii) If $f''(c) > 0$, then $f(c)$ is a local maximum.

Example

1. Use the second derivative test to find the nature of the local extrema of function $f(x) = x^4 - 8x^2 + 10$.

Solution

$$f'(x) = 4x^3 - 16x$$
$$= 4x(x-2)(x+2)$$

Thus the critical numbers are $x = 0$; $x = 2$; $x = -2$.

We also have : $f''(x) = 12x^2 - 16$

$$f''(0) = -16 < 0$$
$$f''(-2) = 32 > 0$$
$$f''(2) = 32 > 0$$

So by the second derivative test, $f(0)$ is a local maximum and $f(-2)$ and $f(2)$ are local minima.

2. Determine where the graph of $f(x) = x^4 - 6x^2 + 1$ is concave up and concave down, and find the inflection point.

Solution

$$f'(x) = 4x^3 - 12x$$
$$= 4x(x - \sqrt{3})(x + \sqrt{3})$$

$$f''(x) = 12x^2 - 12$$
$$= 12(x-1)(x+1)$$

x	$-\infty$		-1		1		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
			I_1		I_2		

The graph is concave up if $x \in]-\infty, 1[\text{u}]1,+\infty[$, and concave down if $x \in]-1, 1[$. The graph has two inflection points: $I_1(-1, -4)$ and $I_2(1, -4)$.

Exercises

1. Determine the intervals where the graph of the given function is concave up and concave down:
   (a) $f(x) = x^3 - 3x^2 + 4x - 1$ (b) $f(x) = x^4 - 6x^2 + 2x + 3$
2. Find the local extrema of function:
   (a) $y = 3x^4 - 4x^3$. (b) $y = \frac{1}{3}x^3 - 2x^2 + 3x + 1$.
3. Given the function $f(x) = -x^3 + 3x + 4$, find:
   (a) the first and second derivatives, study their signs.
   (b) the intervals where f is increasing, decreasing.
   (c) the local extrema and precise the nature of each of them.
   (d) the intervals where the graph of f is concave up, concave down.
   (e) the inflection point.

Theoretical learning Activity

- ✓ Discuss on concavity and inflection point of a function?

Practical learning Activity

- ✓ Given the function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

  (a) State the values of x for which f is increasing

  (b) Find the x-coordinate of each extreme point of f.

  (c) State the values of x for which the curve of f is concave upwards.

  (d) Find the x-coordinate of each point of inflection.

  (e) Sketch the general shape of the graph of f indicating the extreme points and points of inflection.

Points to Remember (Take home message)

- ✓ A curve is said to be concave downwards (or concave) in an interval $]a, b[$ If $f''(x) < 0$ for all $x \in]a,b[$.
- ✓ A curve is said to be concave upwards (or convex)
- ✓ A point of inflection is a point on a graph $y = f(x)$ at which the concavity changes. If f' is continuous at a , then for $y = f(x)$ to have a point of inflection at a it is necessary that $f''(a) = 0$, and so this is the usual method of finding possible points of inflection. in an interval $]a, b[$ if $f''(x) > 0$ for all $x \in]a,b[$.

Learning outcome 2.3: Formative Assessment

1. Find the inflection point of the function defined by: $f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x$
2. State the values of x for which the curve of f is concave upwards

Learning outcome 2.4. Sketch graph of a given function.

Duration: 5 hrs

Learning outcome 3.4. Objectives:

By the end of the learning outcome, the trainees will be able to:

1. Discuss clearly how to establish required parameters as applied in basic mathematical analysis.

2. Describe appropriately how to sketch a graph as applied in basic mathematical analysis.

Resources

Equipment	Tools	Materials
Reference books **Internet** **Scientific calculator**	**Didactic materials such as manila paper** **Geometric instruments** **(Ruler, T-square)**	**Handouts on worked examples**

Advance preparation:

. Refer to a manual describing how to sketch a graph of a function.

2.4.1 : Establishing required parameters

Introduction

A rule that defines a function can be given by: an equation connecting the independent variable x and the dependent variable y = ƒ(x) or a graph: the set of all points (x, y) in the xy– plane that satisfy the equation y = ƒ(x).

• <u>**Parameters required**</u>

✓ **Variation table**

By definition, a variation table is the table which contains the results from the first derivative and second derivative.

✓ **Additional points**

Additional points are other points which shows where the curve of a function must pass

Theoretical learning Activity

- ✓ Discuss on the parameters required for graphing a given function

Practical learning Activity

- ✓ Describe on a variation table and addition points for graphing a function f(x)

Points to Remember (Take home message)

- ✓ **Variation table:** is the table which contains the results from the first and second derivative.
- ✓ **Additional** other points which shows where the curve of a function must pass points**.**

2.4.2 : Sketching graph

● <u>**Curve sketching**</u> **x**

When trying to draw the graph of **y** = ƒ(**x**), start by gathering information about the graph through the following tests:

1. Domain
2. Limits at the end points of the domain and asymptotes
3. First derivative information
4. Second derivative.
5. Variation table
6. Intercepts
7. Supplementary points
8. Sketch the curve.

✓ Curve sketching of a polynomial function

Example 1

Investigate fully the graph of function if: $\mathbb{R} \to \mathbb{R}: x \mapsto f(x) = -x^4 + 5x^2 - 4$:

Solution

1. Domain:

$D_f =]-\infty, +\infty[$

$\forall x \in D_f, -x \in D_f$ and $f(-x) = -(-x)^4 + 5(-x)^2 - 4$

$= -x^4 + 5x^2 - 4$

$= f(x)$

Therefore , f is an even function.

The graph of $y = f(x)$ is symmetrical about the y–axis: it is sufficient to graph the function on $\mathbb{R}^+ = [0, +\infty[$ and complete it on $]-\infty, 0[$ from symmetry.

2. Limits at the end points of the domain

$$= \lim_{x \to +\infty} (-x^4 + 5x^2 - 4)$$
$$= \lim_{x \to +\infty} (-x^4)$$
$$= -(+\infty)^4$$
$$= -\infty$$

From symmetry, $\lim_{x \to -\infty} (-x^4 + 5x^2 - 4) = -\infty$.

The graph has no asymptote.

3. First derivative information.

$f'(x) = -4x^3 + 10x = -2x(2x^2 - 5)$

$f''(x) = 0$ if and only if $x = 0$ or $x = \frac{\sqrt{10}}{2}$(on$[0, +\infty[$)

x	0		$\frac{\sqrt{10}}{2}$		$+\infty$
y'	0	$+$	0	$-$	
y	-4	$\nearrow$	$\frac{9}{4}$	$\searrow$	$-\infty$

Function f is increasing for $x \in]0, \frac{\sqrt{10}}{2}[$ and f is decreasing for $x \in]\frac{\sqrt{10}}{2}, +\infty[$.

