TVET CERTIFICATE III

BASIC ALGEBRA AND TRIGONOMETRY

GENAM301

Apply Basic Algebra and Trigonometry

Competence

Credits: 10

Sector: All

Sub-sector: All

Learning hours: 100

Module Note Issue date: July, 2020

Purpose statement

This General module gives essential knowledge in Apply Basic Algebra and Trigonometry required to learners in order to perform very well in their engineering career.

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fundamentals of trigonometry	to terminal considered position.	
	2.2. Appropriate determination of trigonometric ratios based on isosceles right-angled triangle and equilateral triangle.	
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	2.4. Perfect solving of trigonometric equations based on trigonometric ratios.	
	2.5. Appropriate solving of a given triangle based on Pythagorean Theorem and trigonometric ratios	
Learning Unit3 - Apply fundamentals of complex numbers	3.1. Adequate description of complex numbers based on calculation process.	
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Learning Unit 1 – Solve algebraically or graphically linear and quadratic equations or inequalities

LO 1.1 – Correct solving algebraically or graphically a linear equation and inequality in accordance with the required steps

- <u>Solve linear equation</u>

Definition:

A linear equation is an equation of a straight line. For example:

$y = 2x + 1$

$5x = 6 + 3y$

$\frac{y}{2} = 3 - x$

The general form of the linear equation with one variable is $\mathbf{ax + b = 0}$
Where a, b $\in \mathbb{R}$: and $a \neq 0$ the value of x in which the equality is verified is called the **root.** (solution of the equation).

How to solve a linear equation?

- A linear equation is a polynomial of degree 1.
- In order to solve for the unknown variable, you must isolate the variable.
- In the order of operation, multiplication and division are completed before addition and subtraction.

The linear equation can be solved **algebraically** or **graphically.**

✓ **<u>Algebraic method</u>**

Example

Solve the equation $4x - 7 = 9$

<u>Solution</u>

(i) Isolate x to one side of the equation

$4x - 7 + 7 = 9 + 7$

$4x = 16$

$(ii) Divide$ *both side by 4*

$\frac{4x}{4} = \frac{16}{4}$

$x = 4$

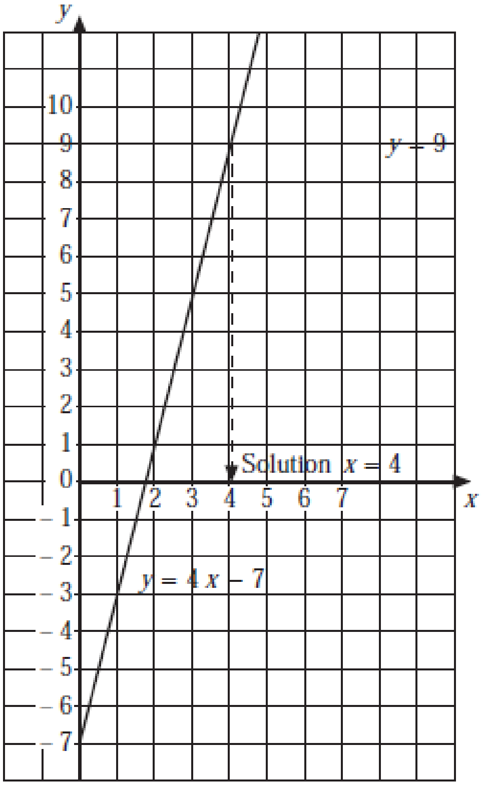
The root of the equation is **4**

This method used is known as **algebraic method**

✓ **Graphical method**

$4x - 7 = 9$

Draw the lines $y = 4x - 7$ and $y = 9$.



The solution is given by the value on the x-axis immediately below the point where $y = 4x - 7$ and $y = 9$ cross.

The solution is $x = 4$.

- Solving a linear inequality

✓ Algebraic method

They are solved as linear equations except that:

(a) When we multiply an inequality by a negative real number the sign will be reversed

(b) When we interchange the right side and the left side, the sign will be reversed.

Example

Solve: $-2(x + 3) < 10$.

Solution

$-2x - 6 < 10$

$-2x - 6 + 6 < 10 + 6$

$-2x < 16$

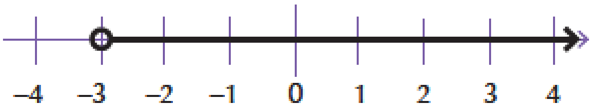
$\frac{-2x}{-2} > \frac{16}{-2}$

$x > -8$

✓ **<u>Graphical method</u>**

The graph of a linear inequality in one variable is a number line. We use an unshaded circle for < and > and a shaded circle for ≤ and ≥.

The graph for $x > -3$:



Solved example

Solve the following inequality

$2x - 6 < 2.$

Solution

Add 6 to both sides: $2x - 6 + 6 < 2 + 6$

Divide both sides by 2: $\frac{2}{2}x < \frac{8}{2}$

$x < 4$

Open circle at 4 (since x cannot equal 4) and an arrow to the left (because we want values less than 4).

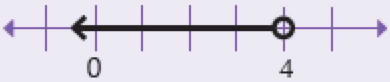


Fig 5.4

Exercises

Solve and graph the following:

1. $y \le 3$
2. $y = \frac{1}{2}x - 3$
3. $y = -3x + 2$
4. $y \ge -2$
5. $2y - x \le 6$
6. $\frac{y}{2} + 2 > x$

LO1.2 Discuss on parametric equations

- <u>Definitions:</u>

✓ Parameter

✓ Parametric equation

In case certain coefficients of equations contain one or several letter variables, the equation is called **parametric** and the letters are called **real parameters**. In this case, we solve and discuss the equation (for parameters only).

- <u>Solving steps</u>

Solve and discuss the equation $(2 - 3m)x + 1 = m^2(1 - x)$

Solution

$(2 - 3m)x + 1 = m^2(1 - x)$

$2x - 3mx + 1 = m^2 - m^2x$

$2x - 3mx + m^2x - m^2 + 1 = 0$

$x(2 - 3m + m^2) - m^2 + 1 = 0$

$x(2 - 3m + m^2) = m^2 - 1$

$$x = \frac{m^2 - 1}{2 - 3m + m^2} = \frac{(m-1)(m+1)}{(m-1)(m-2)} = \frac{m+1}{m-2}$$

If $m = 2$, then there is no solution.

If $m \neq 2$, then the solution is $x = \frac{m+1}{m-2}$

Notes

In the example above we can see that after finding the value of x, it follows a discussion so that we can validate the solution

Parametric equations in one unknown

If at least one of the coefficients a, b and c depend on the real parameter which is not determined, the root of the parametric quadratic equation depends on the values attributed to that parameter

Example

Find the values of k for which the equation $x^2 + (k + 1)x + 1 = 0$ has:

(a) two distinct real roots

(b) no real roots.

Solution

$\Delta = (k + 1)^2 - 4(1)(1) = (k + 1)^2 - 4$

$= k^2 + 2k + 1 - 4 = k^2 + 2k - 3 = (k + 3)(k - 1) = 0$ then $k = -3$ or $k = 1$.

Table of sign of $\Delta = k^2 + 2k - 3 = (k + 3)(k - 1)$

k / Factors	$-\infty$		-3		1		$+\infty$
$k + 3$		$-$	0	$+$		$+$	
$k - 1$		$-$		$-$	0	$+$	
$(k + 3)(k - 1)$		$+$	0	$-$	0	$+$	

Fig. 6.7

a) For two distinct real roots; $\Delta > 0$ and so $k < -3$ or $k > 1$.

b) For no real roots $\Delta < 0$ and so $-3 < k < 1$.

Exercises

1. Find the range of values of m for which the equation $(m - 3)x^2 - 8x + 4 = 0$ has
   (a) two real roots (b) no real root
   (c) one double root.

2. Find the range of values of k for which the equation $x^2 - 2(k + 1)x + k^2 = 0$ has:
   (a) two real roots (b) no real root
   (c) one double root.
3. Find the range of values of m for which the equation $2x^2 - 5x + 3m - 1 = 0$ has
   (a) two real roots (b) no real root
   (c) one double root.
4. Find the set of values of m for which $x^2 + 3mx + m$ is positive for all real values of x.

LO1.3: Solve algebraically or graphically two simultaneous linear equations

- Solving algebraically two simultaneous linear equations

What are simultaneous linear equations? How do we solve them?

A linear equation in two variables x and y is an equation of the form **$ax + by = c$** where $a \neq 0$, $b \neq 0$ and a, b, c are real numbers.

Let us consider such equation $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ where a_1, b_1, c_1, a_2, b_2 and c_2 are constants.

We say that we have two simultaneous linear equation in two unknowns or a system of two linear equation in two unknowns.

The pair (x, y) satisfying both equations is the solution of the given equation.

Consider the following example:

Claire and Laura are sisters; we know that

(i) *Claire is the elder sister,*

(ii) *their ages added together give 20 years,*

(iii) *the difference between their ages is 2 years.*

Let $x =$ Claire's age, in years and $y =$ Laura's age, in years.

$$x + y = 20$$
$$x - y = 2$$

This is an example of a pair of simultaneous equations.

We can solve such systems of linear equations by using one of the following methods:

1. substitution method
2. Elimination method
3. Comparison method
4. Cramer's rule

1. Substitution method

This method is used when one of the variables is given in terms of the other.

Example

Find the simultaneous solution of the following pair of equations: $y = 2x - 1$, $y = x + 3$.

Solution

Note that the system can also be written as $\begin{cases} y = 2x - 1 \\ y = x + 3 \end{cases}$, then

$2x - 1 = x + 3$

$x = 4$

And so $y = 4 + 3$

$y = 7$

So, the simultaneous solution is $x = 4$ and $y = 7$.

2. Elimination method

Elimination method is used to solve simultaneous equations where neither variable is given as the subject of another.

Solve simultaneously, by elimination: $\begin{cases} 5x + 3y = 12 \\ 7x + 2y = 19 \end{cases}$

Solution

$\begin{cases} 5x + 3y = 12 \text{ ... . (1)} \\ 7x + 2y = 19 \text{ ... . (2)} \end{cases}$

We multiply (1) by 2 and (2) by -3:

$\begin{cases} 10x + 6y = 24 \\ -21x - 6y = -57 \end{cases}$

Adding the two equations term by term gives:

$-11x = -33$

$x = 3$

Substituting $x = 3$ into (1) gives:

$5(3) + 3y = 12$

$15 + 3y = 12$

$3y = -3$

$y = -1$

Hence $x = 3$, $y = -1$ is the solution to the system of equations.

3. Comparison method

Let's consider the following simultaneous equations

$3x - 2y = 2$

$7x + 3y = 43$

Steps to solve the system of linear equations by using the comparison method to find the value of **x** and **y**.

$3x - 2y = 2$ ---------- (i)

$7x + 3y = 43$ --------- (ii)

Now for solving the above simultaneous linear equations by using the method of comparison follow the instructions and the method of solution.

Step I: From equation $3x - 2y = 2$ --------- (i), express **x** in terms of **y**.

Likewise, from equation $7x + 3y = 43$ -------- (ii), express **x** in terms of **y**.

From equation (i) $3x - 2y = 2$ we get;

$3x - 2y + 2y = 2 + 2y$ (adding both sides by 2y)

or, $3x = 2 + 2y$

or, $3x/3 = (2 + 2y)/3$ (dividing both sides by 3)

or, $x = (2 + 2y)/3$

Therefore, $x = (2y + 2)/3$ ---------- (iii)

From equation (ii) $7x + 3y = 43$ we get;

$7x + 3y - 3y = 43 - 3y$ (subtracting both sides by 3y)

or, $7x = 43 - 3y$

or, $7x/7 = (43 - 3y)/7$ (dividing both sides by 7)

or, $x = (43 - 3y)/7$

Therefore, $x = (-3y + 43)/7$ ---------- (iv)

Step II: Equate the values of **x** in equation (iii) and equation (iv) forming the equation in **y**

From equation (iii) and (iv), we get;

$(2y + 2)/3 = (-3y + 43)/7$ ---------- (v)

Step III: Solve the linear equation (v) in **y**

(2y + 2)/3 = (−3y + 43)/7 ---------- (v) Simplifying we get;

or, 7(2y + 2) = 3(−3y + 43)

or, 14y + 14 = −9y + 129

or, 14y + 14 − 14 = −9y + 129 − 14

or, 14y = -9y + 115

or, 14y + 9y = −9y + 9y + 115

or, 23y = 115

or, 23y/23 = 115/23

Therefore, y = 5

Step IV: Putting the value of **y** in equation (iii) or equation (iv), find the value of **x**

Putting the value of **y** = 5 in equation (iii) we get;

x = (2 × 5 + 2)/3

or, x = (10 + 2)/3

or, x = 12/3

Therefore, x = 4

Step V: Required solution of the two equations

Therefore, x = 4 and y = 5

Therefore, we have compared the values of **x** obtained from equation (i) and (ii) and formed an equation in **y**, so this method of solving simultaneous equations is known as the comparison method. Similarly, comparing the two values of **y**, we can form an equation in **x**.

04. Cramer's method

Step 1:	Find the determinant, D, by using the x and y values from the problem.
Step 2:	Find the determinant, D_x, by replacing the x-values in the first column with the values after the equal sign leaving the y column unchanged.
Step 3:	Find the determinant, D_y, by replacing the y-values in the second column with the values after the equal sign leaving the x column unchanged.
Step 4:	Use Cramer's Rule to find the values of x and y.

Example 1: Use Cramer's Rule to solve: $\begin{array}{l} 3x - 2y = 17 \\ 4x + 5y = -8 \end{array}$

Step 1: Find the determinant, D, by using the x and y values from the problem.

−8

$$D = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 15 - (-8) = 23$$

15

Step 2: Find the determinant, D_x, by replacing the x-values in the first column with the valu after the equal sign leaving the y column unchanged.

16

$$D_x = \begin{vmatrix} 17 & -2 \\ -8 & 5 \end{vmatrix} = 85 - 16 = 69$$

85

Step 3: Find the determinant, D_y, by replacing the y-values in the second column with the values after the equal sign leaving the x column unchanged.

68

$$D_y = \begin{vmatrix} 3 & 17 \\ 4 & -8 \end{vmatrix} = -24 - 68 = -92$$

−24

Step 4: Use Cramer's Rule to find the values of x and y.

$$x = \frac{D_x}{D} = \frac{69}{23} = 3$$

$$y = \frac{D_y}{D} = \frac{-92}{23} = -4$$

The answer written as an ordered pair is (3, −4).

- Solving graphically two simultaneous linear equations

Let's consider the following example:

Use a graph to solve the simultaneous equations:

$$x + y = 20$$
$$x - y = 2$$

Solution

We can rewrite the first equation to make y the subject:

$$\begin{aligned} x + y &= 20 \\ y &= 20 - x \end{aligned}$$

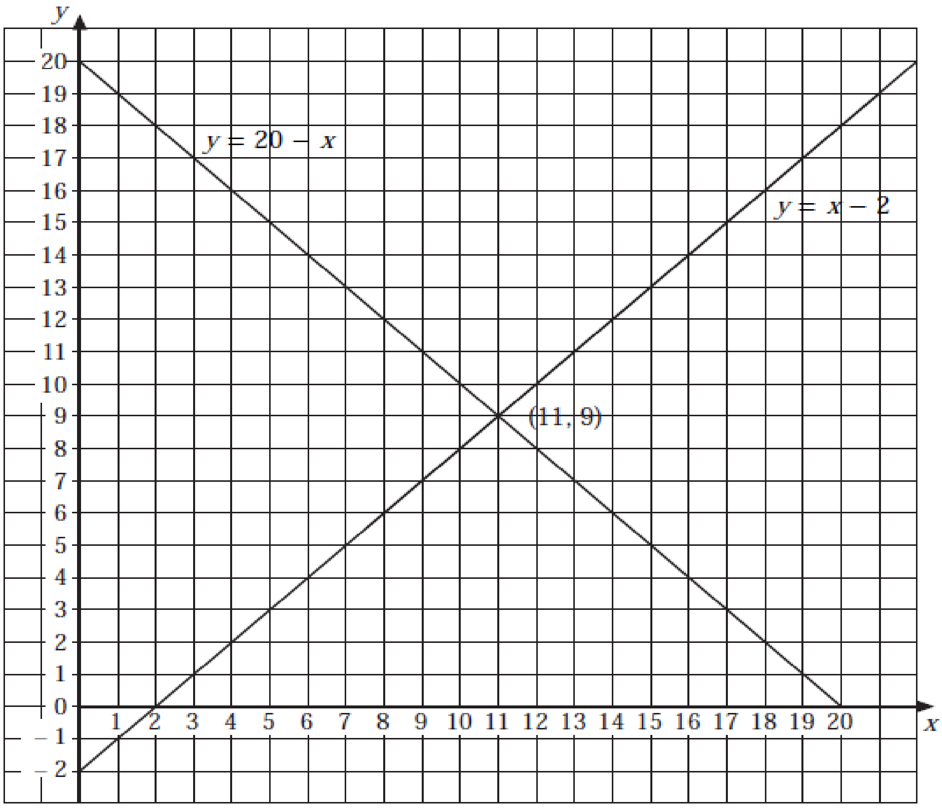
For the second equation,

$$\begin{aligned} x - y &= 2 \\ x &= y + 2 \\ x - 2 &= y \end{aligned}$$

or

$$y = x - 2$$

Now draw the graphs $y = 20 - x$ and $y = x - 2$.