The graph has a maximum at $M(\frac{\sqrt{10}}{2}, \frac{9}{4})$.

4. Second derivative information:

$f''(x) = -12x^2 + 10$

$f''(x) = 0$ if and only if $x = \frac{\sqrt{30}}{6}$

x	0	$\frac{\sqrt{30}}{6}$	$+\infty$
y''	$+$	0	$-$
y	$\smile$	$-\frac{19}{36}$	$\frown$

The graph is concave up if $x \in \left]0, \frac{\sqrt{30}}{6}\right[$ and concave down if $x \in \frac{\sqrt{30}}{6}, +\infty[$.

The graph has an inflection point at $I_1\left(\frac{\sqrt{30}}{6}, -\frac{19}{36}\right)$.

The first and second derivatives information can be summarized in the table below:

x	0		$\frac{\sqrt{30}}{6}$		$\frac{\sqrt{10}}{2}$	$+\infty$
y'	0	$+$		$+$	0	$-$
y''	$+$		0	$-$	$-$	$-$
y	-4		$-\frac{19}{36}$		$\frac{9}{4}$	$-\infty$

Intercepts of the functions:

The x- intercepts:

$$\begin{cases} y = -x^4 + 5x^2 - 4 \\ y = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x = 1 \\ y = 0 \end{cases} \begin{cases} x = 2 \\ y = 0 \end{cases}$$

the x-intercepts are $(1, 0)$ and $2, 0)$ on $[0, +\infty[$

the y–intercept

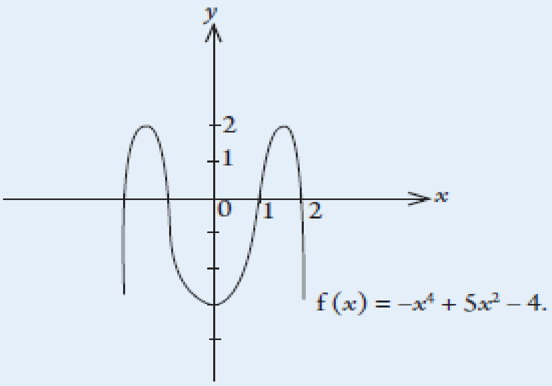
$$\begin{cases} x = 0 \\ y = -x^4 + 5x^2 - 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = -4 \end{cases}$$

The y-intercept is $(0, -4)$.

Some other points on the graph are;

x	y	point
0.5	−2.8	(0.5; −2.8)
1.5	2.2	(1.5; 2.2)
2.5	−11.8	(2.5; −11.8)



✓ Curve sketching of a rational function

Example 2

Investigate fully the graph of function if $\mathbb{R} \to \mathbb{R}$: $x \mapsto f(x) = \dfrac{2x-3}{x+1}$.

Solution

(1) $D_f =]-\infty, -1\,[\cup]\,-1, +\infty[$

f is neither even, nor odd, nor periodic.

(2) Limits at the end points of the domain.

$$\lim_{x \to +\infty} \frac{2x-3}{x+1} = 2$$

$$\lim_{x \to -1^-} \frac{2x-3}{x+1} = +\infty$$

$$\lim_{x \to -1^+} \frac{2x-3}{x+1} = -\infty$$

$$\lim_{x \to +\infty} \frac{2x-3}{x+1} = 2$$

Asymptotes : $x = -1$ and $y = 2$.

(3) First derivative information:

$f'(x) = \dfrac{5}{(x+1)^2}$; $f'(x) \neq 0$, for all x .

x	$-\infty$		-1		$+\infty$
y'		+	‖	+	
y	2	↗ $+\infty$	‖	$-\infty$ ↗	2

f is increasing for $x \in\ ]-\infty, -1[\cup]-1, +\infty[$.

No maximum, no minimum.

4. Second derivative information:

$$f''(x) = -\frac{10}{(x+1)^3}$$

x	$-\infty$	-1	$+\infty$
y''	$+$	$\|\|$	$-$
y	$\cup$	$\|\|$	$\cap$

The graph is concave up if $x \in \,]-\infty, -1[$ and concave down if $x \in \,]-1, +\infty[$.

No inflection point.

The first and second derivatives information can be summarised in the table below:

x	$-\infty$	-1	$+\infty$
y'	$+$	$\|\|$	$+$
y''	$+$	$\|\|$	$-$
y	$2 \nearrow +\infty$ ($\cup$)	$\|\|$	$-\infty \nearrow 2$ ($\cap$)

5. Intercepts

x-intercept: $\begin{cases} y = \dfrac{2x-3}{x+1} \\ y = 0 \end{cases}$ $\qquad \begin{cases} x = \dfrac{3}{2} \\ y = 0 \end{cases}$

The x-intercept is $(\frac{3}{2}, 0)$

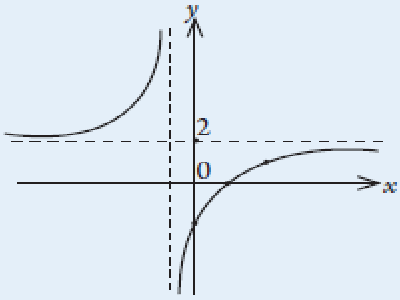
y-intercept $\begin{cases} y = \dfrac{2x-3}{x+1} \\ x = 0 \end{cases}$ $\qquad \begin{cases} y = -3 \\ x = 0 \end{cases}$

The y-intercept is $(0, -3)$.

Some other points on the graph are;

x	y	point
-3	$\frac{9}{2}$	$(-3, \frac{9}{2})$
-2	7	$(-2, 7)$
1	$-\frac{1}{2}$	$(1, -\frac{1}{2})$
2	$\frac{1}{3}$	$(2, \frac{1}{3})$
3	$\frac{3}{4}$	$(3; \frac{3}{4})$
4	1	$(4, 1)$

The needed graph is;



✓ Curve sketching of an irrational function

Example

Investigate fully the graph of the function:

$f: \mathbb{R} \to \mathbb{R} : x \longmapsto f(x) = \sqrt{4x^2 - 2x - 6}$

Solution

1. Domain: $D_f =]-\infty, -1] \cup [\frac{3}{2}, +\infty[$

2. Limits at the boundaries of the domain and asymptotes:

$$\lim_{x \to -\infty} \sqrt{4x^2 - 2x - 6} = +\infty$$

$$\lim_{x \to -1^-} \sqrt{4x^2 - 2x - 6} = 0$$

$$\lim_{x \to \frac{3}{2}^+} \sqrt{4x^2 - 2x - 6} = 0$$

$$\lim_{x \to +\infty} \sqrt{4x^2 - 2x - 6} = +\infty$$

There are two oblique asymptotes:

$y = 2x - \frac{1}{2}$ (at $+\infty$)

and $y = -2x + \frac{1}{2}$ (at $-\infty$).