The lines cross at the point with coordinates $(11, 9)$, so the solution of the pair of simultaneous equation is $x = 11,\ y = 9$.

Example2

Use a graph to solve the simultaneous equations:

$$x + 2y = 18$$
$$3x - y = 5$$

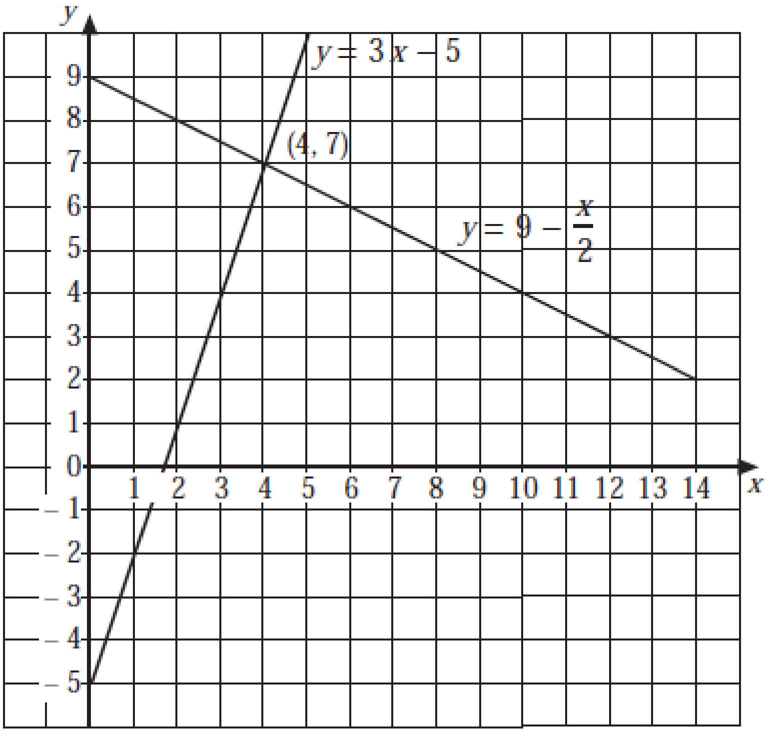
Solution

First rearrange the equations in the form $y = \ldots$

$$\begin{aligned} x + 2y &= 18 \\ 2y &= 18 - x \\ y &= \frac{18 - x}{2} \\ y &= 9 - \frac{x}{2} \end{aligned}$$

$$3x - y = 5$$
$$3x = y + 5$$
$$3x - 5 = y$$
or
$$y = 3x - 5$$

Now draw these two graphs:



The lines cross at the point with coordinates $(4, 7)$, so the solution is $x = 4$, $y = 7$.

LO1.4: Solve algebraically or graphically a quadratic equation .

What is a quadratic equation?

The term **quadratic** comes from the word *quad* meaning square, because the variable gets squared (like x^2). It is also called an "equation of degree 2" because of the "2" on the x.

The **standard form** of a quadratic equation looks like this:

$\mathbf{ax^2 + bx + c = 0}$

where **a**, **b** and **c** are known values and **a** cannot be 0.

"**x**" is the **variable** or the unknown.

Here are some more examples of quadratic equations:

$\mathbf{2x^2 + 5x + 3 = 0}$ In this one, $\mathbf{a = 2}$, $\mathbf{b = 5}$ and $\mathbf{c = 3}$

$\mathbf{x^2 - 3x = 0}$ For this, $\mathbf{a = 1}$, $\mathbf{b = -3}$ and $\mathbf{c = 0}$, so 1 is not shown.

$\mathbf{5x - 3 = 0}$ Thisoneis**not**aquadraticequation.Itismissingavalueinx²i.ea=0, which means it cannot be quadratic.

- Different method of solving algebraically a quadratic equation

✓ Factorizing method

Let us use an example, $x^2 - 5x + 6$. To solve $x^2 - 5x + 6 = 0$ we must first factorise $x^2 - 5x + 6$. To do this we have to find two numbers with a sum of –5 and a product of 6. The numbers required are –2 and –3, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

In solving this, we use the following law:

When the product of two or more numbers is zero, then at least one of them must be zero. So if ab = 0 then a = 0 or b = 0.

Example

Solve for x:

$x^2 - 3x + 2 = 0$

Solution

$x^2 - 3x + 2 = 0$

We need two numbers with sum –3 and product 2. These are –1 and –2.

$x^2 - 3x + 2 = (x - 1)(x - 2) = 0$

$x - 1 = 0$ or $x - 2 = 0$

$x = 1$ or 2

✓ Square root property

This property states: if A and B are algebraic expressions such that $A^2 = B$**, then** $A = \pm\sqrt{B}$**. This method is used if the form of the equation is:** $x^2 = k \ or (ax + b)^2 = k$ **where k represents a constant**

Steps to solve quadratic equations by the square root property:

1. Transform the equation so that a perfect square is on one side and a constant is on the other side of the equation.

2. Use the square root property to find the square root of each side. REMEMBER that finding the square root of a constant yields positive and negative values.

3. Solve each resulting equation. (If you are finding the square root of a negative number, there is no real solution and imaginary numbers are necessary.)

Example

Solve the quadratic equation $(x + 1)^2 = 49$

$(x + 1)^2 = 49$

1. $(x + 1)^2 = 49$
2. $\sqrt{(x + 1)^2} = \pm\sqrt{49}$
3. $x + 1 = 7$ or $x + 1 = -7$

   $x = 6$ or $x = -8$

✓ Completing the square method

Let's consider the equation $x^2 + 2x - 8 = 0$

Step 1- find the completed square form of $x^2 + 2x - 8$

$x^2 + 2x - 8$

Halve the coefficient of x (which here is 2) and add to x in a bracket squared

$(x + 1)^2$

Expand out the bracket

$(x + 1)^2 = x^2 + 2x + 1$

Subtract the 1 from both sides

$(x + 1)^2 - 1 = x^2 + 2x$

Now substitute this back into $x^2 + 2x - 8$ for the first two terms

$x^2 + 2x - 8 = (x + 1)^2 - 1 - 8 = 0$

$(x + 1)^2 - 9 = 0$

Step 2- solve this quadratic equation for x

$(x + 1)^2 - 9 = 0$

Add 9 to both sides

$(x + 1)^2 = 9$

Square root both sides

$x + 1 = \pm 3$

Subtract 1 from both sides

$x = - 1 \pm 3$

$x = - 1 - 3$ or $x = - 1 + 3$

Solutions x = -4 or 2

✓ Quadratic formula

Example

Use the quadratic formula to solve the equation $x^2 + 2x - 8 = 0$

Step 1- get the values of a, b and c to use in the formula

$ax^2 + bx + c = 0$
$x^2 + 2x - 8 = 0$

Therefore

$a = 1, b = 2, c = -8$

Step 2- substitute these values for a, b and c into the quadratic formula and go on to simplify and solve for x

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{-2 \pm \sqrt{((2)^2 - ((4)(1)(-8))}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{(4 - (-32))}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2}$$

$$x = \frac{-2 \pm 6}{2}$$

$x = \frac{-2 - 6}{2}$ or $x = \frac{-2 + 6}{2}$

Solutions x = -4 or 2

Exercises

Solve by factoring and then solve by completing the square.

1. $x^2 + 2x - 8 = 0$
2. $x^2 - 8x + 15 = 0$
3. $y^2 + 2y - 24 = 0$

4. $y^2 - 12y + 11 = 0$

5. $t^2 + 3t - 28 = 0$

6. $t^2 - 7t + 10 = 0$

7. $2x^2 + 3x - 2 = 0$

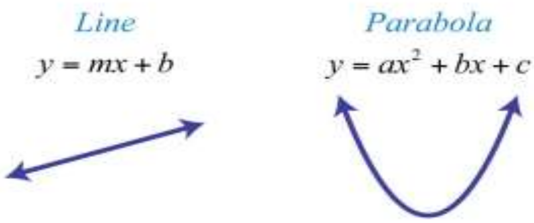
8. $3x^2 - x - 2 = 0$

9. $2y^2 - y - 1 = 0$

- <u>Graphical resolution of a quadratic equation</u>

✓ Construction of a parabola

We know that any linear equation with two variables can be written in the form $y = mx + b$ and that its graph is a line. In this section, we will see that any quadratic equation of the form $y = ax^2 + bx + c$ has a curved graph called a **parabola**[8].



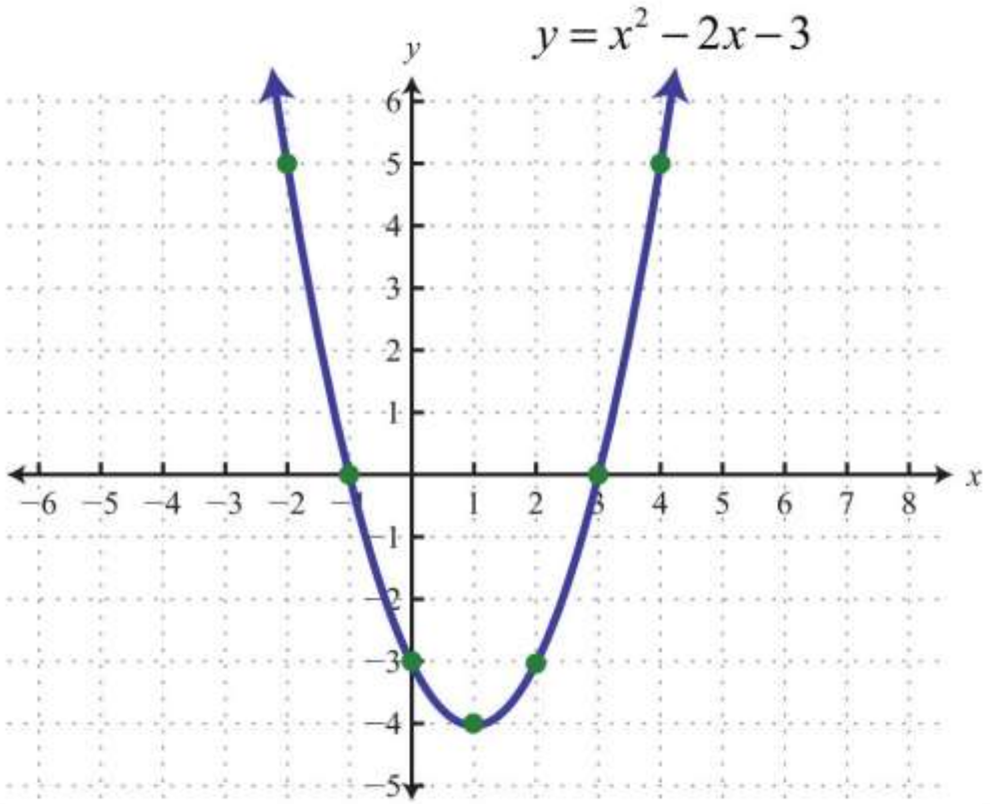
Example 1: Graph by plotting points: $y = x^2 - 2x - 3$.

Solution: In this example, choose the x-values {-2, -1, 0, 1, 2, 3, 4} and calculate the corresponding y-values.

x	y		Points
−2	5	$y=(-2)^2-2(-2)-3=4+4-3=5$	(−2, 5)
−1	0	$y=(-1)^2-2(-1)-3=1+2-3=0$	(−1, 0)
0	−3	$y=(0)^2-2(0)-3=0-0-3=-3$	(0, −3)
1	−4	$y=(1)^2-2(1)-3=1-2-3=-4$	(1, −4)
2	−3	$y=(2)^2-2(2)-3=4-4-3=-3$	(2, −3)
3	0	$y=(3)^2-2(3)-3=9-6-3=0$	(3, 0)
4	5	$y=(4)^2-2(4)-3=16-8-3=5$	(4, 5)

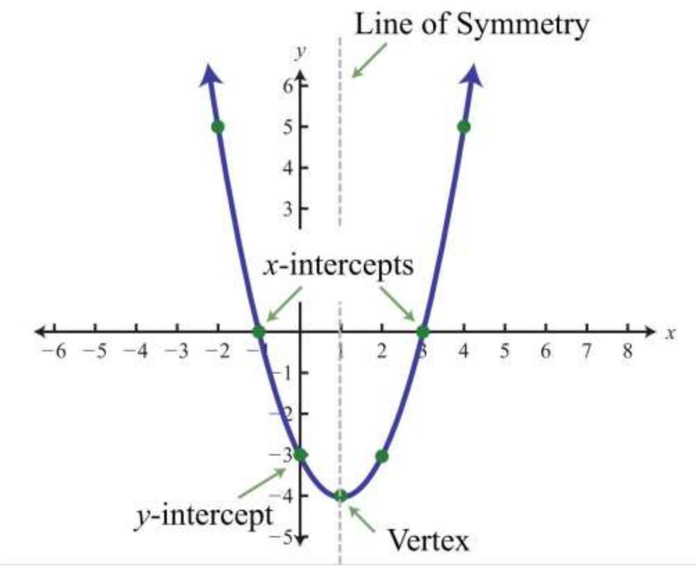
Plot these points and determine the shape of the graph.

Answer:



When graphing, we want to include certain special points in the graph.

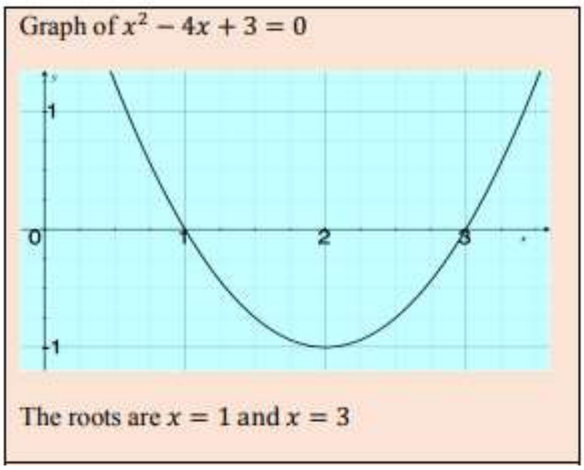
The $y-$ **intercept** is the point where the graph intersects the y-axis. **The** $x-$**intercept** is the point where the graph intercects the x-axis. The **vertex** is the point that defines the **minimum** or **maximum** of the graph. Lastly , **the line of symmetry** (also called the axis of symmetry) is the vertical line through the vertex about which the parabola is symmetric.



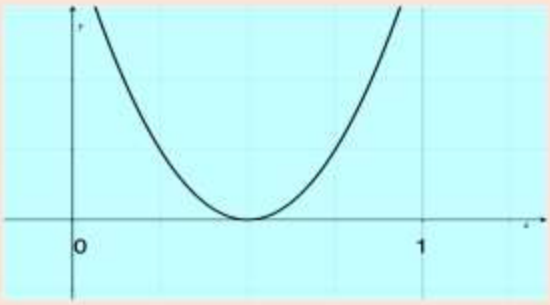
✓ Determination of solution

In using this method, we draw a graph of a quadratic function by creating a table of values. The solutions or roots are obtained by reading-off the x-coordinates of the point of intersection of the curve and the horizontal axis (when the equation = 0) Recall that the quadratic can have a maximum of two roots – this occurs when the graph cuts the x-axis at two distinct points.

If the x-axis is a tangent to the curve, then the two roots are equal to each other and so there is just one solution. If the curve does not cut or touch the x-axis, there are no solutions. These cases are illustrated below.

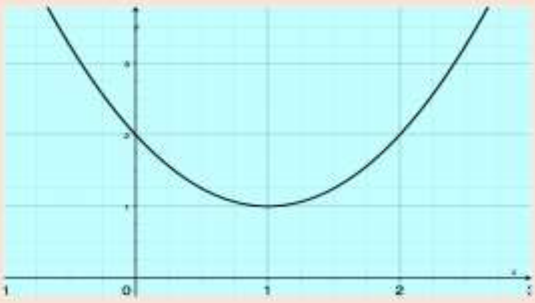


Graph of $4x^2 - 4x + 1 = 0$



There is only one root, $x = \frac{1}{2}$

Graph of $x^2 + 2x + 8 = 0$



There are no solutions – the graph does not cut or touch the x-axis.

LO 1.5: Solve algebraically or graphically a quadratic inequality

- Solving algebraically a quadratic inequality

  - ✓ Factorization of the given inequality
  - ✓ Determination of roots
  - ✓ Study of sign
  - ✓ Determination of interval of solutions

Form of a quadratic inequality:

After rearrangement, quadratic inequality has the following standard form

$$ax^2 + bx + c > 0 \quad (\geq,\ <,\ \leq)$$

Example1

Solve the inequality $(x + 3)(x - 2) > 0$.

Solution

The critical values are $x = -3$, $x = 2$.