3. First derivative information:

$$f'(x) = \frac{4x - 1}{\sqrt{4x^2 - 2x - 6}}$$

$$f'(x) = 0 \Leftrightarrow x = \frac{1}{4}$$

x	$-\infty$	-1		$\frac{3}{2}$	$+\infty$
$f'(x)$	$-$	$\Vert$		$\Vert$ $+$	
$f(x)$	$+\infty \searrow$	0		$0 \nearrow$	$+\infty$

Function f is increasing if $x > \frac{3}{2}$ and decreasing if $x < -1$.

The graph of f is:

Concave down for $x < -1$ or $x > \frac{3}{2}$.

4. Intercepts

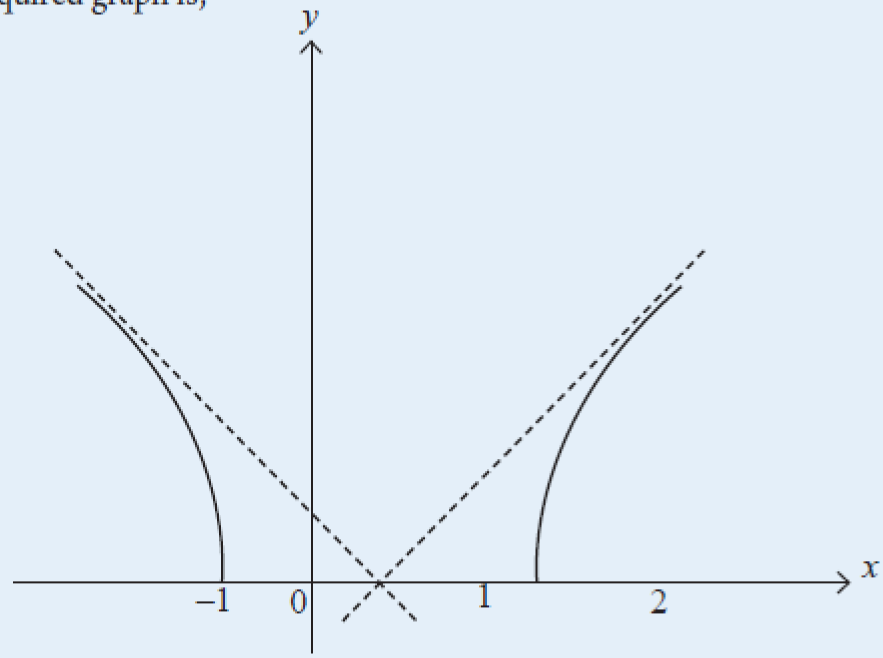
the x – intercepts are $(-1, 0)$ and $(\frac{3}{2}, 0)$.

No y-intercepts exist.

Some other points are;

x	y	point
–3	6	(–3, 6)
–2	$\sqrt{14}$	$(-2, \sqrt{14})$
3	$\sqrt{24}$	$(3, \sqrt{24})$
4	$\sqrt{50}$	$(4, \sqrt{50})$

The required graph is;



Exercises

Investigate fully each of the following:

1. $f(x) = -x^2 - 3x - 2$
2. $f(x) = \frac{x+2}{x-4}$
3. $f(x) = \frac{x^2 - 2x - 8}{x - 1}$
4. $f(x) = \frac{2x^2 + 1}{x^2 - 3x + 2}$

Theoretical learning Activity

- ✓ Discuss on solving and graphing a function?

Practical learning Activity

- ✓ Given the function $y = f(x) = \frac{x^3}{3} + \frac{x^2}{2}$

  (a) State the values of x for which f is increasing

  (b) Find the x-coordinate of each extreme point of f.

  (c) State the values of x for which the curve of f is concave upwards.

  (d) Find the x-coordinate of each point of inflection.

  (e) Sketch the general shape of the graph of f indicating the extreme points and points of inflection.

Points to Remember (Take home message)

- ✓ When trying to draw the graph of $\mathbf{y} = f(\mathbf{x})$, start by gathering information about the graph through the following tests:

  1. Domain
  2. Limits at the end points of the domain and asymptotes
  3. First derivative information
  4. Second derivative.
  5. Variation table
  6. Intercepts
  7. Supplementary points
  8. Sketch the curve.

Learning outcome 2.4: Formative Assessment

1.Given the function. Determine: $f(x) = \frac{x+1}{(x-3)^2}$

a) the domain of f

b) the x-intercept(s) and y-intercept(s)

c) all asymptotes to the curve of the function f.

d) the first derivative f'(x) and the second derivative f''(x)

e) the extrema point(s) (local minimum or local maximum)

f) interval(s) on which f is increasing or decreasing.

g) inflection point(s)
h) interval(s) of upward or downward concavity.

Learning Unit 3: Apply exponential functions

Structure of the Learning unit :

DURATION : 20HOURS

LEARNING OUTCOMES :

3.1 Determine the domain of exponential function

3.2 Calculate limit of exponential functions

3.3 Solve equations involving exponentials

Learning outcome 3.1: Determine the domain of exponential function

Duration: 3hours		
Objectives: **By the end of the learning outcome the learner will be able to:** 3 **Identify exponential functions,** **3 Write exponential functions,** **3 Evaluate the domain of definition of exponential functions**.		
Resources		
Equipment	**Tools**	**Materials**
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

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3.1.1 Definition:

An exponential function is defined as follows:

$y = Exp(x) = e^x \leftrightarrow x = \ln y$

We have that the graph $y = \exp(x)$ is one-to-one and continuous with domain $(-\infty, \infty)$ and range $(0, \infty)$. Note that $\exp(x) > 0$ for all values of x. We see that

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$

$$\begin{aligned} \exp(1) = e &\quad \text{since} \quad \ln e = 1, \\ \exp(2) = e^2 &\quad \text{since} \quad \ln(e^2) = 2, \\ \exp(-7) = e^{-7} &\quad \text{since} \quad \ln(e^{-7}) = -7. \end{aligned}$$