The required sign diagram is:

Factors \ x	$-\infty$	-3		2		$+\infty$
$x + 3$	$-$	0	$+$		$+$	
$x - 2$	$-$		$-$	0	$+$	
$(x + 3)(x - 2)$	$+$	0	$-$	0	$+$	

Fig. 6.3

The answer is $\mathbf{x} < -3$ **or** $\mathbf{x} > 2 \Leftrightarrow x \in]-\infty, -3[\cup]2, +\infty[$

- Graphical resolution of quadratic inequality

  - ✓ Plotting a parabola
  - ✓ Shading the region satisfying the given inequality
  - ✓ Determination of interval of solutions

Example

Solve graphically the following quadratic inequality

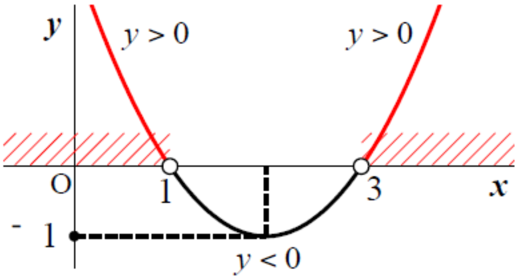
$x^2 - 4x + 3 > 0$

Solution

The standard form $y = (x-2)^2 - 1$

By factoring, we have $y = (x-1)(x-3)$
therefore, the roots are $x = 1,\ x = 3$

The inequality is satisfied in the shaded domain.



The solution is
$x < 1, \quad x > 3$

Exercises

Solve the following quadratic inequalities

1.	$y^2 - 17y + 70 < 0$	**2.**	$x^2 + 9x + 13 > -7$
3.	$x(x+1) > 112 - 5x$	**4.**	$a^2 + 3a + 2 < -3(a+2)$
5.	$2x^2 \le 5x - 2$	**6.**	$10 - 9y \ge -2y^2$
7.	$b(b+3) \ge -2$	**8.**	$a^2 \le 4(2a-3)$
9.	$y^2 - 17y + 70 < 0$	**10.**	$x^2 + 9x + 13 > -7$
11.	$x(x+1) > 112 - 5x$	**12.**	$a^2 + 25 < 10a$

Learning Unit2- Apply fundamentals of trigonometry

LO2.1. Describe angle

- <u>Angle definition</u>

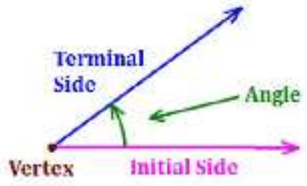
Basic Definitions

Angle: A measure of the space between rays with a common endpoint. An angle is typically measured by the amount of rotation required to get from its initial side to its terminal side.

Initial Side: The side of an angle from which its rotational measure begins.

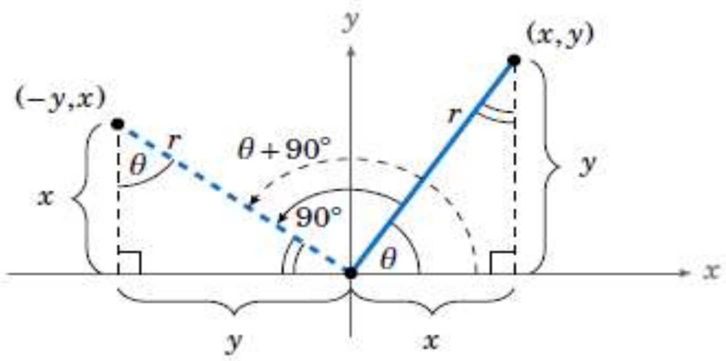
Terminal Side: The side of an angle at which its rotational measure ends.

Vertex: The vertex of an angle is the common endpoint of the two rays that define the angle.



✓ Rotation

To **rotate an angle** means to rotate its terminal side around the origin when the angle is in standard position.



Rotation of an angle θ by 90°

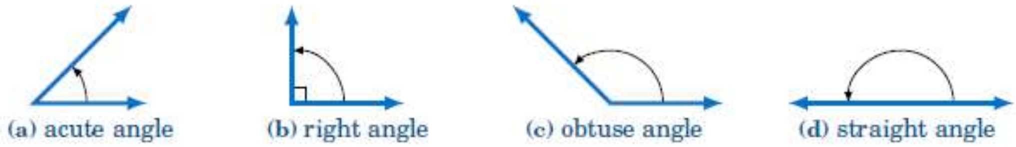
Recall

(a) An angle is **acute** if it is between 0° and 90°.

(b) An angle is a **right angle** if it equals 90°.

(c) An angle is **obtuse** if it is between 90° and 180°.

(d) An angle is a **straight angle** if it equals 180°.

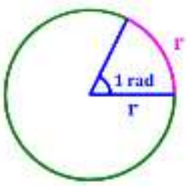


- Angles measurement
  - ✓ Radians
  - ✓ Degrees

So far we have been using degrees as our unit of measurement for angles. However, there is another way of measuring angles that is often more convenient, the **radian.**

What is a radian?

One radian is the measure of the angle made from wrapping the radius of a circle along the circle's exterior.



- Units conversion

Degrees to radians: x degrees $= \left(\frac{\pi}{180} \cdot x\right)$ radians

Radians to degrees: x radians $= \left(\frac{180}{\pi} \cdot x\right)$ degrees

$$\boxed{360^\circ = 2\pi \text{ radians}}$$

Example

a) Convert 65° to radians. b) Convert 1.75 radians to degrees.

Solution

a)

$$\begin{aligned} 1^\circ &= \frac{\pi}{180} \text{ radians} \\ 65^\circ &= 65 \times \frac{\pi}{180} \\ &= 1.134 \text{ radians} \end{aligned}$$

b)

$$\begin{aligned} 1 \text{ radian} &= \frac{180}{\pi} \text{ degrees} \\ 1.75 \text{ radians} &= 1.75 \times \frac{180}{\pi} \\ &= 100.268^\circ \end{aligned}$$

- Pythagorean theorem

Theorem 1.1. Pythagorean Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

$$a^2 + b^2 = c^2$$

Example

Suppose we wish to find the length of the hypotenuse of the right-angled triangle shown in Figure 4. We have labelled the hypotenuse c.

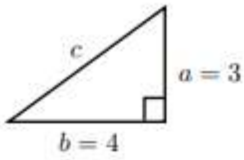


Figure 4.

Using the theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ 5 &= c \end{aligned}$$

So 5 is the length of the hypotenuse, the longest side of the triangle.

So, 12 is the length of the unknown side.

Example

Suppose we wish to find the length q in Figure 6. The statement of the theorem is now

$$p^2 + q^2 = r^2$$

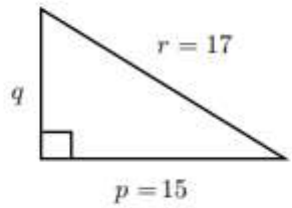


Figure 6.

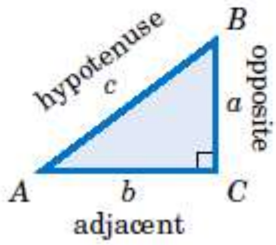
$$\begin{aligned} p^2 + q^2 &= r^2 \\ 15^2 + q^2 &= 17^2 \\ 225 + q^2 &= 289 \\ q^2 &= 64 \\ q &= 8 \end{aligned}$$

So, 8 is the length of the unknown side.

LO2.2:Determine the trigonometric ratios

- Definition of trigonometric ratios

Consider a right triangle $\triangle ABC$, with the right angle at C and with lengths a, b, and c, as in the figure on the right. For the acute angle A, call the leg $\overline{BC}$ its **opposite side**, and call the leg $\overline{AC}$ its **adjacent side**. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the six **trigonometric functions** of A as follows:

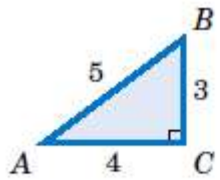


Name of function	Abbreviation		Definition		
sine A	$\sin A$	=	$\frac{\text{opposite side}}{\text{hypotenuse}}$	=	$\frac{a}{c}$
cosine A	$\cos A$	=	$\frac{\text{adjacent side}}{\text{hypotenuse}}$	=	$\frac{b}{c}$
tangent A	$\tan A$	=	$\frac{\text{opposite side}}{\text{adjacent side}}$	=	$\frac{a}{b}$
cosecant A	$\csc A$	=	$\frac{\text{hypotenuse}}{\text{opposite side}}$	=	$\frac{c}{a}$
secant A	$\sec A$	=	$\frac{\text{hypotenuse}}{\text{adjacent side}}$	=	$\frac{c}{b}$
cotangent A	$\cot A$	=	$\frac{\text{adjacent side}}{\text{opposite side}}$	=	$\frac{b}{a}$

Example

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles A and B.

Solution: The hypotenuse of $\triangle ABC$ has length 5. For angle A, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

For angle B, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

$$\csc B = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec B = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot B = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

- Calculation of trigonometric ratios of special angles, 30^0, 45^0, **60^0**, …

Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	**0^0**	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	**30^0**	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	**45^0**	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	**60^0**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	**90^0**	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	*undefined*

LO 2.3: Apply trigonometric identities

- Relationship between trigonometric ratios of some angles

✓ Complementary angles

(a) Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \le \angle A, \angle B \le 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.

✓ Supplementary angles

(b) Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \le \angle A, \angle B \le 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.

- Trigonometric ratios of Sum or difference of two angles**:** Sine, cosine, tangent

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\ \tan\beta}$$

- Trigonometric ratios of double angle: sine, cosine, tangent

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

LO2.4: Solve trigonometric equations

- Solutions of equations reducible to the form
  - ✓ $\sin(x+\alpha)=k,\ |k|\le 1$
  - ✓ $\cos(x+\alpha)=k,\ |k|\le 1$
  - ✓ $\tan(x+\alpha)=b$
  - ✓ $\sin nx=k$
  - ✓ $\cos nx=k$

$$\sin a=\sin b \Leftrightarrow a=\begin{cases} b+2k\pi \\ \pi-b+2k'\pi \end{cases}$$

$$\cos a=\cos b \Leftrightarrow a=\begin{cases} b+2k\pi \\ -b+2k'\pi \end{cases}$$

Example1

Solve the following trigonometric equation

$$\sin\left(x+\frac{\pi}{3}\right)=\frac{\sqrt{2}}{2}$$

Solution

We know that

$$\frac{\sqrt{2}}{2}=\sin\frac{\pi}{4}$$

$$\sin(x+\frac{\pi}{3})=\sin\frac{\pi}{4}$$

$$\Leftrightarrow x+\frac{\pi}{3}=\frac{\pi}{4}+2k\pi \text{ or } x+\frac{\pi}{3}=\pi-\frac{\pi}{4}+2k'\pi$$

$$\Leftrightarrow x=-\frac{\pi}{12}+2k\pi \text{ or } x=\frac{5\pi}{12}+2k'\pi$$

$$S=\left[x=-\frac{\pi}{12}+2k\pi\right]\cup\left[x=\frac{5\pi}{12}+2k'\pi\right]$$

Some important equations and their solutions

$$\cos x = 1 \Leftrightarrow x = 2k\pi \quad (k \in \mathbb{Z})$$
$$\cos x = -1 \Leftrightarrow x = \pi + 2k\pi \quad (k \in \mathbb{Z})$$
$$\cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$$
$$\sin x = 1 \Leftrightarrow x = \frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z})$$
$$\sin x = -1 \Leftrightarrow x = -\frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z})$$
$$\sin x = 0 \Leftrightarrow x = k\pi \quad (k \in \mathbb{Z})$$

<u>Example of trigonometric equations with solutions</u>

<u>Example 1</u>

Solve for $x: \sqrt{3}\sin x - 2\sin x\cos x = 0,\ 0 \le x < 2\pi$.

<u>Solution</u>: Factor the expression on the left and set each factor to zero.

$$\sin x\sqrt{3} - 2\sin x\cos x = 0$$
$$(\sin x)(\sqrt{3} - 2\cos x) = 0$$

$\sin x = 0$ or $\sqrt{3} - 2\cos x = 0$

$x = 0, \pi$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

Answers: $x = 0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$

<u>Example 2</u>

Solve for $x: \sin^2 x - \sin x - 2 = 0,\ 0 \le x < 2\pi$.

<u>Solution</u>: Factor the quadratic expression on the left and set each factor to zero.

$$\sin^2 x - \sin x - 2 = 0$$
$$(\sin x - 1)(\sin x + 2) = 0$$

$\sin x - 1 = 0$ or $\sin x + 2 = 0$

$\sin x = 1$ $\qquad$ $\sin x = -2$

$x = \frac{\pi}{2}$ $\qquad$ No solution. (Since the minimum value of $\sin x$ is -1, it cannot equal -2.)

Answer: $x = \frac{\pi}{2}$

Example 3

Solve for $x: \tan 2x = 1,\ 0 \le x < 2\pi$.

Solution: Solving $\tan\theta = 1$ first, we know that $\tan\frac{\pi}{4} = 1$ (QI) and $\tan\frac{5\pi}{4} = 1$ (QIII). So $\theta = \frac{\pi}{4} + \pi n$, where πn is integer multiples of the period of the tangent function.

For our problem:

$\theta = 2x = \frac{\pi}{4} + \pi n$ for $n = \ldots -1, 0, 1, 2, \ldots$

$x = \frac{\pi}{8} + \frac{\pi n}{2}$ (dividing by 2)

$x = \frac{\pi}{8}$ (if $n = 0$), $\frac{5\pi}{8}$ (if $n = 1$), $\frac{9\pi}{8}$ (if $n = 2$), $\frac{13\pi}{8}$ (if $n = 3$)

Note: If $n < 0$ or $n > 3$, the resulting x values are not in the interval of $0 \le x < 2\pi$.

Answer: $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

- Solutions of equation of the form asinx+bcosx=c

Dividing each term by $\sqrt{a^2+b^2}$, we get the given equation

$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$

Now for solving such equation, assuming $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$.

Observe that it is compatible as $\cos^2\alpha + \sin^2\alpha = 1$ and $\tan\alpha = \frac{a}{b}$ or $\alpha = \tan^{-1}\left(\frac{a}{b}\right)$

and then the given equation becomes

$$\cos x \cos \alpha + \sin x \sin \alpha = \frac{c}{\sqrt{a^2 + b^2}}$$

or $\cos(x - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}$

and hence $x - \alpha = 2n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right)$

and $x = 2n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) + \alpha$

or $x = 2n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) + \tan^{-1}\left(\frac{a}{b}\right)$

Example

Solve the following equation $\sin 2x + \sqrt{3}\cos x = 0$

$$\begin{aligned} \sin(2x) &= \sqrt{3}\cos(x) \\ 2\sin(x)\cos(x) &= \sqrt{3}\cos(x) \\ 2\sin(x)\cos(x) - \sqrt{3}\cos(x) &= 0 \\ \cos(x)\left(2\sin(x) - \sqrt{3}\right) &= 0 \end{aligned}$$

Then $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$ for integers k. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for integers k.

The answers which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

LO 2.5: Solve triangle

- Methods of solving triangle

✓ Sine Law

Used when we have a non-right angle triangle **(Oblique triangle)** where we are given at least one "set" of information-an angle and its opposite side

$$\frac{a}{SinA}=\frac{b}{SinB}=\frac{c}{SinC} \quad \textbf{OR} \quad \frac{SinA}{a}=\frac{SinB}{b}=\frac{SinC}{c}$$

Example

In triangle ABC, $B = 21°$, $C = 46°$ and $AB = 9$cm. Solve this triangle.

Solution

We are given two angles and one side and so the sine rule can be used. Furthermore, since the angles in any triangle must add up to 180° then angle A must be 113°. We know that $c = AB = 9$. Using the sine rule

$$\frac{a}{\sin 113°}=\frac{b}{\sin 21°}=\frac{9}{\sin 46°}$$

So,

$$\frac{b}{\sin 21°}=\frac{9}{\sin 46°}$$

from which

$$b = \sin 21° \times \frac{9}{\sin 46°} = 4.484\text{cm.} \qquad (3\text{dp})$$

Similarly

$$a = \sin 113° \times \frac{9}{\sin 46°} = 11.517\text{cm.} \qquad (3\text{dp})$$

✓ Cosine Law

We use the cosine law if we are given two sides and a contained angle OR if you are given all three sides and asked to find one or all of the angles

Cosine Law:	
To find the length of a side: **OR**	***to find the measure of an angle:***
$a^2 = b^2 + c^2 - 2bcCos\,A$	$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

E

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S

W $a = 37$,
$b = 26$ and $c = 42$ we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

from which

$$37^2 = 26^2 + 42^2 - 2(26)(42)\cos A$$

$$\cos A = \frac{26^2+42^2-37^2}{(2)(26)(42)} = \frac{1071}{2184} = 0.4904$$

and so

$$A = \cos^{-1} 0.4904 = 60.63°$$

You should apply the same technique to verify that $B = 37.76°$ and $C = 81.61°$. You should also check that the angles you obtain add up to 180°.

LU3 Apply fundamentals of complex numbers

LO 3.1 Conceptualize complex numbers

- Description of complex number

✓ Definition and properties of '' i

✓ Real part

✓ Imaginary part

A complex number is a number of the form $a + bi$, where a and b are real and $i^2 = -1$ or $i = \sqrt{-1}$. The letter 'a' is called the real part and 'b' is called the imaginary part of $a + bi$. If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$, the number a is real. Hence, real numbers and pure imaginary numbers are special cases of complex numbers. The complex numbers are denoted by Z , i.e.,

$$Z = a + bi.$$

In coordinate form, $Z = (a, b)$.