In fact for any rational number r, we have

$$\exp(r) = e^r \quad \text{since} \quad \ln(e^r) = r \ln e = r,$$

Existance condition

The function $y = e^x \ni \leftrightarrow x \in [-\infty, +\infty]$

- Boundaries

Here we have two boundaries $-\infty$ and $+\infty$

- Domain of definition

The domain of definition of the function $y = e^x$ is $[-\infty, +\infty]$

Theoretical learning Activity

✓ Define exponential function

✓ Domain of definition of exponential function

Practical learning Activity

- ✓ Find the domain of definition of $F(x) = e^{2x} + e$

Points to Remember (Exponential function; Domain of definition ; Derivative;)

Learning out come 3.1: formative assessment :

Assessment: 5 Marks

Time : 10 minutes

Evaluate the domain of definition of $f(x) = e^{2x}$

Solution:

Here we have two boundaries -∞ and +∞ /**2Marks**

The domain of definition of the function $f(x) = e^{2x}$ is [-∞, +∞];**3Marks**

Learning outcome 3.2: **Calculate limit of exponential functions**

Duration: 4hours		
Objectives: **By the end of the learning outcome the learner will be able to:** **1.Understand the properties related to finding the limit of exponential functions** **2.Understand and combine the properties of limits to simplify and evaluate limit of exponential functions** **3.Finding limits by direct substitution of exponential functions.**		
Resources		
Equipment	**Tools**	**Materials**

Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

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3.2.1 limit of exponential functions

Finite limit

Example Find the limit $\lim_{x\to-\infty} \frac{e^x}{10e^x-1}$ and $\lim_{x\to 0} \frac{e^x}{10e^x-1}$.

$$\lim_{x\to-\infty} \frac{e^x}{10e^x-1} = \frac{\lim_{x\to-\infty} e^x}{\lim_{x\to-\infty}(10e^x)-1} = \frac{0}{0-1} = 0.$$

$$\lim_{x\to 0} \frac{e^x}{10e^x-1} = \frac{\lim_{x\to 0} e^x}{\lim_{x\to 0}(10e^x)-1} = \frac{1}{10-1} = \frac{1}{9}.$$

- **Limits at infinity**

$$\boxed{\lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} e^x = \infty.}$$

- **Deduction/ calculation of asymptotes**

As $\lim_{x\to-\infty} e^x = 0$,

$HA \equiv Y = 0$

Theoretical learning Activity

✓ Definition of limits exponential limits

Practical learning Activity

- ✓ Find the $\lim\limits_{x \to -\infty} e^{2x}$ +e^x

Points to Remember (**Finite limit , Limits at infinity**)

Learning out come 3.2: formative assessment

Assessment: 5 Marks

Time : 10 minutes

Evaluate the following limits

$\lim\limits_{x \to -\infty} e^{2x}$ +e^x

SOLUTION:

we know $e^{-\infty} = 0$ **2Marks**

then $\lim\limits_{x \to -\infty} e^{2x} + \boldsymbol{e^x}$=$\boldsymbol{e^{-2\infty} + e^{-\infty} = 0}$ **3Marks**

Learning outcome 3.3: Solve equations involving exponentials equestions

Duration: 4hours
By the end of the learning outcome the learner will be able to: 1.**Use the rules of exponents to solve exponential equations where the variable appears as part of the exponent.** **2.Identify the rules of exponents(product, quotient, power, product of power/quotient)** **3.Solve exponential equations using logarithms.**
Resources

Equipment	**Tools**	**Materials**

Books Handouts		Blackboard Chalk Marker pen Flip chart
Advance preparation: . .		

CONTENT

3.3.1 Exponentials equestions

- **Properties of exponentials**

$$\boxed{e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}.}$$

- **Domain of validity**

The domain of validity of an exponential equation is the condition for which the equation has roots.

- **Solving equations involving exponentials**

Examples

Solve the following equation

$$e^{2x} + e^x - 12 = 0$$

Solution:

Step 1: In this problem our equation, $e^{2x} + e^{x} - 12 = 0$, is quadratic. We can isolate the exponential term by factoring.

$$e^{2x} + e^{x} - 12 = 0$$

$$\left(e^{x}\right)^{2} + e^{x} - 12 = 0 \qquad \text{Law of Exponents}$$

$$\left(e^{x} + 4\right)\left(e^{x} - 3\right) = 0 \qquad \text{Factor (a quadratic in } e^{x}\text{)}$$

$$e^{x} + 4 = 0 \quad \text{or} \quad e^{x} - 3 = 0 \qquad \text{Zero-Product Property}$$

$$e^{x} = -4 \qquad\qquad e^{x} = 3$$

Step 2: Since we now have two equations, we have a possibility of two solutions. We should perform the rest of our steps on each equation. Notice though that $e^{x} = -4$ has no solution because $e^{x} > 0$ for all x, so we can discard this equation. Now we will take the natural logarithm of both sides of $e^{x} = 3$, and use the Laws of Logarithms to "bring down the exponent."

$$e^{x} = 3$$

$$\ln e^{x} = \ln 3 \qquad \text{Take the logarithm of each side}$$

$$x \ln e = \ln 3 \qquad \text{Bring down the exponent}$$

Step 3: Now we solve for the variable.

$$x \ln e = \ln 3$$

$$x = \ln 3 \qquad \ln e = 1$$

$$x \approx 1.0986 \qquad \text{Use a calculator}$$

Step 4: Check the answer by substituting $x = 1.0980$ into the original equation and using a calculator. We get

$$e^{2(1.0986)} + e^{1.0986} - 12 \approx 0 \qquad \checkmark$$

- **Set of solutions**

In the example above the set of solution is *S*= [ln 3]

Theoretical learning Activity

✓ Step of solving exponential function

Practical learning Activity

- ✓ Solve: e^{2x} +e^{x} -6=0

- **Points to Remember**:

$(Domain\ of\ definition\ of\ exponential\ function; Equation\ of\ exponential\ function,$
$propeties\ of\ exponentials\ )$

Learning outcome 3.4: Differentiate exponential functions

Duration: 4hours

By the end of the learning outcome the learner will be able to:

1.**Differentiate exponential functions where the base is Euler's number.**
2.Differentiate exponential functions where the base is a constant.
3.Differentiate exponential functions with linear exponents.
4.Differentiate exponential functions with quadratic exponents.
5.Evaluate the differential of an exponential function at a given point

Resources

Equipment	Tools	Materials
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

.

CONTENT

3.4.1 Differentiation

Derivatives

$$\boxed{\frac{d}{dx}e^{x} = e^{x}} \qquad \boxed{\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}}$$

Examples

Find the derivative of each of the following

(i) e^{5x} (ii) e^{ax} (iii) $e^{-\frac{3x}{2}}$

Soution : (i) Let $y = e^{5x}$.