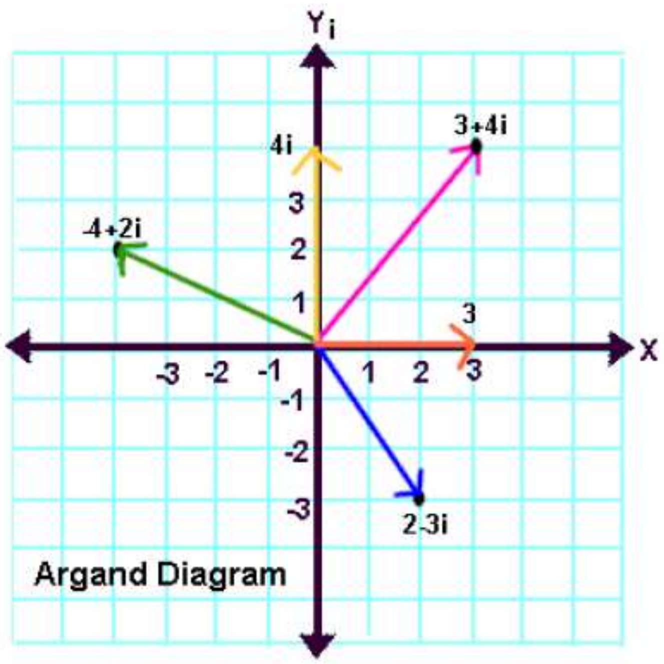
✓ Set of complex number

By $\mathbb{C}$ we denote the set of all complex numbers, that is,
$\mathbb{C} = \{z : z = x + iy, x \in \mathbb{R}, y \in \mathbb{R}\}$.

- Geometric representation of a complex number

✓ **Argand diagram**

- We can represent the complex number $z = x + iy$ by a position vector in the XY−plane whose tail is at the origin and head is at the point (x, y).
- When XY−plane is used for displaying complex numbers, it is called **Argand plane** or **Complex plane** or **z plane**.
- The X-axis is called as the real axis where as the Y-axis is called as the imaginary axis.



LO 3.2 Operate on complex numbers

- <u>Calculations in (□ ,+,·)</u>

(i) **Addition:**

If $Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$, then

$$Z_1 + Z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$

(ii) **Subtraction:**

$$Z_1 - Z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

(iii) **Multiplication:**

$$Z_1\ Z_2 = (a_1 + b_1 i)\ .\ (a_2 + b_2 i)$$
$$= a_1\ a_2 + b_1 b i^2 + a_1 b_2 i + b_1 a_2 i$$
$$= (a_1\ a_2 - b_1 b_2) + i(\ a_1 b_2 + b_1 a_2)$$

(iv) **Division:**

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply Numerator and denominator by the number $a_2 - b_2 i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\,\frac{b_1 a_2 - b_1 b_2}{a_2^2 + b_2^2}$$

Generally result will be expressed in the form a + ib.

Examples

Add and subtract the numbers 3 + 4i and 2 – 7i.

Solution:

Addition: $(3 + 4i) + (2 - 7i) = (3 + 2) + i(4 - 7) = 5 - 3i$

Subtraction: $(3 + 4i) - (2 - 7i) = (3 - 2) + i(4 + 7) = 1 + 11i$

Example 2: Find the product of the complex numbers: 3 + 4i and 2 – 7i.

Solution: $(3 + 4i)(2 - 7i) = 6 - 21i + 8i - 28i^2$

$$= 6 + 28 - 13i$$

$$= 34 - 13i$$

Example 3: Divide 3 + 4i by 2 – 7i.

Solution: $\frac{3 + 4i}{2 - 7i} = \frac{3 + 4i}{2 - 7i} \times \frac{2 + 7i}{2 + 7i}$

$$= \frac{6 + 28i^2 + i(21 + 8)}{4 + 49}$$

$$= \frac{-22 + 29i}{53}$$

$$= \frac{-22}{53} + i\,\frac{29}{53}$$

Example 4: Express $\frac{(2+i)(1-i)}{4-3i}$ in the form of a + ib.

Solution:

$$\frac{(2+i)(1-i)}{4-3i} = \frac{(2+1)+i(1-2)}{4-3i} = \frac{3-i}{4-3i}$$

$$= \frac{3-i}{4-3i} \times \frac{4+3i}{4+3i} = \frac{(12+3)+i(9-4)}{16+9}$$

$$= \frac{15+i(5)}{25} = \frac{15}{25} + \frac{5}{25}i$$

$$= \frac{3}{5} + \frac{1}{5}i$$

Example 5: Separate into real and imaginary parts: $\frac{1+4i}{3+i}$.

Solution :

$$\frac{1+4i}{3+i} = \frac{1+4i}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{(3+4)+i(12-1)}{9+1} = \frac{7+11i}{10}$$

$$= \frac{7}{10} + \frac{11}{10}i = \frac{7}{10} + \frac{11}{10}i$$

Here, real part = a = $\frac{7}{10}$

And imaginary part = b = $\frac{11}{10}$

✓ Modulus of a complex number

The Modulus or the absolute value of the complex number Z = a + ib is denoted by r, |Z| or |a + ib| and is given by,

$$r = |Z| = |a+ib| = \sqrt{a^2+b^2}$$

Thus the modulus |a + ib| is just the distance from the origin to the point (a , b)

Example

For $z = 2 + 3i$ and $w = 6 - i$, determine each of the following.

(a) $|z|$ (b) $|w|$ (c) $|zw|$

Solution (a) $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

(b) $|w| = \sqrt{6^2 + (-1)^2} = \sqrt{37}$

(c) Because $zw = (2 + 3i)(6 - i) = 15 + 16i$, we have

$$|zw| = \sqrt{15^2 + 16^2} = \sqrt{481}.$$

✓ Square roots of a complex number

Example: Find the square roots of the complex number $21 - 20i$

Solution:

Let $a + ib = \sqrt{21 - 20i}$

Squaring both sides

$(a + ib)^2 = 21 - 20\,i$

$a^2 - b^2 + 2abi = 21 - 20\,i$

Comparing both sides

$a^2 - b^2 = 21$ ....(1)

$2ab = -20$ ....(2)

From (2) $b = -\dfrac{10}{a}$ Put b in equation (1),

$a^2 - \dfrac{100}{a^2} = 21$

$a^4 - 21a^2 - 100 = 0$

$(a^2 - 25)(a^2 + 4) = 0$

$a^2 = 25$ or $a^2 = -4$

$a = \underline{+}5$ or $a = \underline{+}\sqrt{-4} = \underline{+}\,2i$

But a is not imaginary, so the real value of a is

$a = 5$ or $a = -5$

The corresponding value of b is

$b = -2$ or $b = 2$

Hence the square roots of $21 - 20i$ are:

$5 - 2i$ and $-5 + 2i$

Example 2

Find the square roots of $z = 8 - 6i$.

Solution

Let $x + yi$ be a square root of z.

then $\begin{cases} x^2 - y^2 = 8 \\ x^2 + y^2 = 10 \\ xy = -3 \end{cases} \Leftrightarrow \begin{cases} x = 3 \\ y = -1 \end{cases}$ or $\begin{cases} x = -3 \\ y = 1 \end{cases}$

Therefore, the square roots of $8 - 6i$ are $z_1 = 3 - i$ and $z_2 = -3 + i$.

- Solving equations in complex number
  - ✓ Quadratic equations

Quadratic equations

Type: $az^2 + bz + c = 0$, where $a \neq 0$, a, b, c, are constant complex numbers, z : unknown.

Solving equation $az^2 + bz + c = 0$

Case 1: All the coefficients a, b, c, are real numbers.

$\Delta = b^2 - 4ac \in \mathbb{R}$

If $\Delta > 0$ then the equation has two district real roots $z_1 = \frac{b + \sqrt{\Delta}}{2a}$ and $z_2 = \frac{-b - \sqrt{\Delta}}{2a}$

If $\Delta = 0$ then the equation has a repeated real root $z_1 = z_2 = -\frac{b}{2a}$.

If $\Delta < 0$, then the equation has two complex conjugate roots:

$$z_1 = \frac{-b + i\sqrt{-\Delta}}{2a} \text{ and } z_2 = \frac{-b - i\sqrt{-\Delta}}{2a}$$

Example1

Solve, in the set of complex numbers, the equation $4z^2 - 4z + 5 = 0$.