Then $y = e^{t}$ where $5x = t$

$\therefore \quad \frac{dy}{dt} = e^{t}$ and $5 = \frac{dt}{dx}$

We know that, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^{t} \cdot 5 = 5e^{5x}$

Alternatively $\frac{d}{dx}\left(e^{5x}\right) = e^{5x} \cdot \frac{d}{dx}(5x) = e^{5x} \cdot 5 = 5e^{5x}$

(ii) Let $y = e^{ax}$.

Then $y = e^{t}$ when $t = ax$

$\therefore$ $\frac{dy}{dt} = e^{t}$ and $\frac{dt}{dx} = a$

We know that, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = e^{t} \cdot a$

Thus, $\frac{dy}{dx} = a \cdot e^{ax}$

(iii) Let $y = e^{\frac{-3x}{2}}$

$\therefore$ $\frac{dy}{dt} = e^{\frac{-3}{2}x} \frac{d}{dx}\left(\frac{-3}{2}x\right)$

Thus, $\frac{dy}{dt} = \frac{-3}{2} e^{\frac{-3x}{2}}$

- Increasing and decreasing intervals

The graph of the function $y = e^x$ is **increasing** for all real numbers

- Concavity

The graph of the exponential function is **concave up**

Theoretical learning Activity

✓ Increase and decrease interval
✓ Differentiation.
✓ Direction of the concavity

Practical learning Activity

✓ Evaluate the following function $f(x) = e^{2x} + e^{x}$

Points to Remember:

(Differentiation of exponential function; increasing and decreasing interval of exponential functions, concavity of exponential function)

Learning outcome 3.5: Sketch the curve of exponential functions

Duration: 5hours

By the end of the learning outcome the learner will be able to:

1.Sketch the graph of an exponential function,
2.Identify the domain and range of an exponential function using its graph,
3.Find where a given exponential graph intersects the axes,
4.Find a missing value in an exponential equation given a point it passes through.

Resources

Equipment	**Tools**	**Materials**
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

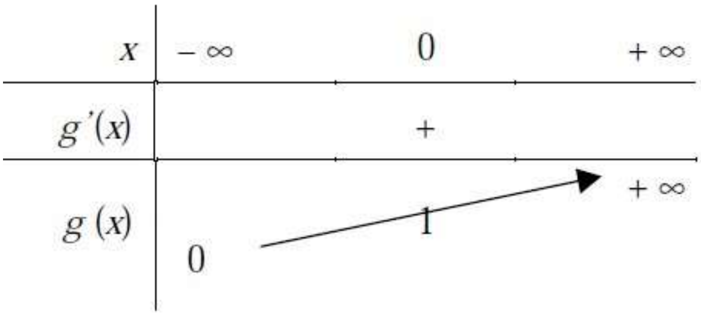
.

CONTENT

3.5.1 curve sketching

- <u>**Parameters required**</u>

  ✓ **Variation table**

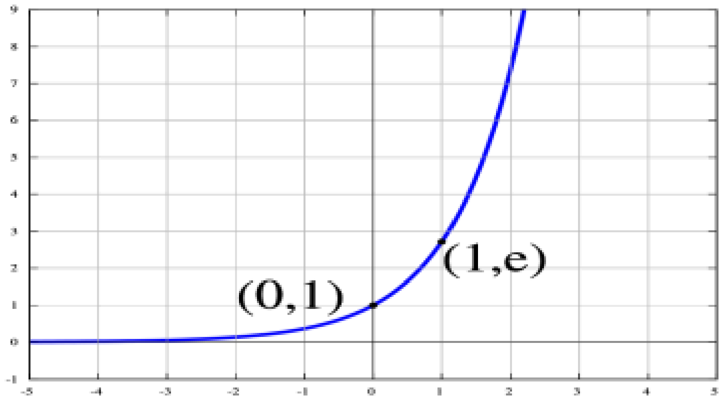


Additional points

x	0.5	1	1.5	2	2.5	3
ex	1.65	e	4.5	7.4	12	20

- <u>**Curve sketching**</u>

The following is the graphical representation of the function $y = e^x$



Example of curve sketching containing exponential function. Example1

Given the numerical function

$f(x) = x\,e^{1-x}$, find:

(a) the domain;

(b) the *lim*its at the boundaries of the domain

(c) the asymptotes

(d) the first and second derivatives and their signs and the conclusion about intervals where f is increasing, decreasing, concave up, concave down, statitionary points and their nature, inflection points.

(e) the graph

Solution

(a) Domain; $Df =]-\infty, +\infty[$

(b) *Lim*its at the boundaries of the domain

* $\lim\limits_{x \to +\infty} x\,e^{1-x} = -\infty;$

* $\lim\limits_{x \to +\infty} x\,e^{1-x} = \lim\limits_{x \to +\infty} \dfrac{x}{e^{x-1}} = \lim\limits_{x \to +\infty} \dfrac{1}{e^{x-1}} = 0$

(c) Horizontal asymptote: $y = 0$

(d) $f'(x) = e^{1-x} - x\,e^{1-x}$

$= (1 - x)\,e^{1-x}$

$f'(x) = 0 \Leftrightarrow x = 1$

x	$-\infty$		1		$+\infty$
$f'(x)$		$+$	$+0$	$-$ $-$ $-$	
$f(x)$	$-\infty$	$\nearrow$	1	$\searrow$	0

f is increasing on $]-\infty, 1[$,

f is decreasing on $]1, +\infty[$

Maximum point $M(1, 1)$.

$$f''(x) = -e^{1-x} - (1-x)\, e^{1-x}$$

$$= (x-2)\, e^{1-x}$$

$$f''(x) = 0 \Leftrightarrow x = 2$$

x	$-\infty$		2		$+\infty$
$f''(x)$	$-$	$-$	0	$+$	$+$
$f(x)$					

The curve is:

Concave down if $x \in]-\infty, 2[$

Concave up if $x \in]2, +\infty[$

The curve has inflection point $I\left(2, \frac{2}{e}\right)$

(e) Graph

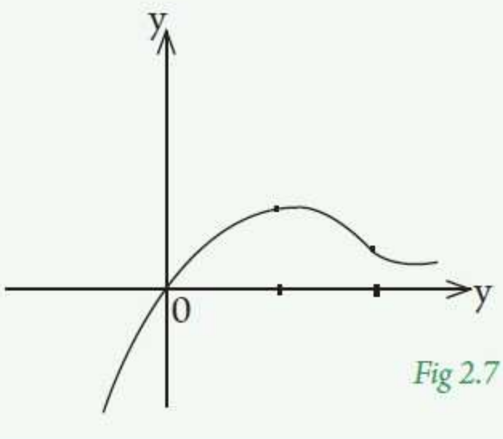


Fig 2.7

Theoretical learning Activity

- ✓ Step of sketching exponential function

Practical learning Activity

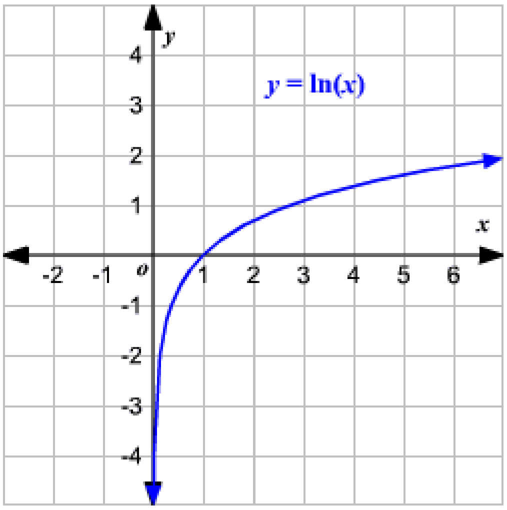
- ✓ Sketch the following function $f(x) = e^{2x}$

Points to Remember (Parametrics required for sketching exponential functions, and it's Sketching)

CONTENT

Learning out come 3.5: formative assessment

Learning Unit 4: Apply natural logarithmic functions



Structure of learning unit

Duration: 19hours

Learning outcomes:

4.1 Determine the domain of definition of natural logarithmic functions

4.2 Calculate limits of natural logarithmic functions

4.3 Solve equations involving logarithms

4.4 Differentiate natural logarithmic functions

4.5 Sketch the curve of logarithmic function

Learning outcome 4.1: Determine the domain of definition of natural logarithmic functions

Duration: 5hours

By the end of the learning outcome the learner will be able to:

1.**Identify logarithmic functions,**

2.Write logarithmic functions.

3.Evaluate the domain of definition of logarithmic functions.

Resources

Equipment	**Tools**	**Materials**
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

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CONTENT

4.1.1 Definition

The **natural logarithm**, also called neperian **logarithm**, is noted **ln**. The **domain** is D=] 0, +∞ [because **ln**(x) exists if and only if x>0. The range is I=R=] −∞, +∞ [because **ln** is strictly increasing.

- ✓ Existence conditions

The function $y = \ln x$ **exists if and only if** $x > 0$

- ✓ Boundaries

In the case of our function y=ln x, we have two boundaries which are 0^{+} *and* $+\infty$

- ✓ Domain of definition

The domain of our function y=ln x is D=] 0, +∞

Theoretical learning Activity

- ✓ Domain of definition of natural logarithm
- ✓ Existence condition.

Practical learning Activity

- ✓ Evaluate the domain of definition of $f(x) = ln2x + 2$

Points to Remember :(Domain of definition of logarithmic function, Existence condition)

CONTENT

Learning out come 4.1: formative assessment

Assessment: 5 Marks

Time : 10 minutes

Evaluate the domain of our function y=ln2x

SOLUTION:

In the case of our function y=ln2x, we have two boundaries which are 0^{+} *and* $+\infty$

The domain of our function y=ln $2x$ is Dmf=] 0, +∞[

Learning outcome 4.2: Calculate limits of natural logarithmic functions

Duration: 5hours

By the end of the learning outcome the learner will be able to:

1. **Understand the properties related to finding the limit of logarithmic functions**
2. **Understand and combine the properties of limits to simplify and evaluate limit of logarithmic functions.**
3. **Finding limits by direct substitution of logarithmic functions.**

Resources

Equipment	Tools	Materials
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

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4.2.2 Properties of logarithms

Properties of Logarithms (Recall that logs are only defined for positive values of x.)

For the natural logarithm	For logarithms base a
1. $\ln xy = \ln x + \ln y$	1. $\log_a xy = \log_a x + \log_a y$
2. $\ln \frac{x}{y} = \ln x - \ln y$	2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\ln x^y = y \cdot \ln x$	3. $\log_a x^y = y \cdot \log_a x$
4. $\ln e^x = x$	4. $\log_a a^x = x$
5. $e^{\ln x} = x$	5. $a^{\log_a x} = x$

Useful Identities for Logarithms

For the natural logarithm	For logarithms base a
1. $\ln e = 1$	1. $\log_a a = 1$, for all $a > 0$
2. $\ln 1 = 0$	2. $\log_a 1 = 0$, for all $a > 0$

- **Finite limits**

Evaluate:

a. $\lim_{x \to 2} (\ln x - 3)$

b. $\lim_{x \to 2} (\ln x - 1)$

SOLUTIONN

a. $\lim_{x \to 2} (\ln x - 3)$ **=ln2-3**

b. $\lim_{x \to 2} (\ln x - 1)$ **=ln1-1=0-1=-1**

- **Limits at infinity**

The log function $\ln x$ *has range* $(-\infty, \infty)$ *and*

$$\lim_{x\to 0^+} \ln x = -\infty, \quad \lim_{x\to\infty} \ln x = \infty.$$

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = 0 \quad \textit{for any } r > 0.$$

$\ln x$ *grows slower than any positive power as* $x \to \infty$.

Deduction asymptotes

As:

$$\lim_{x\to 0^+} \ln x = -\infty,$$

The line x=0 is the vertical asymptote

Theoretical learning Activity

- ✓ Limit of natural logarithm.
- ✓ Existence condition.

Practical learning Activity

- ✓ Evaluate the following limit $\lim_{x\to 2}$ (ln 6x -3)

Points to Remember (Finity and Limits at infinity of logarithmics functions)

Learning out come 4.2: formative assessment

Assessment: 5 Marks

Time : 10 minutes

Evaluate the following limits

a. $\lim_{x\to 4}$ (ln 3x-3) **/2.5Marks**
b. $\lim_{x\to 3}$ (ln 10x -1) / **2.5Marks**

Solution:

a. $\lim\limits_{x \to 4}$ (ln3x -3) **=ln12-3** **/2.5Marks**

b. $\lim\limits_{x \to 3}$ (ln10x-1) **=ln10-1** **/2.5Marks**

Learning outcome 4.3: Solve equations involving logarithms

Duration: 5hours

By the end of the learning outcome the learner will be able to:

1.**Use the rules of logarithms to solve logarithmic equations where the variable appears as part of the logarithm.**

2.Identify the rules of logarithmic.