Solution

$\Delta = (-4)^2 - 4(4)(5)$

$= -64$

$= (8i)^2$

$$z_1 = \frac{4+8i}{8} = \frac{1}{2} + i;\ z_2 = \frac{4-8i}{8} = \frac{1}{2} - i$$

Example2

Solve, in the set of complex numbers, the equation;

(i) $iz^2 + (1+i)z + \frac{1}{2} = 0$ (ii) $z^2 + (5-i)z + 8 - i = 0$

Solution

(i) $\Delta = (1+i)^2 - 4(i)\left(\frac{1}{2}\right) = 0$

$$z_1 = z_2 = \frac{-(1+i)}{2i}$$

$$= -\frac{1}{2} + \frac{1}{2}i$$

(ii) $\Delta = (5-i)^2 - 4(1)(8-i)$

$= -8 - 6i$

A square root of Δ is $\delta = 1 - 3i$

$$z_1 = \frac{-(5-i)+(1-3i)}{2} = -2 - i$$

$$z2 = \frac{-(5-i)-(1-3i)}{2} = -3 + 2i$$

LO 3.3 Perform calculations of complex numbers in polar form

- Definitions

✓ Argument of a complex number

- A complex number is written in the form

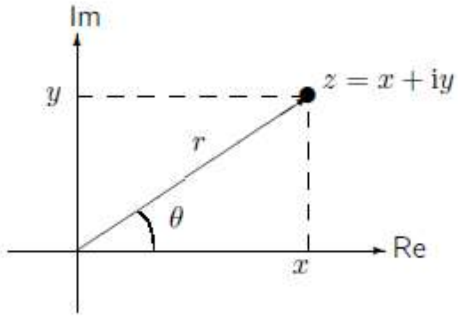
$$z = x + iy.$$

- The **modulus** of z is

$$|z| = r = \sqrt{x^2 + y^2}.$$

- The **argument** of z is

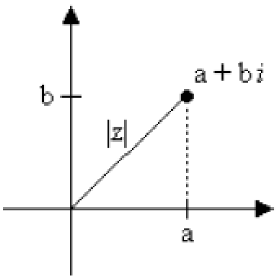
$$\arg z = \theta = \arctan\left(\frac{y}{x}\right).$$



Note: When calculating θ you must take account of the quadrant in which z lies - if in doubt draw an Argand diagram.

✓ Polar form of a complex number

A complex number in the form of a + bi, whose point is (a, b), is in rectangular form and can therefore be converted into polar form just as we need with the points (x, y). The relationship between a complex number in rectangular form and polar form can be made by letting θ be the angle (in standard position) whose terminal side passes through the point (a, b).



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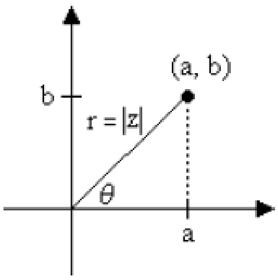


$\sin\theta = \dfrac{b}{r}$ $\qquad\cos\theta = \dfrac{a}{r}$ $\qquad\tan\theta = \dfrac{b}{a}$

$r\ \sin\theta = b$ $\qquad r\ \cos\theta = a$

$$r = |z| = \sqrt{a^2 + b^2}$$

Using these relationships, we can convert the complex number z from its rectangular form to its polar form.

$$z = a + bi$$
$$z = (r\cos\theta) + (r\sin\theta)i$$
$$z = r\cos\theta + r\,i\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$

Finally, the polar form of the complex number is $Z = r(\cos\theta + i\sin\theta)$

Examples

01. Write $Z = 2 + 2i$ in polar form.

Solution

Here $x = 2$ and $y = 2$, so that $r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ and

$$\alpha = \arctan\left|\frac{2}{2}\right| = \pi/4.$$

Here z lies in the 1st quadrant, therefore $\arg z = \theta = \pi/4$

Using the formula for polar form we get $Z = 2 + 2i = 2\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

- Calculations

✓ Polar form of product and quotient of two complex numbers

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then their product z_1z_2 is given by:

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then their quotient $\frac{z_1}{z_2}$ is given by:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

Example1

Find the product of the complex numbers $z_1 = 4(\cos 32^\circ + i\sin 32^\circ)$ and $z_2 = 3(\cos 61^\circ + i\sin 61^\circ)$. Leave the answer in polar form.

Solution

$$z_1z_2 = r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$
$$z_1z_2 = (4)(3)\left[\cos(32^\circ+61^\circ) + i\sin(32^\circ+61^\circ)\right]$$
$$z_1z_2 = 12\left[\cos 93^\circ + i\sin 93^\circ\right]$$

Example2

Find the quotient of the complex numbers $z_1 = 12(\cos 84° + i \sin 84°)$ and $z_2 = 3(\cos 35° + i \sin 35°)$. Leave the answer in polar form.

Solution

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{12}{3}\left[\cos(84° - 35°) + i\sin(84° - 35°)\right]$$

$$\frac{z_1}{z_2} = 4\left[\cos 49° + i\sin 49°\right]$$

- ✓ Power of complex number in polar form and De Moivre's theorem

If $z = r(\cos\theta + i\sin\theta)$ then raising the complex number to a power is given by DeMoivre's Theorem:

$$z^n = r^n(\cos n\theta + i\sin n\theta);\ \text{where n is a positive integer}$$

Example 1

Use DeMoivre's Theorem to find the 5^{th} power of the complex number $z = 2(\cos 24° + i \sin 24°)$. Express the answer in the rectangular form a + bi.

Solution

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$z^5 = 2^5\left[\cos 5(24°) + i\sin 5(24°)\right]$$

$$z^5 = 32(\cos 120° + i\sin 120°)$$

$$z^5 = 32\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z^5 = 16\sqrt{3} + 16i$$

Example 2

Express each of the following in terms of cos x or sin x

(a) $\cos 2x$ (b) $\sin 2x$ (c) $\cos 3x$ (d) $\sin 3x$

Solution

$$\cos 2x + i \sin 2x = (\cos x + i \sin x)^2$$
$$= (\cos^2 x - \sin^2 x) + (2 \sin x \cos x)\, i$$

We obtain:

$$\begin{cases} \cos 2x = \cos^2 x - \sin^2 x \\ \sin 2x = 2 \sin x \cos x \end{cases}$$

(a) $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$

$= 2 \cos^2 x - 1$

(b) $\sin 2x = 2 \sin x \cos x$

$$\cos 3x + i \sin 3x = (\cos x + i \sin x)^3$$
$$= (\cos^3 x - 3 \cos x \sin^2 x) + (3 \cos^2 x \sin x - \sin^3 x)i$$

(c) $\cos 3x = \cos^3 x - 3 \cos x \sin^2 x$

$= \cos^3 x - 3 \cos x\, (1 - \cos^2 x)$

$= 4 \cos^3 x - 3 \cos x$

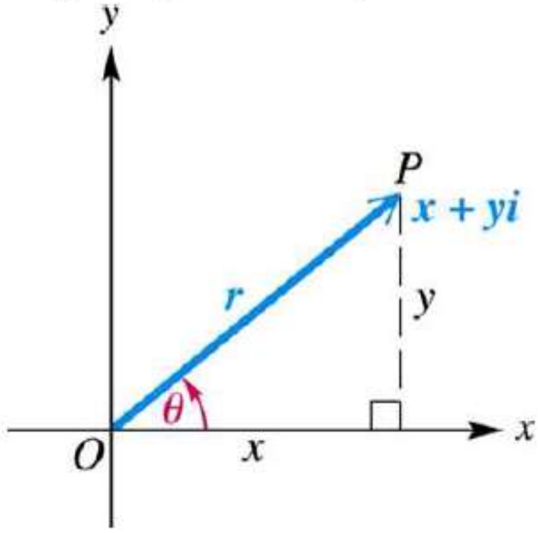
(d) $\sin 3x = 3 \cos^2 x \sin x - \sin^3 x$

$= 3\, (1 - \sin^2 x) \sin x - \sin^3 x$

$= 3 \sin x - 4 \sin^3 x.$

✓ Changing a complex number from polar form to algebraic form and vice versa

We sketch a vector with initial point $(0,0)$ and terminal point $P(x, y)$. The length r of the vector is the **absolute value** or **modulus** of the complex number and the angle θ with the positive x-axis is the is called the **direction angle** or **argument** of $x + yi$.



Conversions between rectangular and polar form follows the same rules as it does for vectors.

Rectangular to Polar

For a complex number $x + yi$

$$|x + yi| = r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x},\ x \neq 0$$

Polar to Rectangular

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The polar form $r(\cos\theta + i\sin\theta)$ is sometimes abbreviated

$$r\,\mathrm{cis}\,\theta$$

Example

Convert $-\sqrt{3} + i$ to polar form.

Solution

$x = -\sqrt{3}$ and $y = 1$ so that

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$$

and

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Here the reference angle and for θ is $30°$. Since the complex number is in QII, we have

$$\theta = 180° - 30°$$

$$\theta = 150°$$

So that $-\sqrt{3} + i = 2\,\mathrm{cis}\,150°$. In radian mode, we have

$$-\sqrt{3} + i = 2\,\mathrm{cis}\,\frac{5\pi}{6}$$

Polar to rectangular

Example

Converting polar to rectangular form is straightforward.

$$\begin{aligned} 4\operatorname{cis}240^\circ &= 4\cos 240^\circ + i\sin 240^\circ \\ &= 4\left(-\frac{1}{2}\right) + i\left(-\frac{\sqrt{3}}{2}\right) \\ &= -2 - 2i\sqrt{3} \end{aligned}$$

Note that the i follows an integer or fraction but precedes a radical, an "unwritten rule" of mathematical writing style.

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