2. Solve logarithmic equations.

Resources

Equipment	Tools	Materials
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

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4.3.1 Logarithmic equation

Domain of validity

The domain of validity of a logarithmic equation is the condition for which the equation has roots.

- **Solving logarithmic equations**

Steps for solving logarithmic Equations

Step 1: Determine if the problem contains only logarithms. If so, go to step 2. If not, stop and use the steps for solving logarithmic equations containing terms without logarithms

Step 2: Use the Properties of logarithms to simplify the problem if needed. If the problem can be simplified

Step 3: Rewrite the problem without the logarithms.

Step 4: simplify the problem.

Step 5: solve for x.

Step 6: Check your answer**(s).** Remember we cannot take logarithm of a negative number, so we need to make sure that when we plug our answer**(s)** back into the original equation we get a positive number.

Otherwise, we must drop that answer

Examples:

Solve the following equation

01.$\ln(x+2)-\ln(4x+3)=\ln\left(\frac{1}{x}\right)$

$$\boldsymbol{ln}\left(\frac{x+2}{4x+3}\right)=\boldsymbol{ln}\left(\frac{1}{x}\right)$$

$$\frac{x+2}{4x+3}=\frac{1}{x}$$

$x(x+2) = 4x+3$

$x^2 + 2x= 4x+ 3$

$x^2 - 2x-3 = 0$

We get two values of x which are 3 and -1.

The only true solution is (3)

- **Set of solutions**

The set of solutions is the set of all roots satisfying the equation. In the example above (3) is the set of solutions.

Theoretical learning Activity

- Step of solving logarithmic function

Practical learning Activity

- ✓ Solve the following: ln(x+2)+ln(x+3)= $\ln(2x-12)$

Points to Remember (Domain of definition of logarithmic function; existence condition; solving logarithmic function)

Learning out come 4.3: formative assessment

Assessment: 5 Marks

Time : 10 minutes

Solve $\log_2 x + \log_2(x-1) = \log_2(1-x)$

Solution:

Log2 x(x-1) =$\log_{2\,(1-x)}$, **/1MARK**

x(x-1) =1-x, **/1MARK**

x^2 –x =1-x, **/1MARK**

x^2 –x+x=1, **/1MARK**

x^2=1

X=$\pm$1 it means x_{1} =-1, x_2 =1 **/1MARK**

Learning outcome 4.4: Differentiate natural logarithmic functions

Duration: 2hours
By the end of the learning outcome the learner will be able to: 1.**Differentiate logarithmic functions.** **2.Differentiate logarithmic functions.** **3.Evaluate the differential of logarithmic function at a given point.**

Resources		
Equipment	**Tools**	**Materials**
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation:

. Manual

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4.4.1 Differentiation

General rule

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}.$$

Example1

Given the numerical function:

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right)$$

find $f'(\sqrt{2})$

Solution

$$f'(x) = \frac{\left(x+\sqrt{1+x^2}\right)'}{x+\sqrt{1+x^2}} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$= \frac{x+\sqrt{1+x^2}}{x+\sqrt{1+x^2}\ \sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$f'\left(\sqrt{2}\right) = \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{3}$$

Exercises

Find y` if

(a) $y = \ln\left(\frac{1+x}{1-x}\right)$

(b) $f(x) = \frac{\ln x - 2}{\ln x - 1}$

- Increasing and decreasing intervals

The graph of the *y*= ln *x* is increasing, continuous and concave down on the interval **(0,∞)**

Concavity

As it is mentioned above the function y= ln *x* is **concave down**

Theoretical learning Activity

✓ Differentiation of logarithmic function.

✓ Decreasing and increasing interval.

Practical learning Activity

✓ Evaluate the derivative of f(x)=ln(*x*+2)

Points to Remember (Derivative of logarithmic function; existence condition,

Concavity of logarithmic functions, Increasing and decreasing of logarithmic function)

Learning out come 4.4: formative assessment

Assessment: 5 Marks

Time : 10 minutes

Differentiate the following function :

F(x)= ln (*x* – 20)

Solution: $(\ln\,(x-20))' = \frac{(x-20)'}{x-20} = \frac{1}{x-20}$ **/5Marks**

Learning outcome 4.5: Sketch the curve of logarithmic function

Duration: 5hours
By the end of the learning outcome the learner will be able to: **1.Evaluate and graph logarithmic functions with different bases,** **2.Solving logarithmic equations graphically.** **3.Recognize the features and characteristics of logarithmic functions (increasing, decreasing, asymptotes, etc.).**
Resources

Equipment	Tools	Materials
Books Handouts		Blackboard Chalk Marker pen Flip chart

Advance preparation: . Manual .

CONTENT

4.5.1 curve sketching

- **Parameters required** for the function $y = \ln x$

$$x \to +\infty \Rightarrow \ln x \to +\infty$$

$$x \to 0 \Rightarrow \ln x \to -\infty$$

$$x = 1 \Rightarrow \ln x = 0$$

$$x = e \Rightarrow \ln x = 1$$

$$f'(x) = \frac{1}{x} > 0$$

- ✓ **Variation table**

x	0	1	$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	0	$+\infty$

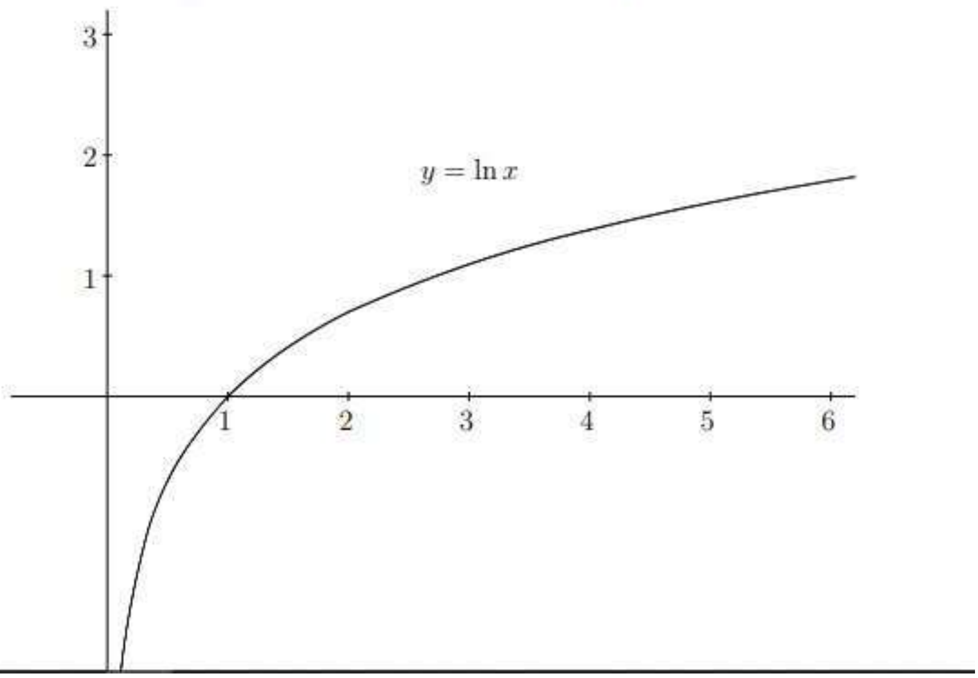
- ✓ **Additional points**

x	-2	-1,5	-1	-0,5	0	0,5	1	1,5	2	2,5	3
$\ln x$						-0,7	0	0,4	0,7	0,9	1,1

- **Curve sketching**

From the information we already have about the logarithmic function y= ln *x*

We now have enough qualitative information to sketch a graph of $y = \ln x$.



Solved example

Given the numerical function $f(x) = \dfrac{1 + \ln x}{-1 + \ln x}$ find,

(a) the domain

(b) the *lim*its at the boundaries of the domain;

(c) the asymptotes

(d) the first and second derivatives, study their signs and draw conclusion about intervals where f is increasing, decreasing, concave up, concave down, stationary points and their nature, inflection point.

(e) the graph

Solution

(a) Domain; $Df = \{x \in \mathbb{R}; x > 0, \ln x - 1 \neq 0\}$

$=]0, e[\cup] e, +\infty[$

(b) *Lim*its at the boundaries of the domain:

* $\lim\limits_{x\to 0} \dfrac{\ln x + 1}{\ln x - 1} = \dfrac{\infty}{\infty}$: indetermine form

$\lim\limits_{x\to 0} \dfrac{\ln x + 1}{\ln x - 1} = \lim\limits_{x\to 0} \dfrac{(\ln x + 1)'}{(\ln x - 1)'} = \lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{\frac{1}{x}} = 1$

* $\lim\limits_{x\to e} \dfrac{\ln x + 1}{\ln x - 1} = -\infty;$

* $\lim\limits_{x\to e} \dfrac{\ln x + 1}{\ln x - 1} = +\infty$

* $\lim\limits_{x\to +\infty} \dfrac{\ln x + 1}{\ln x - 1} = 1$

(c) Asymptotes:

* Vertical asymptote; $x = e$ * Horizontal asymptote $y = 1$

(d) Derivatives:

$y' = \dfrac{-2}{x(\ln x - 1)^2}$

x	0		e		$+\infty$
-2	–	–	–	–	–
$x(\ln x - 1)^2$	||	+	0	+	+
$f'(x)$	||	–	||	–	
$f(x)$	||	$1 \searrow -\infty$	||	$+\infty \searrow 1$	

Function f is decreasing on interval $]0, e[\cup]e, +\infty[$

No stationary point

$$f''(x) = \frac{2(\ln x + 1)}{x^2(\ln x - 1)^3}$$

x	0 $\quad\quad$ $\frac{1}{e}$ $\quad\quad$ e $\quad\quad$ $+\infty$
$2(\ln x + 1)$	$\|\|$ – – – 0 + + + + + +
$x^2(\ln x - 1)^2$	$\|\|$ – – – – – – 0 + + + +
$f''(x)$	$\|\|$ + 0 – – – $\|\|$ + + + +
$f(x)$	$\|\|$ I $\|\|$

The graph of f is

Concave up if $x \in]0, \frac{1}{e}[\cup]e, +\infty[$

Concave down if $x \in]\frac{1}{e}, e[$

The graph has an inflection point at $I\left(\frac{1}{e}, 0\right)$

(e) Graph

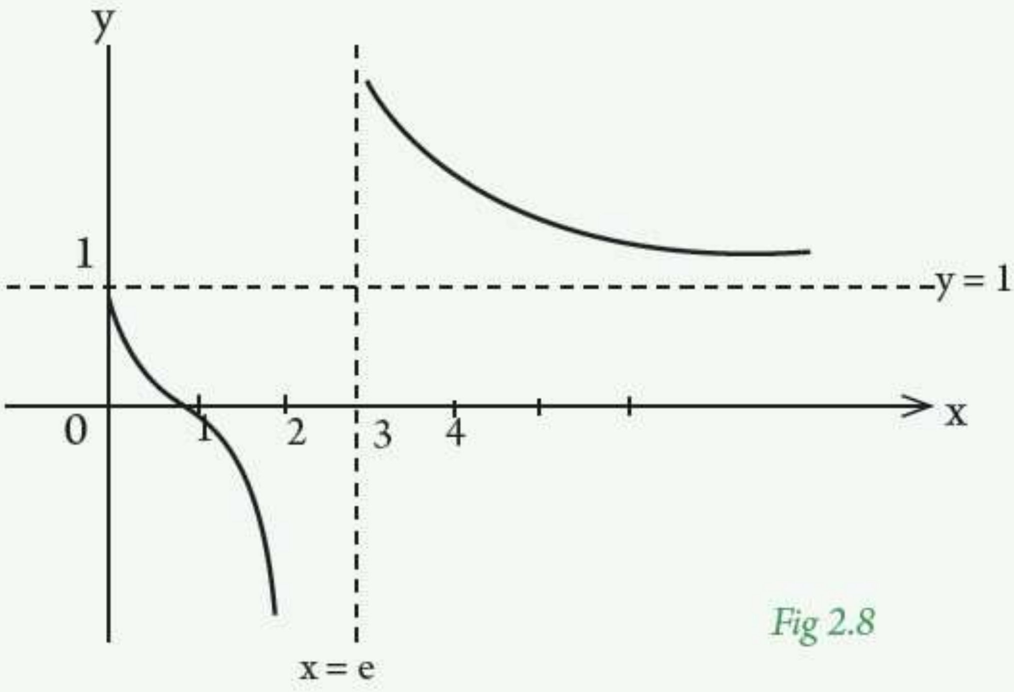


Fig 2.8

Theoretical learning Activity

- ✓ Graph of logarithmic function
- ✓ Step of sketching logarithmic function.

Practical learning Activity

- ✓ Sketch the following function $f(x) = \ln(x+2)$

Points to Remember (Sketching logarithmic function; existence condition; supplementary points)

Learning out come 4.5: formative assessment:

Assessment: 5 Marks

Time : 10 minutes

Sketch the following function $f(x) = \ln(2x + 1)$

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